ON A CLASS OF c-SAMPLE WEIGHTED RANK-SUM TESTS FOR LOCATION AND SCALE[1]

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Summary

A class of c-sample ($c \geqq 2$) non-parametric tests for the homogeneity of location or scale parameters is proposed and their various properties studied. These tests are based on a family of congruent interquantile numbers, and may be regarded as the c-sample extension of a class of two sample tests, proposed and studied by Sen [15]. A useful theorem on the asymptotic distribution of the proposed class of statistics is established. With the aid of this result, the asymptotic power-efficiency of the proposed class of test is studied and comparison is made with other test procedures. Location-free scale tests are also considered.

1. Introduction

Let $X_{i1}, \cdots, X_{in_i}$ be n_i independent and identically distributed random variables constituting the ith sample, drawn from a population having a continuous cumulative distribution function (cdf) $F_i(x)$ $(i=1, \cdots, c)$; all these c samples are again assumed to be mutually independent. It is desired to test the null hypothesis

$$H_0: F_1(x) \equiv \cdots \equiv F_c(x) \equiv F(x) \text{ (say)}, \tag{1.1}$$

against translation or scale type of alternatives, which we may pose as follows.

In the translation type of alternatives, we let

$$F_i(x) = F(x+\theta_i), \quad i=1, \cdots, c \tag{1.2}$$

where $\theta_1, \cdots, \theta_c$ are all real and finite. We are then interested in testing

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H_0 in (1.1), against the class of alternatives

$$H_L : \sup_{(i,j)} |\theta_i - \theta_j| > 0. \tag{1.3}$$

Again, for scale type of alternatives, we let

$$F_i(x) = F([x-\mu]/\delta_i), \ i=1, \cdots, c, \tag{1.4}$$

where μ is any real quantity and $\delta_1, \cdots, \delta_c$ are positive constant. Then, we frame the null hypothesis H_0 as in (1.1), against the class of alternatives

$$H_S : \sup_{(i,j)} |\delta_i - \delta_j| > 0. \tag{1.5}$$

In spite of quite significant growth of the literature on non-parametric methods in the various types of two sample problems, the development of the theory in the general case of c sample problems seems to be comparatively inadequate. Only a few non-parametric contenders of standard parametric procedures are available in this case. Mention may be made, in particular, to the works of Kruskal and Wallis [10], Mood and Brown [11], Terpestra [18], Kiefer [7], Dwass [6], Bhapkar [2], Despande [5], Sen [14], Puri [12], among others. Most of the works relate specifically to the c-sample location problem, and the efficiency aspects of these tests have been studied by Andrews [1], Bhapkar [2], Sen [14], and Puri [12], among others. The c-sample scale problem seems to be rather inadequately investigated, and the two tests considered by Sen [14] and Terpestra [18] can not be regarded as very satisfactory from the power-efficiency standpoint. Very recently, Puri [13] has considered a class of c-sample scale tests of the Chernoff-Savage [3] type and has also studied their asymptotic efficiency. Kiefer [7] has considered the multisample analogue of Kolmogorov-Smirnov and Cramér-von Mises tests, which are applicable for testing any divergence of the c different cdf's. The extended class of quantile tests by Sen [14] has also been shown to be applicable in this situation, with slight modifications.

The object of the present investigation is to provide a class of c-sample non-parametric tests which are applicable for testing the homogeneity of either locations or scales, and to study the asymptotic power properties of these tests. Certain asymptotic power-equivalence relations of our tests with the extended Chernoff-Savage type of tests (considered by Puri ([12], [13])) are also established here for both location and scale alternatives.

2. Preliminary notions and the proposed test

Let us arrange the n_1 first sample observations in order of magnitude and denote these ordered variables by $X_{1(1)}, \cdots, X_{1(n_1)}$. Also for the sake of convenience, we let

$$X_{1(0)}=-\infty \quad \text{and} \quad X_{1(n_1+1)}=\infty. \tag{2.1}$$

Let us then define (as in Sen [15]) a set of (n_1+1) non-overlapping and contiguous cells $\{I_j : j=0, \cdots, n_1\}$ by

$$I_j : X_{1(j)}<x\leqq X_{1(j+1)},\ j=0, \cdots, n_1. \tag{2.2}$$

Also, let $r_{k,j}$ denote the number of observations of the kth sample belonging to the jth cell I_j, for $k=1, \cdots, c, j=0, \cdots, n_1$; so that

$$\begin{aligned} &\sum_{j=0}^{n_1} r_{k,j}=n_k \text{ for } k=1, \cdots, c; \\ &r_{1,j}=\begin{cases} 1, & \text{for } j=0, \cdots, n_1-1 \\ 0, & \text{for } j=n_1. \end{cases} \end{aligned} \tag{2.3}$$

Finally let us define a sequence of real numbers $\{a(j, n),\ j=0, \cdots, n\}$ for each positive integer n, and assume that this sequence satisfies the following three conditions.

(C. 1) For each n and all $j=0, \cdots, n$,

$$|a(j, n)|\leqq K\left\{\frac{(j+1)(n+1-j)}{(n+1)^2}\right\}^{\delta-1/2}, \text{ for some } \delta>0,\ K>0. \tag{2.4}$$

Then, let

$$\begin{aligned} &\bar{a}_n=\sum_{j=0}^{n} a(j, n)/(n+1), \\ &\sigma^2_{a_n}=\sum_{j=0}^{n} \{a^2(j, n)/(n+1)\}-\bar{a}^2_n. \end{aligned} \tag{2.5}$$

It, then, follows from Sen [15, p. 118] that $\bar{a}_n$ and $\sigma^2_{a_n}$ are both bounded, even when $n\to\infty$.

(C. 2) $\bar{a}_n$ as well as $\sigma^2_{a_n}$ converges to some finite limit as $n\to\infty$, and we denote by

$$\bar{a}=\lim_{n=\infty} \bar{a}_n, \quad \sigma^2_a=\lim_{n=\infty} \sigma^2_{a_n}. \tag{2.6}$$

(C. 3) For each finite n as well as when $n\to\infty$, $\sigma^2_{a_n}$ is positive, that is,

$$\sigma^2_{a_n}>0 \text{ for all } n, \text{ and } \sigma^2_n>0. \tag{2.7}$$

Condition (C. 3) implies that $a(j, n)$ is not a constant.

Let us now define

$$S_{N,k}=\sum_{j=0}^{n_1} a(j, n_1) r_{k,j}/n_k, \quad k=1, \cdots, c, \tag{2.8}$$
$$N=\sum_{k=1}^{c} n_k ,$$

and

$$\bar{S}_N=\sum_{j=0}^{n_1} a(j, n_1)\Big(\sum_{k=1}^{c} r_{k,j}\Big)\Big/N \tag{2.9}$$
$$=\sum_{k=1}^{c} n_k S_{N,k}/N.$$

Then, our proposed test is based on

$$T_N=\Big[\sum_{k=1}^{c} n_k[S_{N,k}-\bar{S}_N]^2/\sigma^2_{a_{n_1}}. \tag{2.10}$$

In the particular case of two samples, the test based on T_N reduces to the two-tail test based on $S_{N,2}$ alone (considered by Sen [15]). It may be noted here that if we write

$$b(j, n_1)=a(j, n_1)-a(j+1, n_1) \text{ for } j=0, \cdots, n_1-1, \tag{2.11}$$
$$=a(n_1, n_1) \text{ for } j=n_1;$$

$$R_{k,j}=\sum_{l\leq j} r_{k,l},\ j=0, \cdots, n_1,\ k=1, \cdots, c, \tag{2.12}$$

it is then readily seen that

$$S_{N,k}=\frac{1}{n_k}\sum_{j=0}^{n_1} b(j, n_1)\ R_{k,j},\ k=1, \cdots, c. \tag{2.13}$$

Now, if we consider the pooled sample of size n_1+n_k, comprising the first and the kth samples, and denote by $l_{k,1}, \cdots, l_{k,n_1}$ the ranks of the n_1 first sample observations, then we have

$$l_{k,j}=j+R_{k,j-1}, \text{ for } j=1, \cdots, n_1+1, \tag{2.14}$$

where conventionally, we take $l_{k,(n_1+1)}=n_1+n_k$. Hence, it follows from (2.12), (2.13) and (2.14) that

$$S_{N,k}=\frac{1}{n_k}\sum_{j=0}^{n_1} b(j, n_1) l_{k,(j+1)}-\frac{n_1+1}{n_k}\bar{a}_{n_1},\ k=1, \cdots, c. \tag{2.15}$$

The first term on the right hand side of (2.15) represents (apart from the factor $1/n_k$) a weighted sum of the ranks of the first sample observations with respect to the kth sample, and the second term is a non-random constant. Further, it can be readily shown that T_N defined in (2.10) can be expressed as a positive definite quadratic form in $S_{N,2}, \cdots, S_{N,c}$, each of which is a weighted rank-sum of the first sample observations with respect to the corresponding sample being pooled with the first one. Hence, the test based on T_N has been termed here as c-sample weighted rank-sum tests.

It may be noted further that if the weight-function $\{a(j, n_1)\}$ be chosen appropriately then T_N or $S_{N,k}$ reduces to some well-known statistics. For example, if we take $a(j, n_1)=j/(n_1+1)$, for $j=0, \cdots, n_1$, $S_{N,k}$ reduces to the usual Wilcoxon statistic for the kth sample with respect to the first sample, for $k=2, \cdots, c$. Again, if we let

$$a(j, n_1)=\begin{cases}1 & \text{if } j\leqq[n_1/2] \\ 0 & \text{otherwise,}\end{cases} \tag{2.16}$$

($[s]$ being the largest integer contained in s); $S_{N,k}$ reduces to the median test criterion by Mathisen (cf. [14]) for the kth sample with respect to the first sample, and T_N reduces to the median test considered by Sen [14]. The class of c-sample interquantile test considered by Sen [14] also belongs to the type of tests proposed here.

3. Null distribution of T_N

The statistic T_N is based on $(c-1)n_1$ random variables $\{r_{k,j}, j=1, \cdots, n_1, k=2, \cdots, c\}$. Let us first consider the joint probability function of these integer valued random variables, under the null hypotheses (1.1). If we denote the usual multinominal coefficient by

$$\binom{N}{n_1, \cdots, n_c}=\frac{N!}{n_1!\cdots n_c!}, \tag{3.1}$$

it can be shown following a few simple steps that the joint probability function of $\{r_{k,j}, k=2, \cdots, c, j=1, \cdots, n_1\}$ is given by

$$\binom{N}{n_1, \cdots, n_c}^{-1}\prod_{j=0}^{n_1}\binom{\sum_{k=2}^{c} r_{k,j}}{r_{2,j}, \cdots, r_{c,j}}, \tag{3.2}$$

where $\sum_{k=1}^{c} n_k=N$, $\sum_{j=0}^{n_1} r_{k,j}=n_k$, $k=2, \cdots, c$.

Thus, if the probability function (3.2) be evaluated completely, the null distribution of T_N can be traced by direct computations, and this makes T_N to be exactly distribution-free. However, the labour of this numerical evaluation of the distribution of T_N increases prohibitively with the increase of $n_1, \cdots, n_c$ and c, and we are practically forced to adopt some simple limiting form of the distribution of T_N, which has been accomplished in the next section. It has been shown there that under the conditions (C. 1), (C. 2) and (C. 3), T_N has asymptotically a χ^2 distribution with $c-1$ degrees of freedom. Thus, in large samples, the test based on T_N may be carried out using the tail of a χ^2 distribution with $(c-1)$ d.f.

We consider here the first two moments of $S_{N,k}$, $k=2, \cdots, c$ and the expectation of T_N, as these will be subsequently required. It follows by more or less straight forward computations that

$$S_{N,1}=\bar{a}_{n_1}-\{a(n_1, n_1)-\bar{a}_{n_1}\}/n_1, \tag{3.3}$$
$$E(S_{N,k}|H_0\}=\bar{a}_{n_1}, \text{ for } k=2, \cdots, c;$$

$$\mathrm{Cov}(S_{N,k}\,;\, S_{N,q}|H_0)=\frac{\sigma^2_{a_{n_1}}}{n_1+2}\left\{\frac{n_k+\delta_{k,q}(n_1+1)}{n_k}\right\}, \tag{3.4}$$
$$k, q=2, \cdots, c;$$

where $\delta_{k,q}$ is the Kronecker delta.

Hence, by simple algebraic manipulations, we get that

$$E(T_N|H_0)=\frac{(c-1)(n_1+1)}{(n_1+2)}+\frac{(N-n_1)}{N}\left[\frac{\{a(n_1, n_1)-\bar{a}_{n_1}\}^2}{n_1\sigma^2_{a_{n_1}}}-\frac{1}{n_1+2}\right] \tag{3.5}$$
$$=(c-1)+0(n_1^{-2\delta}),$$

by condition (C. 1).

4. Joint asymptotic normality of $S_{N,k}$, $k=2, \cdots, c$

Let us define

$$u_{k,j}=F_k(X_{1(j+1)})-F_k(X_{1(j)}),\ j=0, \cdots, n_1,\ k=1, \cdots, c;$$
$$\bar{a}_k(X_1)=\sum_{j=0}^{n_1} a(j, n_1)u_{k,j},\ k=1, \cdots, c; \tag{4.1}$$
$$\mathscr{S}^2_k(X_1)=\sum_{j=0}^{n_1} a^2(j, n_1)u_{k,j}-[\bar{a}_k(X_1)]^2,\ k=1, \cdots, c.$$

Also, let

$$U_{N,k}=n_k^{1/2}[S_{N,k}-\bar{a}_k(X_1)]/\mathscr{S}_k(X_1),\ k=1, \cdots, c. \tag{4.2}$$

Finally, let us assume that asymptotically $n_1, n_2, \cdots, n_c$ are such that

$$\lim_{n=\infty} n_k/N=\lambda_k : 0<\lambda_1, \cdots, \lambda_c<1, \sum_{k=1}^{c}\lambda_k=1. \tag{4.3}$$

THEOREM 4.1. *If $F_1(x), \cdots, F_c(x)$ are all continuous and if $\{a(j, n_1), 1\leqq j\leqq n_1\}$ satisfies conditions (C. 1), (C. 2) and (C. 3) given in (2.4) through (2.7), then $U_N=(U_{N,1}, \cdots, U_{N,c})$ has asymptotically a c-variate* *normal distribution with a null mean vector and a unit dispersion matrix.*

PROOF. To the random variable $X_{k,\alpha}$, we attach a counter function $\phi(X_{k,\alpha}|X_1)$ defined in the following manner:

$$\phi(X_{k,\alpha}|X_1)=a(j,\, n_1) \text{ if } X_{k,\alpha}\in I_j, \quad \text{for } j=0,\cdots,n_1,\ \alpha=1,\cdots,n_k,\ k=1,\cdots,c. \tag{4.4}$$

Then, conditioned on $X_1=(X_{11},\cdots,X_{1n_1})$,

$$\{\phi(X_{k,\alpha}|X_1),\ \alpha=1,\cdots,n_k\} \tag{4.5}$$

forms a sequence of independent and identically distributed random variables, and for different $k(=2,\cdots,c)$ these $(c-1)$ sequences of random variables are row wise stochastically independent. Then, let us define

$$\psi(X_{k,\alpha}|X_1)=\{\phi(X_{k\alpha}|X_1)-\bar{a}_k(X_1)\}/\mathscr{S}_k(X_1), \tag{4.6}$$

for $\alpha=1,\cdots,n_k$, $k=1,\cdots,c$, and let

$$\bar{\psi}_k(X_1)=\sum_{\alpha=1}^{n_k}\psi(X_{k,\alpha}|X_1)/n_k,\ k=1,\cdots,c. \tag{4.7}$$

Now, it is easily seen that $n_k^{1/2}\bar{\psi}_k(X_1)=U_{N,k}$, for $k=1,\cdots,c$. Then from (4.6), we have

$$\begin{aligned}&E[\psi(X_{k,\alpha}|X_1)|X_1]=E[\psi(X_{k,\alpha}|X_1)]=0,\\ &V[\psi(X_{k,\alpha}|X_1)|X_1]=1,\\ &E\{|\psi(X_{k,\alpha}|X_1)^3|X_1\}\leqq Kn_1^{(1/2-\delta)},\ \text{uniformly in } X_1;\end{aligned} \tag{4.8}$$

for $\alpha=1,\cdots,n_k$, $k=1,\cdots,c$.

For any given X_1, $\bar{\psi}_1(X_1)$ is fixed, while $\bar{\psi}_2(X_1),\cdots,\bar{\psi}_c(X_1)$ are mutually independent random variables. Let now $t'=(t_1,\cdots,t_c)$ be a vector of real and finite quantities, and let $\mu(t)$ denote the joint characteristic function of $\{n_k^{1/2}\bar{\psi}_k(X_1),\ k=1,\cdots,c\}$. Then, we have

$$\begin{aligned}\mu(t)&=E\left\{\exp\left[i\sum_{k=1}^{c}t_k n_k^{1/2}\bar{\psi}_k(X_1)\right]\right\}\\ &=E\{\exp[it_1n_1^{1/2}\bar{\psi}_1(X_1)]\prod_{k=2}^{c}[\{\exp[it_kn_k^{1/2}\bar{\psi}_k(X_1)]|X_1\}]\}.\end{aligned} \tag{4.9}$$

Now

$$
\begin{aligned}
&E\{\exp[it_k n_k^{1/2}\bar{\phi}_k(X_1)]|X_1\} \\
&\quad=\prod_{\alpha=1}^{n_k} E\{\exp[i(t_k/n_k^{1/2})\phi(X_{k\alpha}|X_1)]|X_1\} \\
&\quad=\Big[E\Big\{1+\frac{it_k}{n_k^{1/2}}\phi(X_{k1}|X_1)-\frac{t_k^2}{2n_k}\phi^2(X_{k1}|X_1) \\
&\qquad+\frac{(it_k)^3}{n_k^{3/2}}\phi^3(X_{k1}|X_1)\exp[i(\theta_k t_k/n_k^{1/2})\phi(X_{k1}|X_1)]\Big|X_1\Big\}\Big]^{n_k}, \\
&\qquad(0<\theta_k<1),\ k=2,\cdots,c.
\end{aligned}
\tag{4.10}
$$

Using (4.8) and after some simplifications it can readily be shown that (4.10) reduces to

$$
\exp\left\{-\frac{1}{2}t_k^2+R_{k,N}(t)\right\},
\tag{4.11}
$$

where $|R_{k,N}(t)|=0(t_k^3/n_1^\delta)$ for any real and finite t_k and uniformly in X_1. Thus, from (4.9) and (4.11), we get that

$$
\mu(t)=\exp\left\{-\frac{1}{2}\sum_{k=2}^{c}t_k^2\right\}[E\{it_1n_1^{1/2}\bar{\phi}_1(X_1)\}+R_N^*(t)],
\tag{4.12}
$$

where for any real and finite t, $R_N^*(t)=0(n_1^{-\delta})\to 0$ as $n_1\to\infty$. Now

$$
\begin{aligned}
n_1^{1/2}\bar{\phi}_1(X_1)&=n_1^{1/2}\left\{\sum_{j=0}^{n_1-1}a(j,n_1)/n_1-\sum_{j=0}^{n_1}a(j,n_1)u_{1,j}\right\}\Big/\mathscr{S}_1(X_1) \\
&=-\frac{\sigma_{a_{n_1}}}{\mathscr{S}_1(X_1)}\left\{n_1^{1/2}\sum_{j=0}^{n_1}u_{1,j}\left[a(j,n_1)-\frac{1}{n_1}\sum_{j=0}^{n_1-1}a(j,n_1)\right]\Big/\sigma_{a_{n_1}}\right\}. \\
&=-\frac{\sigma_{a_{n_1}}}{\mathscr{S}_1(X_1)}n_1^{1/2}\sum_{j=0}^{n_1}a^*(j,n_1)u_{1,j},
\end{aligned}
\tag{4.13}
$$

where $a^*(j,n_1)=[a(j,n_1)-(1/n)\sum_{j=0}^{n_1-1}a(j,n_1)]/\sigma_{a_{n_1}}$ for $j=0,\cdots,n_1$. Thus, it is readily seen that

$$
\begin{aligned}
&\sum_{j=0}^{n_1}a^*(j,n_1)=\frac{n_1+1}{n_1}[a(n_1,n_1)-\bar{a}_{n_1}]/\sigma_{a_{n_1}}=0(n_1^{1/2-\delta}), \\
&\sum_{j=0}^{n_1}[a^*(j,n_1)]^2=(n_1+1)[1+0(n_1^{-\delta})], \\
&\sum_{j=0}^{n_1}|a^*(j,n_1)|^r=0(n_1^{1+(r-2)(1/2-\delta)}),\ \text{for } r>2.
\end{aligned}
\tag{4.14}
$$

Again, it is easily shown that

(4.15) $E\left\{\prod_{j=1}^{k} u_{1,\alpha_j}^{\mathscr{S}_j}\right\}=\left\{\prod_{j=1}^{k}\mathscr{S}_j!\right\}\Big/(n_1+\mathscr{S})^{[\mathscr{S}]}$; $p^{[q]}=p(p-1)\cdots(p-q+1)$,

where $\mathscr{S}_j(\geqq 0)$ are non-negative integers, $\mathscr{S}=\sum_{j=1}^{k}\mathscr{S}_j$, and $\alpha_1,\cdots,\alpha_k$ are any $k(\geqq 1)$ numbers from $(0, 1, \cdots, n_1)$. Hence, using (4.14) and (4.15) and proceeding precisely on the same line as in the proof of lemma 3.2 and the results in (4.14) and (3.15) of Sen [15], it can be shown that

$$E\left\{\left[n_1^{1/2}\sum_{j=0}^{n_1} a^*(j, n_1)u_{1j}\right]^{2k}\right\}=\frac{(2k)!}{k!\,2^k}+0(n_1^{-\delta}),$$

(4.16)

$$E\left\{\left[n_1^{1/2}\sum_{j=0}^{n_1} a^*(j, n_1)u_{1j}\right]^{2k+1}\right\}=0(n_1^{-\delta}),\ \text{for } k=0, 1, \cdots;$$

and hence, $n_1^{1/2}\sum_{j=0}^{n_1} a^*(j, n_1)u_{1,j}$ has asymptotically a normal distribution with zero mean and unit standard deviation. Again,

$$\mathscr{S}_1^2(X_1)=\sum_{j=0}^{n_1} a^2(j, n_1)-[\bar{a}_1(X_1)]^2, \tag{4.17}$$

and from the asymptotic normality of $n^{1/2}\sum_{j=0}^{n_1} a^*(j, n_1)u_{1,j}$, we get, following a few simple steps that

(4.18) $n_1^{1/2}|\bar{a}_1(X_1)-\bar{a}_{n_1}|$ is bounded in probability.

Also

$$V\left\{\sum_{j=0}^{n_1} a^2(j, n_1)u_{1,j}\right\}$$

(4.19)

$$=\sum_{j=0}^{n_1} a^4(j, n_1)/(n_1+1)(n_1+2)-\left(\sum_{j=0}^{n_1} a^2(j, n_1)/(n_1+1)\right)^2$$

$$=0(n_1^{-2\delta}),\ \text{by condition (C. 1)};$$

and hence

$$\sum_{j=0}^{n_1} a^2(j, n_1)\left[u_{1,j}-\frac{1}{n_1+1}\right]\xrightarrow{P}0. \tag{4.20}$$

Consequently, from (4.17), (4.18), and (4.20), we get that

$$\mathscr{S}_1^2(X_1)\xrightarrow{P}\sigma_{a_{n_1}}^2\longrightarrow\sigma_a^2>0. \tag{4.21}$$

Hence, from (4.13), (4.16) and (4.21), we get, on applying a well-known convergence theorem by Cramér [4, p. 253] that $n^{1/2}\bar{\phi}(X_1)$ has asymptotically a normal distribution with zero mean and unit standard deviation. Thus, from (4.12), we get that

(4.22) $$\mu(\boldsymbol{t})=\exp\left\{-\frac{1}{2}\sum_{k=1}^{c} t_k^2\right\}+R_N(\boldsymbol{t}),$$

where $R_N(\boldsymbol{t})\to 0$ as $N\to\infty$, for any real and finite $\boldsymbol{t}$. Thus, $\{n_k^{1/2}\bar{\phi}_k(\boldsymbol{X}_1), k=1,\cdots,c\}$ has asymptotically a c-variate normal distribution with a null mean vector and a unit dispersion matrix. The proof of the theorem is then completed by noting that $U_{N,k}=n_k^{1/2}\bar{\phi}_k(\boldsymbol{X}_1)$ for $k=1,\cdots,c$.

THEOREM 4.2. *If $F_1(x)\equiv\cdots\equiv F_c(x)\equiv F(x)$ is continuous and if $\{a(j,n_1), 1\leqq j\leqq n_1\}$ satisfies the conditions (C. 1), (C. 2) and (C. 3) in (2.4) through (2.7), then T_N, defined in (2.10), has asymptotically chi-square distribution with $(c-1)$ degrees of freedom.*

PROOF. From $\boldsymbol{U}_N$ we transform to a new vector valued variable $\boldsymbol{V}_N$ by means of an orthogonal transformation, where we take

(4.23) $$V_{N,1}=\sum_{k=1}^{c}(n_k/n)^{1/2}U_{N,k},$$

and $V_{N,2},\cdots,V_{N,c}$ arbitrarily. Let us also apply the co-gradient transformation to $\boldsymbol{t}$ and denote the transformed vector by $\boldsymbol{\tau}$. Then the characteristic function of $\boldsymbol{V}_N$ can be shown to be (using (4.22))

(4.24) $$\exp\left\{-\frac{1}{2}\sum_{k=1}^{c}\tau_k^2\right\}+R_N(\boldsymbol{\tau}),$$

where $R_N(\boldsymbol{\tau})\to 0$ as $N\to\infty$, for any real and finite $\boldsymbol{\tau}$. Consequently, it follows from a well-known theorem on the limit distribution of a continuous function of random variables (cf. Sverdrup [17]) that

(4.25) $$\sum_{k=2}^{c}V_{N,k}^2=\sum_{k=1}^{c}U_{N,k}^2-V_{N,1}^2=\sum_{k=1}^{c}[U_{N,k}-(n_k/N)^{1/2}V_{N,1}]^2$$

has asymptotically a chi-square distribution with $(c-1)$ d.f. Also, when $F_1\equiv\cdots\equiv F_c$, we have

(4.26) $$\mathscr{S}_k^2(\boldsymbol{X}_1)=\mathscr{S}_1^2(\boldsymbol{X}_1)\xrightarrow{P}\sigma_{a_{n_1}}^2 \quad\text{for all } k=1,\cdots,c.$$

Hence, from (4.2), (4.25) and (4.26), we get that

(4.27) $$\sum_{k=2}^{c}V_{N,k}^2\xrightarrow{P}\sum_{k=1}^{c}n_k[S_{N,k}-\bar{S}_N]^2/\sigma_{a_{n_1}}^2=T_N.$$

Consequently, T_N has asymptotically a chi-square distribution with $(c-1)$ d.f.

5. Asymptotic non-null distribution of T_N

We shall now consider the limiting distribution of T_N under a sequence of alternative hypotheses which relate to heterogeneity of locations and scales, and for which the power of the test based on T_N lies in the open interval (0, 1). Thus, in this section, we will take c sequences of distributions, each converging to a common distribution, in the following manner:

$$H_N : F_{k,N}(x)=F(x+N^{-1/2}[\alpha_k+\beta_k x]),\ k=1, \cdots, c, \tag{5.1}$$

where $\boldsymbol{\alpha}=(\alpha_1, \cdots, \alpha_c)$ and $\boldsymbol{\beta}=(\beta_1, \cdots, \beta_c)$ are real and finite vectors, at least one of them being non-null, N is the total sample size, and $F_{k,N}$ replaces F_k, $k=1, \cdots, c$.

Also, let $\mathscr{F}$ be the class of all absolutely continuous cumulative distribution functions with continuous density functions, which are assumed to be positive over the range of variation of x, and for which either (a) the range of x is finite, or (b) if it extends to infinity in at least one of the extremities (say right), it satisfies the following two conditions:

(i) *Smoothness condition.* Let

$$f(x)=-\frac{d[1-F(x)]}{dx} \quad \text{and} \quad \phi(x)=\frac{d\log[1-F(x)]}{dx}. \tag{5.2}$$

Then, in the right hand tail, $f(x)$ and $\phi(x)$ are both monotonic. Obviously, $f(x)$ is $\downarrow$ while $\phi(x)$ may be $\uparrow$ or $\downarrow$ or a constant.

(ii) *Gradient conditions.* $x\phi(x)$ either tends to a non-zero limit $a(>0)$ or it is non-decreasing in the tail and tends to ∞ as $x\to\infty$.

If the range extends to $-\infty$ in the lower extremity, we assume that similar conditions hold for $f(x)=F'(x)$ and $\phi(x)=f(x)/F(x)$.

It may be noted that the gradient condition is satisfied by all cdf's having a finite δth order moment, for some $\delta>0$, and in fact, we have the following lemma.

LEMMA 5.1. *If for any absolutely continuous cdf $F(x)$, $x\phi(x)\to 0$ as $x\to\infty$, then $F(x)$ has no moment of any finite order.*

The proof of this lemma along with that of the following lemma is given in the appendix.

LEMMA 5.2. *If $X_{(1)}<\cdots<X_{(n)}$ be the order statistic in a random sample of size n drawn from a population with the cdf $F(x)$, and if $F(x)\in\mathscr{F}$, then, for all real and finite (α, β)*

$$f(x+n^{-1/2}[\alpha+\beta x])/f(x)=1+o_p(1),$$

for all $x \in [x_{(1)}, x_{(n)}]$.

Let us define $\{b(j, n_1),\ 0 \leqq j \leqq n_1\}$ as in (2.11) and assume further that the following condition holds:

(C. 4) For all $0 < j \leqq n_1$, if we let $n_1 \to \infty$ subject to $j/(n_1+1) \to u$ $0 < u < 1$, then

$$n_1 b(j, n_1) \to b(u), \tag{5.3}$$

where $b(u)$ is continuous and integrable with respect to u.

Let now $\mathscr{F}_0 \subset \mathscr{F}$ be a subclass of cdf's $F(x) \in \mathscr{F}$, for which

$$\left| \int_{-\infty}^{\infty} b(F(x)) x f^2(x) dx \right| < \infty, \text{ if } \beta \text{ is non-null}, \tag{5.4}$$

and

$$\left| \int_{-\infty}^{\infty} b(F(x)) f^2(x) dx \right| < \infty, \text{ if } \beta \text{ is null}. \tag{5.5}$$

Let us then define

$$\theta_k = \int_{-\infty}^{\infty} (\alpha_k + \beta_k x) b(F(x)) f^2(x) dx,\ k = 1, \cdots, c; \tag{5.6}$$

$$\lambda_k = \lim_{N=\infty} (n_k/N) : 0 < \lambda_1, \cdots, \lambda_c < 1, \sum_1^c \lambda_k = 1, \tag{5.7}$$

and

$$\Delta = \sum_{k=1}^{c} \lambda_k (\theta_k - \bar{\theta})^2 / \sigma_a^2. \tag{5.8}$$

Then, we have the following theorem.

THEOREM 5.3. *Under the sequence of alternatives* $\{H_N\}$ *given by (5.1),* T_N *has asymptotically a non-central chi-square distribution with (c−1) degrees of freedom and the non-centrality parameter* Δ *(defined in (5.8)), provided* $\{a(j, n_1);\ j = 0, \cdots, n_1\}$ *satisfies the conditions (C. 1), (C. 2), (C. 3) and (C. 4) (which are stated in (2.4), (2.5), (2.6), (2.7) and (5.3)) and* $F(x) \in \mathscr{F}_0$.

PROOF. Let us define $\mathscr{S}_k^2(X_1)$ as in (4.1) and prove that under the conditions stated in the theorem

$$|\mathscr{S}_k^2(X_1) - \mathscr{S}_1^2(X_1)| \xrightarrow{P} 0, \text{ for all } k = 2, \cdots, c. \tag{5.9}$$

To prove (5.9), it follows from (4.1) that it is sufficient to show that

(5.10) $$\left|\sum_{j=0}^{n_1} a^2(j, n_1)(u_{k,j}-u_{i,j})\right| \xrightarrow{P} 0, \quad \text{for all } k=2, \cdots, c\,; \text{ i.e.,}$$

(5.11) $$\left|\sum_{j=1}^{n_1} \{a^2(j-1, n_1)-a^2(j, n_1)\}\{F_{k,N}(X_{1(j)})-F_{1,N}(X_{1(j)})\}\right| \xrightarrow{P} 0,$$

for all $k=2, \cdots, c$.

It now follows from (2.11), (5.1), (5.3) and lemma 5.2 after some essentially simple adjustments that the left hand side of (5.10) or (5.11) reduces to

(5.12) $$\frac{c}{N^{\delta}}\left\{\int_0^1 |b(F_n(x))(\alpha_k^* + \beta_k^* x)| f(x) dF_n(x) + 0_p(1)\right\}, \quad c<\infty,$$

where $\alpha_k^* = \alpha_k - \alpha_1$, $\beta_k^* = \beta_k - \beta_1$, $k=2, \cdots, c$ and where $F_{n_1}(x)$ is the first sample empirical cdf. Consequently, by (5.5) and (5.12), we get that (5.11) is valid and hence, from (4.21) we obtain

(5.13) $$\mathscr{S}_k^2(X_1) \overset{P}{\sim} \mathscr{S}_1^2(X_1) \xrightarrow{P} \sigma_a^2 .$$

Also, it follows more or less similarly that

(5.14) $$\begin{aligned} N^{1/2}\{\bar{a}_k(X_1)-\bar{a}_1(X_1)\} &\xrightarrow{P} \int(\alpha_k^* + \beta_k^* x) f(x) b(F(x)) dF(x) \\ &= \theta_k - \theta_1, \qquad \text{for } k=2, \cdots, c. \end{aligned}$$

Now, looking at the statistic $\sum_{k=2}^{c} V_{N,k}^2$, defined in (4.25), and substituting the expression for $U_{N,k}$ as in (4.2), we get that

(5.15) $$\begin{aligned} \sum_{k=2}^{c} V_{N,k}^2 &= \sum_{k=2}^{c} n_k[S_{N,k} - \bar{S}_N - (a_k(\tilde{X}_1) - \bar{a}_1(X_1))]^2 / \mathscr{S}_k^2(X_1) \\ &\xrightarrow{P} \frac{1}{\sigma_a^2} \sum_{k=2}^{c} n_k [S_{N,k} - \bar{S}_N - N^{-1/2}(\theta_k - \theta_1)]^2. \end{aligned}$$

Since, $\sum_{k=2}^{c} V_{N,k}^2$ has (by theorem 4.2) a chi-square distribution with $(c-1)$ degrees of freedom when N is large and (5.7) holds, it follows from a well-known theorem on the limiting distribution of a continuous function of random variables (cf. Sverdrup [17]) that

(5.16) $$(1/\sigma_a^2) \sum_{k=1}^{c} n_k (S_{N,k} - \bar{S}_N)^2$$

has asymptotically a non-central chi-square distribution with $(c-1)$ d.f.

and the non-centrality parameter Δ, defined in (5.8). The proof of the theorem is then completed by noting that T_N converges in probability to the statistic given in (5.16).

It may be noted that there is some similarlity of the regularity conditions pertaining to the asymptotic chi-square distribution of T_N and extended Chernoff-Savage type of statistics (cf. Puri [12]). However, the comparison of these regularity conditions follows more or less on the same line as in the two sample case, considered by Sen [15], and hence, is omitted.

6. Asymptotic efficiency of T_N-test

Since, it has been shown that under H_0, T_N has asymptotically a chi-square distribution with $(c-1)$ degrees of freedom, and under H_N, it has a non-central chi-square distribution with the same degrees of freedom and the non-centrality parameter Δ, defined in (5.8), it follows that if S_N is any other test-statistic having a similar distribution-property (both under H_0 and H_N), the ratio of the two non-centrality parameters will give us the usual measure of the asymptotic efficiency of one test with respect to the other.

We consider first the location problem, and state the sequence of alternatives $\{H_N\}$ as

$$F_{k,N}(x)=F(x+\alpha_k/N^{1/2}),\ k=1,\cdots,c\,. \tag{6.1}$$

In the parametric (normal) case, it is known that the usual analysis of variance test has some optimum properties, and the test criterion (F) has asymptotically a non-central chi-square distribution (under H_N) with $(c-1)$ degrees of freedom and the non-centrality parameter

$$\Delta^0=\sum_{k=1}^{c}\lambda_k(\alpha_k-\bar{\alpha})^2/\sigma^2$$

where

$$\sigma^2=\int x^2dF(x)-\left[\int xdF(x)\right]^2 \text{ and } \bar{\alpha}=\sum_{k=1}^{c}\lambda_k\alpha_k.$$

Hence, it follows from theorem 5.3 that the asymptotic efficiency of T_N-test with respect to the F-test is

$$\frac{\sigma^2}{\sigma_a^2}\left[\int_{-\infty}^{\infty}b(F(x))f^2(x)dx\right]^2, \tag{6.2}$$

which does not depend on c, the number of independent samples. Also, we note that if we let $b(F(x))=(d/dF)J(F(x))$, where $J(F(x))$ has been defined by Chernoff and Savage [3] for their test, then (6.2) becomes identical with the expression for the asymptotic efficiency of their test. Hence, proceeding precisely on the same line as in Puri [12], we can present the expression for the asymptotic efficiency of T_N-test for various suggested or known weight functions $\{a(j, n_1), j=0, \cdots, n_1\}$. For brevity of our discussion here, these are therefore not done in detail.

Let us next consider the scale problem. Here $\{H_N\}$ relates to the sequence of alternatives

$$F_{k,N}(x)=F(x[1+\beta_k/N^{1/2}]),\ k=1, \cdots, c. \tag{6.3}$$

In the parametric case, Bartlett's test for the homogeneity of variances is the mostly used test. This test criterion has asymptotically a non-central chi-square distribution (under H_N) with $(c-1)$ degrees of freedom with the non-centrality parameter Δ^* given by

$$\Delta^*=\frac{4}{\gamma_2+2}\sum_{k=1}^{c}\lambda_k(\beta_k-\bar{\beta})^2,\ \ \bar{\beta}=\sum_{k=1}^{c}\lambda_k\beta_k$$

where $\gamma_2=\mu_4/\mu_2^2-3$, $\mu_k=\int[x-E(x)]^k dF(x)$, $k=1, 2, 3, 4$. Thus, the asymptotic efficiency of T_N-test with respect to Bartlett's test turns out to be

$$\frac{\gamma_2+2}{4\sigma_a^2}\left[\int_{-\infty}^{\infty}xb(F(x))f^2(x)dx\right]^2, \tag{6.4}$$

which is also independent of c and hence, all the efficiency values obtained in the two sample case by Klotz [8], Sen [15] and others, remain valid in the general class of c-samples. Here also, by letting $b(F(x))=(d/dF)J(F(x))$, where $J(F(x))$ is the Chernoff-Savage type of weight-function, we are able to have the same asymptotic power efficiency for both the classes of tests.

It is thus seen that even in the c-sample case, the weighted rank-sum tests (proposed here), based on the first-sample weight functions and the weighted rank-sum tests based on the pooled sample weight functions, (considered by Puri [12]) are both asymptotically power equivalent for both location and scale alternatives, if the same weight functions are used in both the cases. From computational aspects, the ordering of the observations appear to be less tedious in any single sample, than in the pooled sample and hence, T_N test, proposed here, appears to be computationally preferable. The evaluation of the critical value of T_N for very small samples also appears to be relatively easier here.

7. Location-free scale test

Suppose now we are interested in testing the homogeneity of the scale parameters of the c populations, without assuming the equality of the associated location parameters. In the case of two samples, Sen [15] has considered this problem and imposed the regularity conditions of Sukhatme (see Sen [15], p. 131) on the underlying cdf's and the estimates of the population location parameters, under which the weighted rank-sum scale tests when thus modified, will be asymptotically distribution-free. In the c-sample case also, the same regularity conditions apply to the population cdf's and the estimates of the location parameters, for which the modified scale-test will be asymptotically distribution-free. For the intended brevity of our discussion, this is not considered in detail.

8. Appendix

PROOF OF LEMMA 5.1. From (5.2), we get on writing $\phi(x)=\psi'(x)$ that

$$1-F(x)=e^{-\psi(x)} \quad \text{and} \quad f(x)=\phi(x)e^{-\psi(x)}. \tag{8.1}$$

We are to prove that for any $\delta>0$, $E(x^\delta)$ does not exist. Since $x\phi(x)\to 0$ as $x\to\infty$; given any $\delta>0$, we can always find a value of x, say x_δ, such that for $x\geqq x_\delta>0$, $x\phi(x)<\delta/2$. Then for $x>x_\delta$

$$\psi(x)-\psi(x_\delta)=\int_{x_\delta}^{x}\phi(x)dx<\frac{\delta}{2}\log(x/x_\delta)$$

or

$$e^{-\psi(x)}\geqq x^{-\delta/2}\{e^{-\psi(x_\delta)}x_\delta^{\delta/2}\}=kx^{-\delta/2},$$

where k is a finite quantity. Thus, from (8.1), we get

$$x^\delta[1-F(x)]=x^\delta e^{-\psi(x)}\geqq kx^{\delta/2}\to\infty \text{ as } x\to\infty.$$

Consequently, for $x\geqq x^\delta$

$$\int_x^\infty x^\delta dF(x)\geqq x^\delta[1-F(x)]\geqq kx^{\delta/2}$$

does not converge as $x\to\infty$, and hence, $E(x^\delta)$ does not exist. This completes the proof of the lemma.

PROOF OF LEMMA 5.2. If the range of x is finite and $f(x)>0$ in this domain, the proof is trivial. So we consider the case where the range of x tends to ∞ on the right. The case with the infinite lower extremity will follow similarly.

By virtue of the smoothness and gradient conditions, we assume that there exists a value of x, say x_0, such that for all $x\geqq x_0$

(i) $f(x)$ is $\downarrow$ and $\phi(x)(>0)$ is $\downarrow$ or $\uparrow$ or a constant;
(ii) $x\phi(x)\to a$ as $x\to\infty$ or $x\phi(x)$ is $\uparrow$ and it tends to ∞, as $x\to\infty$,
(iii) $F(x_0)=p_0,\ 0<p_0<1$.

As the lemma follows trivially for any fixed value of x, we require to prove it only for tail values of x. Thus, we require to show that for all $x_0<x\leqq X_{(n)}$

(8.2) $$\{\phi(x+n^{-1/2}[\alpha+\beta x])/\phi(x)\}\exp\{-[\psi(x+n^{-1/2}[\alpha+\beta x])-\psi(x)]\}\\ =1+o_p(1).$$

Let us now define a sequence of values of x, say $\{x_n^*\}$, such that

(8.3) $$1-F(x_n^*)=cn^{-(1+\delta)},\ \delta>0,\ c\geqq 1.$$

It is then easily seen that

(8.4) $$P\left\{\bigcup_1^n [X_{(i)}>x_n^*]\right\}\to 0 \text{ as } n\to\infty.$$

We now classify $F(x)$ according to the nature of $\phi(x)$ as follows (cf. Sen [16]):

Type I : $\phi(x)\to\infty,\ x\phi(x)\to\infty$ as $x\to\infty$,
Type II : $\phi(x)\to c<\infty,\ x\phi(x)\to\infty$ as $x\to\infty$,
Type III: $\phi(x)\to 0$ but $x\phi(x)\to\infty$ as $x\to\infty$,
Type IV: $x\phi(x)\to a<\infty$ as $x\to\infty$.

Let us first prove that for the first three types of cdf's

(8.5) $$x_n^*\phi(x_n^*)=o(n^{1/2}),\ x_n^*=o(n^{1/2}).$$

For type I cdf's, $E(xe^{t\phi(x)})$ exists for all $0<t<1$, and this implies that $x\phi(x)e^{-(1-t)\psi(x)}\to 0$ as $x\to\infty$. Thus, for any fixed t, $x\phi(x)$ is at most of the order $[\exp(1-t)\psi(x)]$. Since, $\psi(x_n^*)=(1+\delta)\log n-\log c$ (by (8.3)), we get on choosing t appropriately that

(8.6) $$x_n^*\phi(x_n^*)=0(\log n)=o(n^{1/2}).$$

Also, $\phi(x)\to\infty$ as $x\to\infty$, and hence $x_n^*=o(n^{1/2})$. For type II cdf's, $\phi(x)\to c<\infty$ as $x\to\infty(c>0)$, and hence, it follows similarly that $x_n^*\phi(x_n^*)=o(n^{1/2})$, $x_n^*=o(n^{1/2})$. For type III cdf's, $\phi(x)\to 0$ but $x\phi(x)\uparrow\to\infty$ as $x\to\infty$. Thus we can always select a value of x, say x_1, such that for all $x>x_1$

$$x\phi(x)\geqq c_1>(1+\delta)(2+\delta),\ \delta>0,$$

where δ has been defined in (8.3), and where $F(x_1)=p_1<1$. Then, for any fixed x_1, we have choosing n appropriately

$$\int_{x_1}^{x_n^*} \phi(x)dx \geqq c_1 \log(x_n^*/x_1)$$

or

$$\log(x_n^*/x_1) \leqq \frac{1}{c_1}[\psi(x_n^*) - \psi(x_1)] \leqq \frac{1}{c_1}\psi(x_n^*) \leqq \frac{(1+\delta)}{c_1}\log n$$

$$\text{i.e.}\quad x_n^* = O(n^{1/(2+\delta)}) = o(n^{1/2}) .$$

Further, as $\phi(x) \to 0$ with $x \to \infty$, $x_n^* \phi(x_n^*) = o(n^{1/2})$. Consequently, from the monotonicity of $x\phi(x) \uparrow$ for $x \geqq x_0$, we get that for type I, II and III cdf's

$$n^{-1/2}(\alpha + \beta x) = o(1) , \tag{8.7}$$

for all $x_0 < x \leqq x_n^*$, and hence,

$$\begin{aligned} &\psi(x + n^{-1/2}[\alpha+\beta x]) - \psi(x) \\ &= \frac{n^{-1/2}(\alpha+\beta x)}{x + \eta n^{-1/2}(\alpha+\beta x)} y\phi(y);\ y = x + n^{-1/2}(\alpha+\beta x)\eta , \qquad 0<\eta<1 . \\ &= o(1), \text{ uniformly in } x_0 < x \leqq x_n^* . \end{aligned} \tag{8.8}$$

Hence, from (8.2) and (8.8), we get that for all $x_0 < x \leqq x_n^*$

$$\frac{f(x+n^{-1/2}[\alpha+\beta x])}{f(x)} = \frac{\phi(x+n^{-1/2}[\alpha+\beta x])}{\phi(x)}[1+o(1)] . \tag{8.9}$$

Now the monotonicity of $f(x)\downarrow$ and $x\phi(x)\uparrow$ implies that

$$f(y)/f(x) \gtreqless 1 , \qquad \frac{y\phi(y)}{x\phi(x)} \lesseqgtr 1 , \qquad \text{according as } y \lesseqgtr 1 .$$

These two contrasted inequalities along with (8.9) imply that for all $x_0 < x < x_n^*$

$$\frac{f(x+n^{-1/2}[\alpha+\beta x])}{f(x)} = 1 + o(1) . \tag{8.10}$$

For type IV cdf's, it is easy to show that (8.8) holds for all positive x, since $x\phi(x) \to a$ as $x \to \infty$. Further,

$$\frac{\phi(x+n^{-1/2}[\alpha+\beta x])}{\phi(x)} = \frac{y\phi(y)}{x\phi(x)} \cdot \frac{x}{y} \tag{8.11}$$

where $y = x + n^{-1/2}[\alpha+\beta x]$, and hence it is easy to show that (8.11) can be made arbitrarily close to 1, for large x, and it has the limit 1 as $x \to \infty$. Thus, from (8.9) and (8.11), it follows that (8.10) holds.

Essentially the same proof applies to the lower extremity also. Hence, from (8.4) and (8.10), the lemma follows.

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REFERENCES

[1] F. C. Andrews, "Asymptotic behaviour of some rank tests for analysis of variance," *Ann. Math. Statist.*, 25 (1954), 724-735.

[2] V. P. Bhapkar, "A non-parametric test for the problem of several samples," *Ann. Math. Statist.*, 32 (1961), 1108-1117.

[3] H. Chernoff and I. R. Savage, "Asymptotic normality and efficiency of certain non-parametric test statistics," *Ann. Math. Statist.*, 29 (1958), 972-994.

[4] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.

[5] J. V. Deshpande, "A non-parametric test based on U-statistics for the problem of several samples," *Jour. Ind. Stat. Asso.*, 3 (1965), 20-29.

[6] M. Dwass, "Some k-sample rank order tests," *Contributions to Probability and Statistics*, Essays in honor of Harold Hotelling (Edited by I. Olkin et al) Stanford Univ. Press, 1960.

[7] J. Kiefer, "K-sample analogues of the Kolmogorov-Smirnov and Cramer-Von Mises tests," *Ann. Math. Statist.*, 30 (1959), 420-447.

[8] J. Klotz, "Non-parametric tests for scale," *Ann. Math. Statist.*, 33 (1962), 498-512.

[9] W. H. Kruskal, "A non-parametric test for the several sample problem," *Ann. Math. Statist.*, 23 (1952), 525-540.

[10] W. H. Kruskal and W. A. Wallis, "Use of ranks in one criterion analysis of variance," *Jour. Amer. Stat. Asso.*, 47 (1952), 583-621.

[11] A. M. Mood, *Introduction to the Theory of Statistics*, McGraw Hill, New York, 1950.

[12] M. L. Puri, "Asymptotic efficiency of a class of c-sample tests," *Ann. Math. Statist.*, 35 (1964), 102-121.

[13] M. L. Puri, "On some tests of homogeneity of variances," *Tech. Report* IMM-NYU 338, New York Univ. Courant Inst. Math. Sc., 1964.

[14] P. K. Sen, "On the role of a class of quantile tests in some non-parametric problems," *Calcutta Stat. Assoc. Bull.*, 11 (1962), 125-143.

[15] P. K. Sen, "On weighted rank-sum tests for dispersion," *Ann. Inst. Stat. Math.*, 15 (1963), 117-135.

[16] P. K. Sen, "On stochastic convergence of sample extreme values from distributions with infinite end-points," *Jour. Ind. Soc. Agri. Stat.*, 16 (1964), 189-201.

[17] E. Sverdrup, "The limit distribution of a continuous function of random variables," *Skandinavisk Aktuarietidskrift*, 35 (1952), 1-10.

[18] T. J. Terpestra, "A non-parametric k-sample test and its connection with H-test," Report S(92) (VP2) of *Stat. Dept. Math. Centre*, Amsterdam, (1951).