# SOME ECONOMIC PARTIALLY BALANCED $2^m$ FACTORIAL FRACTIONS

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## Summary

This paper offers plans mainly for $2^m$ factorial fractions with $(6 \leqq m \leqq 10)$, each of which possesses the following properties : ( i ) the general mean and all the main effects and two factor interactions are estimable, assuming the higher order interactions to be zero; ( ii ) the number of assemblies involved is kept at a minimum, with the condition that the available number of degrees of freedom for error is not too few; (iii) the fraction is subdivided into blocks (size 8 for $6 \leqq m \leqq 7$, size 16 for $8 \leqq m \leqq 10$); (iv) the estimates of the main effects and interactions can be obtained without any special adjustment needed for eliminating the block effects; ( v ) structurally, the fraction possesses a partial balance; and (vi) subject to the above conditions, the fraction has been chosen so that for any pair of effects, the correlations between the corresponding estimates may not be too large. (In each fraction, the general mean has zero correlation with the other effects.) Furthermore, the inverses of the matrices involved in the normal equations have also been presented so that a complete analysis is available for each fraction.

Also for high-precision experiments where the error variance is very low, but the cost of observation per assembly is very high, a whole series of $2^m$ fractions has been presented. These fractions may also be useful for situations where a good estimate of error variance is available from past experience. Each of these fractions possesses the property ( i ) above. Furthermore the fraction requires only a minimum of $\nu$ assemblies, where $\nu$ is the number of effects to be estimated. To this an optional $m+1$ assemblies could be added for the estimation of error, by including a dummy factor.

## 1. Introduction

Starting with the pioneering work of Finney [9], the theory of factorial fractions has received considerable and increasing attention by a large number of researchers, and the fractions themselves have found increasing use in agricultural, biological and industrial experimentation. For detailed introduction to and up to date information on the subject, reference may be made, for example, to [12, 16, 10, 7, 8, 1, 11, 2, 15].

In the beginning, the theory largely concerned itself with orthogonal fractions, in which the estimates of the effects of interest to us are all uncorrelated. However the number of assemblies required in such fractions is rather large, i.e. they are uneconomic. Attention has therefore lately (for example, [1, 3, 4, 5, 19]) shifted more to the consideration of fractions which are economic but which may give rise to correlated estimates. We shall call such fractions as irregular.

Obviously, the properties outlined in the summary are the ones that in *general*, one may like to have in a 'good' fraction.

An explanation of the term "partially balanced fraction" is available in [3, 5], to which the interested reader may refer. Although some remarks may not be out of place here, a full appreciation of this concept is not necessary for reading this paper and using the designs developed herein. Suppose for example that our interest lies in *all* the two-factor and lower order interactions. Then we call a fraction $T$ '*completely balanced*' provided the covariance matrix of the estimates is 'symmetrical with respect to the factors'. To elaborate, let $\mu$ denote the general mean, and $A_i$ and $A_{ij}$ respectively the main effect of the $i$th factor, and the interaction between $i$th and $j$th factors. Let the corresponding estimates, obtained from $T$, be $\hat{\mu}$, $\hat{A}_i$, etc. Then we require that the variances of these, and the covariances $\text{cov}(\hat{\mu}, \hat{A}_i)$, $\text{cov}(\hat{A}_i, \hat{A}_j)$, $\text{cov}(\hat{A}_i, \hat{A}_{ij})$, $\text{cov}(\hat{A}_i, \hat{A}_{jk})$, $\text{cov}(\hat{A}_{ij}, \hat{A}_{ik})$ and $\text{cov}(\hat{A}_{ij}, \hat{A}_{kl})$, be independent of the suffixes $i$, $j$, $k$, $l$ (which are assumed to be distinct). Thus we must have: $\text{var}(\hat{A}_{12}) = \text{var}(\hat{A}_{34})$, $\text{cov}(\hat{A}_{12}, \hat{A}_{13}) = \text{cov}(\hat{A}_{23}, \hat{A}_{35})$ etc., though *not necessarily*: $\text{var}(\hat{A}_1) = \text{var}(\hat{A}_{12})$, or $\text{cov}(\hat{A}_{12}, \hat{A}_{13}) = \text{cov}(\hat{A}_{12}, \hat{A}_{34})$. However if the covariance matrix has some other structure, e.g. like that of the covariance matrix of the estimates for a PBIB design or an intra and intergroup balanced design or some generalisations of these two, then we call the fraction partially balanced. Roughly speaking, for a partially balanced fraction the set of all the effects of interest to us is divisible into one or more groups (as in an intra and intergroup balanced design) such that the covariance matrices of the estimates of the effects within any group, and also between groups has a structure similar to that of the covariance matrix for a PBIB design. Indeed, because of this feature, a partially balanced fraction possesses the same advantage in the ease of analysis and interpretation relative to a general unbalanced fraction as a PBIB does compared to an arbitrarily chosen design. Needless to say that a PB fraction would be called for when complete balance is not feasible.

Special efforts have been made to ensure that the correlation between no two estimates is very large, since it would imply that the corresponding effects are almost confounded. So far as the authors are aware, the fractions presented here are better than the corresponding known ones from the overall point of view outlined in the summary. Similar investigations are in progress regarding other factorials and shall be reported later.

## 2. Some general remarks

Consider a $2^m$ experiment. Denote by $\phi$, the assembly (treatment combination) in which all factors are at level 0, by $a_{i_1}a_{i_2}\cdots a_{i_r}(1\leqq r\leqq m)$ the one in which factors $i_1, i_2, \cdots, i_r$ are at level 1 and the rest at level 0, and let

$$(1)\qquad \boldsymbol{a}'=(\phi;\ a_1, a_2, \cdots, a_m;\ a_1a_2, a_1a_3, \cdots, a_{m-1}a_m;\ a_1a_2a_3, \cdots;\ a_1a_2\cdots a_m)\ ,$$

be the vector containing all the $2^m$ assemblies in their natural order. For simplicity the expected response for any assembly $a_{i_1}a_{i_2}\cdots a_{i_r}$ shall also be denoted by the same symbol. Also, as usual, let $A_{i_1}A_{i_2}\cdots A_{i_r}$ denote the $r$-factor interaction between the factors $i_1, i_2, \cdots, i_r$; let these interactions be written in the form of a vector $\boldsymbol{A}'$, having the form (1) with $a$'s replaced by $A$'s. Suppose we are interested in estimating $\mu$, the main effects $A_i(i=1, 2, \cdots, m)$ and the 2-factor interactions $A_iA_j$ (written $A_{ij}$ for short; $i<j$, $i, j=1, 2, \cdots, m$) assuming that the rest of the interactions are zero. These can be arranged as in $\boldsymbol{A}'$ in the form of a vector

$$(2)\qquad \boldsymbol{L}'=(\mu;\ A_1, \cdots, A_m;\ A_{12}, A_{13}, \cdots, A_{m-1, m})$$

of length $\nu=1+m(m+1)/2$. It is well known that each element of $\boldsymbol{A}$ is a certain linear contrast of the elements of $\boldsymbol{a}$, so that we can write

$$(3)\qquad \boldsymbol{A}=\boldsymbol{D}\boldsymbol{a}\ ,$$

where $D$ is a $2^m\times 2^m$ matrix (any two rows of which are orthogonal) which can be easily written down and need not be reproduced here. Since $(2^{-m/2})D$ is an orthogonal matrix we readily obtain

$$(4)\qquad \boldsymbol{a}=(2^{-m})\boldsymbol{D}'\boldsymbol{A}\ .$$

Writing $\boldsymbol{A}'=(\boldsymbol{L}'\ \vdots\ \boldsymbol{l}_0')$, and since we are assuming $\boldsymbol{l}_0$ to be zero, we find that

(5) $$a=(2^{-m})D_0'L\ ,$$

where $D_0'$ is obtained from $D'$ by cutting out the last $(2^m-\nu)$ columns.

Now let $T$ be any fraction, let the observed responses of the $N$ (say) assemblies in $T$ written in the form of a vector be denoted by $\boldsymbol{y}$, and assume that

(6) $$\text{Exp}\,(\boldsymbol{y})=\boldsymbol{y}^*,\ \ \text{Var}\,(\boldsymbol{y})=I_N\sigma^2,$$

$\sigma^2$ being unknown. The problem is to find $\hat{\boldsymbol{L}}$, the best linear unbiased estimate of $\boldsymbol{L}$, given $\boldsymbol{y}$. However, in virtue of (5) it is evident that

(7) $$\boldsymbol{y}^*=(2^{-m})E'\boldsymbol{L}\ ,$$

where $E'(N\times\nu)$ is the matrix obtained from $D_0'$ by cutting out (or repeating) the rows of $D_0'$ corresponding to treatment combinations omitted (or repeated) from $\boldsymbol{a}$ to get $\boldsymbol{y}^*$. We shall also assume the presence of blocks. Therefore let

(8) $$\boldsymbol{y}=\boldsymbol{y}^*+H\boldsymbol{\beta}+\boldsymbol{e}=(2^{-m})E'\boldsymbol{L}+H\boldsymbol{\beta}+\boldsymbol{e}\ ,$$

where $\boldsymbol{\beta}'=(\beta_1, \beta_2, \cdots, \beta_b)$ is the vector of the $b$ (say) block effects, $H(N\times b)$ is a matrix of zeroes excepting for a single unity in each row, and $\boldsymbol{e}$ is the error vector. In the row of $H$ corresponding to any element $y_\alpha$ of $\boldsymbol{y}$, the unity stands in the column which corresponds to the block from which $y_\alpha$ comes. The normal equations for $\boldsymbol{\beta}$ and $\boldsymbol{L}$ are therefore

(9) $$\begin{bmatrix} (2^{-m})E \\ \cdots \\ H' \end{bmatrix} [(2^{-m})E' \ \vdots\ H] \begin{bmatrix} \boldsymbol{L} \\ \cdots \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} (2^{-m})E \\ \cdots \\ H' \end{bmatrix} \boldsymbol{y}\,.$$

We shall now investigate the case where no adjustment for blocks is needed to obtain $\hat{\boldsymbol{L}}$. It is useful to recall a definition first.

An orthogonal array of strength $d$, with $m$ constraints, $N$ assemblies and $s$ symbols is an $(m\times N)$ matrix $M$ whose elements are these symbols, such that each of the $s^d$ possible vectors of length $d$ with these symbols as elements occur an equal number of times as columns in any $(d\times N)$ submatrix of $M$. As an example the following is an orthogonal array of strength 2, with $N=8$, $m=4$ and $s=2$:

(10) $$M=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here each of the four ($2^2$) vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ occur twice as columns in any two-rowed submatrix of $M$. It can be checked that the array is of maximum strength 3. Notice that each column of $M$ could be considered as an assembly from a $2^4$ factorial. In general the columns of the above array $M$ can be considered as the treatment combinations from an $s^m$ factorial.

We next prove

THEOREM. *Let $T$ be a fraction from an $s^m$ factorial, and let the assemblies in $T$ be divided into $b$ blocks, such that the set of assemblies in any block form an orthogonal array of strength 2. Then if all 3-factor and higher order interactions are assumed zero, the BLU estimates of the rest do not change by the assumption of block effects.*

We shall take $s=2$; the general case follows similarly. The result will be established in view of (9), if we show that $EH$ is a zero matrix under the stated conditions. Now take some row of $E$, say the one corresponding to the interaction $A_{12}$. Also take (say) the first column of $H$, corresponding to $\beta_1$ and consider this row by column product (say $p$). Since the assemblies in this block form an orthogonal array of strength 2, they can be divided into 4 equal parts $P_{ij}$, the assemblies in a particular part having the levels $(i, j)$ respectively for factors 1 and 2. The four parts are obtained by taking $i, j=0, 1$. Now, the unity occurs in the first column of $H$ precisely at the rows which correspond to assemblies in one of the sets $P_{11}$, $P_{00}$, $P_{01}$ and $P_{10}$. Also, corresponding to this unity, the element in the row of $E$ under consideration is $(+1)$ corresponding to the assemblies in $P_{11}$ and $P_{00}$, and is $(-1)$ for $P_{01}$ and $P_{10}$. Since the number of assemblies in each $P_{ij}$ is the same, the product $p$ is zero. Thus every row of $E$ is orthogonal to every column of $H$, and the result is proved.

From (9), it therefore follows that under the conditions of the above theorem,

$$\hat{\boldsymbol{L}}=(2^m)(EE')^{-1}E\boldsymbol{y}=(2^m)(EE')^{-1}D_0\boldsymbol{z}\,, \tag{11}$$

say, provided that $(EE')$ is nonsingular, and the vector $\boldsymbol{z}$ is defined as below. The elements of $\boldsymbol{z}$ are put into (1, 1) correspondence with those of $\boldsymbol{a}$, such that the $i$th element of $\boldsymbol{a}$ corresponds to the $i$th element of $\boldsymbol{z}$. Let $\theta$ be any assembly in $\boldsymbol{a}$. Then we define the element $z(\theta)$ in $\boldsymbol{z}$ corresponding to the element $\theta$ in $\boldsymbol{a}$ by

$$z(\theta)=\begin{cases} 0, \text{ if } \theta \notin T \\ \text{total yield of } \theta \text{ (for all repetitions of } \theta \text{ in } T\text{), if } \theta \in T. \end{cases}$$

The advantage in using $z$ is that one does not need to write down $E$, and the computation of $D_0 z$ from $z$ is the same as that of $L$ from $a$ for which various simple rules are known. From (11) and (6), we have

$$\text{Var}(\hat{L})=2^{2m}(EE')^{-1}\sigma^2 . \tag{12}$$

For any fraction $T$, formulae have been developed in [4] for a direct computation of $(EE')$, and in [5] methods for its inversion have been studied. In this paper we shall present $(EE')^{-1}$ for each fraction. This not only provides the variances and the correlations but also makes the calculation of $\hat{L}$ from (11) easy and straightforward.

In case the vector $I_0$ is not zero, the bias $(\hat{L}-L)$ can be obtained in terms of $I_0$ using theorem 4 of [3].

We now come to the construction aspect. For a $2^m$ factorial, consider $EG(m, 2)$. In the classical theory of confounding and (orthogonal) fractional replication, the total fraction itself is an orthogonal array of strength 4, generated by an appropriate flat $\Omega$ of $EG(m, 2)$. Assemblies lying on suitably chosen parallel subflats $\Omega_1, \cdots, \Omega_b$ of $\Omega$ are assigned to the different blocks, $b$ in number, such that the assemblies in any given block form an array of strength 2. If the number of assemblies is $2^{r_1}$ in $\Omega$ and $2^{r_2}$ in any $\Omega_i$, then the number of blocks $b$ is obviously $2^r$ where $r=r_1-r_2$. Before proceeding further, it is useful to define the function $n_d(r, 2)$, more about which could be found in [6]. It denotes the maximum $m$, such that a flat of $EG(m, 2)$ can be chosen such that it contains exactly $2^r$ assemblies, and these assemblies form an orthogonal array of strength $d$. The results

$$n_2(r, 2)=2^r-1,\ n_3(r, 2)=2^{r-1}, \tag{13}$$

have been proved in Bose [2], and are important for our discussion. For example, the first one implies that if we want a block of size 8 to hold an orthogonal array of strength 2 with assemblies from a $2^m$ factorial, then we must have $m\leqq 7$. Similarly a block size 16 will be good enough if $m\leqq 15$. This also explains why in our fractions we use block size 8 for $m=6$, 7, and 16 for $8\leqq m\leqq 10$.

The value of $n_4(r, 2)$ is not available for general $r$, but can be shown for example, to equal 6 and 8 respectively for $r=5, 6$. This means that an orthogonal fraction in $64(=2^6)$ assemblies can be obtained as a flat of $EG(m, 2)$ for $m\leqq 8$, but not for $m>8$. Thus the fraction so obtained, though orthogonal is of a size which is generally too large compared to $\nu$, the number of effects to be estimated. Furthermore, sometimes even though the size of an orthogonal fraction may not be too important, the division of the fractions into blocks which are arrays of strength 2 may not be permissible.

Hence, in order to obtain fractions of more convenient sizes, one has to modify the above method of starting from an $\Omega$ of strength 4, and then trying to break them into the $\Omega_i$. One variation that has been sometimes (for example [14]) used is to take $\Omega^*$ as a fraction, where $\Omega^*$ denotes the set of $(b' \cdot 2^r)$ assemblies lying on $\Omega_{i_1}, \Omega_{i_2}, \cdots, \Omega_{i_{b'}}$, where $b'<b$. This set of parallel flats is to be chosen such that the fraction $\Omega^*$ has various desirable properties. However the experience of the authors is that in general the 'parallelism' of these flats causes high correlation between the estimates of certain effects. In this paper therefore a new approach is used which consists of taking steps in the following sequence: (i) on the basis of the value of $\nu$, find $n$ the number of assemblies which we want to have in the fraction; (ii) knowing $m$, find $r$ such that $b$ orthogonal arrays of strength 2 with $2^r$ assemblies from a $2^m$ factorial exist and $b \cdot 2^r$ nearly equals $n$; (iii) find a set of arrays $\Omega_1^*, \Omega_2^*, \cdots, \Omega_b^*$ of strength 2 each (which may be generated by flats in $EG(m, 2)$ which are not necessarily parallel), such that (a) the total fraction is partially balanced, and (b) no large correlations are present.

While (iii $a$) will be illustrated in the designs to follow, a broad principle for achieving (iii $b$) could be explained here. Firstly, it is well known that a flat $\Omega$ in $EG(m, 2)$ could be expressed by the equation $A\boldsymbol{x}=\boldsymbol{c}$, where the elements of $A(l\times m)$ and $\boldsymbol{c}(l\times 1)$ belong to $GF(2)$ and the $m$ elements of $\boldsymbol{x}$ correspond respectively to the $m$ factors. This flat is called an $(m-l)$-flat of $EG(m, 2)$ provided that rank $(A)=l$, and it then contains $2^{m-l}$ points. Now consider the set $A^*$ of $2^l$ vectors obtained by taking all possible linear combinations of the rows of $A$. If $\boldsymbol{v}$ is any vector, we define the weight $w(\boldsymbol{v})$ of $\boldsymbol{v}$ as the number of nonzero elements in $\boldsymbol{v}$. It is well known that if $A$ is such that $w(\boldsymbol{v})\geqq d+1$ for all $\boldsymbol{v}\in A^*$, then the $2^{m-l}$ elements of $\Omega$ form an orthogonal array of strength $d$. The set $\Omega$ can as well be considered as a fraction with $2^{m-l}$ assemblies. Considering the case $d=2$ suppose for illustration that $w(\boldsymbol{v}_1)=3$, $w(\boldsymbol{v}_2)=4$, where the nonzero coordinates in $\boldsymbol{v}_1$ correspond to the factors $i_1$, $i_2$, and $i_3$, and those in $\boldsymbol{v}_2$ to $j_1, j_2, j_3, j_4$. Then it is well known that for this fraction, the main effect $A_{i_1}$ will be confounded with the interaction $A_{i_2i_3}$, $A_{i_2}$ with $A_{i_1i_3}$, etc. Similarly $A_{j_1j_2}$, $A_{j_3j_4}$ are confounded, as also $A_{j_1j_3}$ and $A_{j_2j_4}$, and $A_{j_1j_4}$ and $A_{j_2j_3}$.

Now let there be a fraction $\Omega$ divided into $b$ blocks, the $g$th containing an array $\Omega_g$ of strength 2 formed by the above method. Let $\alpha_c(i_1, i_2, i_3)=\alpha_c(\boldsymbol{v}_1)$ be the number of flats $\Omega_g$ such that $\boldsymbol{v}_1\in A_g^*$, and $\boldsymbol{v}_1'\boldsymbol{x}=c$ is an equation satisfied by the elements of $\Omega_g$, where $c\in GF(2)$. Let $\beta(\boldsymbol{v}_1)=|\alpha_0-\alpha_1|$. Then it is found that in general, $\hat{A}_{i_1}$ and $\hat{A}_{i_2i_3}$ are correlated if $\beta\neq 0$, and are either uncorrelated or have relatively small correlation if $\beta=0$. Also the correlation increases with the increase in $\beta$

(roughly proportionately, in the authors' experience). Same is true of other pairs like $(\hat{A}_{i_2}, \hat{A}_{i_1 i_3})$, etc. In the case of $\boldsymbol{v}_2$, $\beta(\boldsymbol{v}_2)$ is similarly defined, and the above remarks hold for pairs like $(\hat{A}_{j_1 j_2}, \hat{A}_{j_3 j_4})$, etc.

The technique used in the paper is therefore to choose $\Omega$ in such a way that for most out of the set of possible vectors $\boldsymbol{v}$ (of length $m$) of weight 3 or 4, $\beta(\boldsymbol{v})$ is zero, and is 1 for the rest of them. Having considered a few such competing $\Omega$'s, one is finally selected after an inspection of the different covariance matrices.

## 3. $2^{10}$ design ($\nu=56$, n=80, b=5)

Let the symbols $X_1, X_2, \cdots, X_9, X_0$ correspond respectively to the 10 factors, and consider the flat $\Omega_1$ (with $2^4$ points) defined by the following six linearly independent equations over $GF(2)$:

$$(14) \qquad \begin{array}{lll} x_1+x_2+x_3=1 & x_1+x_7+x_0=1 & x_3+x_4+x_6=0 \\ x_1+x_4+x_5=0 & x_1+x_8+x_9=0 & x_6+x_9+x_0=0 \end{array}$$

The linear combinations of these involving only 3 or 4 $x$'s are:

$$(15) \qquad \begin{array}{lll} x_6+x_7+x_8=1 & x_1+x_2+x_4+x_6=1 & x_2+x_3+x_7+x_0=0 \\ x_2+x_5+x_6=1 & x_1+x_3+x_5+x_6=0 & x_4+x_5+x_8+x_9=0 \\ x_1+x_6+x_7+x_9=1 & x_2+x_3+x_4+x_5=1 & x_4+x_5+x_7+x_0=1 \\ x_1+x_6+x_8+x_0=0 & x_2+x_3+x_8+x_9=1 & x_8+x_9+x_7+x_0=1 \\ x_2+x_5+x_9+x_0=1 & x_3+x_4+x_9+x_0=0 & \\ x_2+x_5+x_7+x_8=0 & x_3+x_4+x_7+x_8=1 & \end{array}$$

No linear combination of (14) involves less than 3 $x$'s, and hence the 16 assemblies satisfying them form an orthogonal array of strength 2. We assign these to one block. Let $B$ denote the above set of 22 equations. Let $TB$ denote the set of equations derived from $B$ after the $x$'s have been shifted cyclically, i.e. $\boldsymbol{x}'=(x_1, x_2, \cdots, x_9, x_0)$ changed to $T\boldsymbol{x}'=(x_2, x_3, \cdots, x_9, x_0, x_1)$. Thus for example in $TB$, the top left hand equation of (14) becomes $x_2+x_3+x_4=1$. By further similar shifts we get sets $T^2B$, $T^3B$, etc. It can be easily checked that $T^5B=B$, so that we get exactly five different sets $B, TB, \cdots, T^4B$, each of which defines a block. This gives the required fraction.

The cyclical shift $T$ imparts to the fraction properties similar to that of a cyclic PBIB design. This accounts for the partial balance possessed by the fraction. Furthermore, in view of the remarks made at the end of the last section, the equation (14) and hence the set $B$

has been so chosen that the correlation between any two estimates may not be too large. In particular $\beta(v)$ is mostly zero, and is not greater than 1 for any $v$ of interest to us.

For ease of presentation, the rows and columns of the matrix $(EE')^{-1}$ for this fraction have been rearranged. In the present form the rows (and columns) of $(EE')^{-1}$ correspond respectively to $(S_1; S_2; S_3; S_4; S_5; S_6)$, where the sets $S_i$ correspond to the treatment effects as follows: $S_1: (A_0, A_1, \cdots, A_9)$, $S_2: (A_{01}, A_{12}, \cdots, A_{89}, A_{90})$, $S_3: (A_{02}, A_{13}, \cdots, A_{91})$, $S_4: (A_{03}, A_{14}, \cdots, A_{92})$, $S_5: (A_{04}, A_{15}, \cdots, A_{93})$, $S_6: (A_{05}, A_{16}, \cdots, A_{49})$. The set $S_6$ has 5 effects, and the rest each have 10. Corresponding to this the matrix $(EE')^{-1}$ can be partitioned into 36 submatrices having the submatrix $M_{ij}$ in the $i$th row and $j$th column block $(i, j=1, 2, \cdots, 6)$. For each $M_{ij} (i \leqq j)$, the first row $c'_{ij}$ is presented below, the other rows being obtained by successively using the cyclic permutation $T^{-1}$, such that if $c'=(c_1, \cdots, c_q)$, then $T^{-1}c'=(c_q, c_1, c_2, \cdots, c_{q-1})$. Thus for example the rows of $M_{36}$ are given by $(c'_{36})$, $(T^{-1}c'_{36})$, $\cdots$, $(T^{-9}c'_{36})$, and those of $M_{66}$ by $(c'_{66})$, $(T^{-1}c'_{66})$, $\cdots$, $(T^{-4}c'_{66})$. The $c'_{ij}$ (multiplied by $16 \cdot 10^4$) are:

$c_{11}=(3192, -254, -314, 64, -180, 362, -180, 64, -314, -254)$

$c_{12}=(69, -1043, 489, 296, -451, -135, -457, 1191, -348, -234)$

$c_{13}=(632, 1163, -21, -406, -680, 480, 488, 52, -309, -1400)$

$c_{14}=(-14, 598, -585, -391, -8, -30, 1085, -673, -495, 825)$

$c_{15}=(367, 235, 412, -299, -808, 263, 407, 184, -309, -452)$

$c_{16}=(664, -23, 618, -1226, -34)$

$c_{22}=(4145, -371, 283, -301, -211, 803, -211, -301, 283, -371)$

$c_{23}=(-285, -572, -562, 1800, 500, -68, -1553, -720, 1211, 250)$

$c_{24}=(-1265, 105, 161, 1043, 586, -1816, 11, 87, 337, 1377)$

$c_{25}=(374, -1014, -290, 695, 371, 155, -905, -486, 644, 457)$

$c_{26}=(-52, -1867, 1091, 247, 581)$

$c_{33}=(5114, 725, -1320, -1347, 51, 1998, 51, -1347, -1320, 725)$

$c_{34}=(637, 92, -2128, -11, 970, 912, 263, -2180, 395, 1049)$

$c_{35}=(1304, 404, -590, -1894, -55, 1916, 518, -431, -1232, 61)$

$c_{36}=(1015, -94, -927, -2199, 2205)$

$c_{44}=(4649, -529, -582, -521, -522, 185, -522, -521, -582, -529)$

$c_{45}=(195, 165, -22, -104, -191, 48, 1131, 27, -1652, -152)$

$c_{46}=(337, 2447, -2442, -2489, -94)$

$c_{55}=(3422, 140, -716, -718, 112, 941, 112, -718, -716, 140)$

$c_{56}=(1182, -929, -506, -929, 1182)$

$c_{66}=(5717, -1353, -506, -506, -1353)$.

Notice that the $M_{ij}(1 \leqq i, j \leqq 5)$ are linear functions of the matrices $K_\theta(1 \leqq \theta \leqq 10)$, where $K_\theta(10 \times 10)$ has 1 in the cell $(\mu, \mu')$, $1 \leqq \mu, \mu' \leqq 10$, if $(1+\mu'-\mu)=\theta$ (mod 10), and 0 otherwise.

The $K_\theta$ have the property

$$K_\theta K_{\theta'} = K_{\theta'} K_\theta = K_\pi ,$$

where $1 \leqq \pi \leqq 10$, and $\pi = \theta + \theta' - 1$ (mod 10). Thus the $K_\theta$ form a commutative linear associative algebra. If $(EE')$ itself is correspondingly partitioned into submatrices $M^*_{ij}$ (say), then the $M^*_{ij}$ as well possess the above properties of the $M_{ij}$. Indeed, the theory developed in [5] tells us that because of the above linear algebras being involved, both $(EE')$ and $(EE')^{-1}$ will necessarily have the same pattern, and that one can be easily obtained from the other by using the algorithm developed therein.

## 4. $2^9$ design ($\nu=46$, n=64, b=4)

The four blocks in this design have been obtained by using respectively the equations

(1) $U_1 \mathbf{1}=\mathbf{0}$, $U_3 \mathbf{1}=\mathbf{0}$, (2) $U_2 \mathbf{1}=\mathbf{0}$, $U_4 \mathbf{1}=\mathbf{0}$, (3) $U_1 \mathbf{1}=\mathbf{1}$, $U_4 \mathbf{1}=\mathbf{1}$,
(4) $U_2 \mathbf{1}=\mathbf{1}$, $U_3 \mathbf{1}=\mathbf{1}$

where $\mathbf{0}'=(0, 0, 0)$, $\mathbf{1}'=(1, 1, 1)$, and $U_2=U_1'$, $U_4=U_3'$, and

$$U_1=\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}, \quad U_3=\begin{bmatrix} x_1 & x_5 & x_9 \\ x_6 & x_7 & x_2 \\ x_8 & x_3 & x_4 \end{bmatrix}.$$

By writing down the above equations in full and inspecting them, one easily finds that the main effects are all orthogonally estimable. Also, the 36 interactions can be divided into 9 sets of four each as follows. Consider the 12 triplets of $x$'s obtained by taking the rows of the matrices $U_i$, i.e. the rows and columns of $U_1$ and $U_3$. These can be considered as the 12 lines of $EG(2, 3)$, the 9 points of this geometry being the $x$'s themselves. Take the point $x_1$, and consider the four lines passing through it, viz. (1, 2, 3), (1, 4, 7), (1, 5, 9) and (1, 6, 8), writing $i$ for $x_i$ for brevity. Note that the four lines come one from each $U_i$. Omitting 1 in any line, we get two other points, and we consider the two factor interaction defined by the corresponding factors. Thus here we get the ordered set of interactions $S_1$: $(A_{23}, A_{47}, A_{59}, A_{68})$, the ordering being such that the interaction term arising out of a line (i.e. a row) from the matrix $U_i$ has been put at the $i$th place. Similarly for $1 \leqq$

$i \leqq 9$, we get an ordered set $S_i$ of 4 interactions by considering the lines passing through $x_i$. Let $M_{ij}(i, j=1, 2, \cdots, 9)$ be the covariance matrix between the estimates of the interactions in the set $S_i$ and those in the set $S_j$. Then it turns out that

$$(16) M_{ij}=\begin{cases} R_1, & \text{if } i=j, \\ R_2, & \text{if } x_i \text{ and } x_j \text{ occur in the same row or} \\ & \quad \text{same column of the matrix } U_1, \\ R_3, & \text{otherwise,} \end{cases}$$

where

$$R_1=\begin{bmatrix} a_1 & a_2 & a_3 & a_3 \\ a_2 & a_1 & a_3 & a_3 \\ a_3 & a_3 & a_1 & a_2 \\ a_3 & a_3 & a_2 & a_1 \end{bmatrix}, \quad R_2=\begin{bmatrix} b_1 & b_2 & b_3 & b_3 \\ b_2 & b_1 & b_3 & b_3 \\ b_3 & b_3 & -a_2 & -b_1 \\ b_3 & b_3 & -b_1 & -a_2 \end{bmatrix}, \quad R_3=\begin{bmatrix} -a_2 & -b_1 & b_3 & b_3 \\ -b_1 & -a_2 & b_3 & b_3 \\ b_3 & b_3 & b_1 & b_2 \\ b_3 & b_3 & b_2 & b_1 \end{bmatrix},$$

$$a_1=0.4292, \quad a_2=0.1042, \quad a_3=-0.1500$$
$$b_1=0.0292, \quad b_2=0.0042, \quad b_3=0.0167.$$

The whole $(36 \times 36)$ covariance matrix of the estimates of the interactions can now be easily written down. This matrix is obviously highly patterned, the partially balanced structure arising from the fact that there are four blocks, one block corresponding to each pencil in $EG(2, 3)$.

## 5. $2^8$ design ($\nu=37$, $n=64$, $b=4$)

The four blocks are respectively obtained by the solutions of the following four sets of equations.

| | | | |
|---|---|---|---|
| $x_1+x_2+x_8=1$ | $x_2+x_3+x_4=1$ | $x_3+x_4+x_5=1$ | $x_5+x_6+x_7=1$ |
| $U_1=x_1+x_2+x_5+x_6=0$ | $U_1=1$ | $U_1=1$ | $U_1=0$ |
| $U_2=x_1+x_3+x_5+x_7=1$ | $U_2=1$ | $U_2=0$ | $U_2=0$ |
| $U_3=x_3+x_4+x_7+x_8=1$ | $U_3=0$ | $U_3=0$ | $U_3=1$ |
| $U_4=x_2+x_3+x_6+x_7=1$ | $U_4=0$ | $U_4=1$ | $U_4=0$ |
| $U_5=x_2+x_4+x_6+x_8=0$ | $U_5=0$ | $U_5=1$ | $U_5=1$ |
| $U_6=x_1+x_4+x_5+x_8=0$ | $U_6=1$ | $U_6=0$ | $U_6=1$ |
| $x_5+x_6+x_8=1$ | $x_2+x_7+x_8=1$ | $x_1+x_4+x_7=1$ | $x_1+x_2+x_7=1$ |
| $x_1+x_4+x_6=1$ | $x_4+x_6+x_7=1$ | $x_5+x_7+x_8=1$ | $x_1+x_3+x_6=1$ |
| $x_2+x_4+x_5=1$ | $x_3+x_6+x_8=1$ | $x_1+x_3+x_8=1$ | $x_2+x_3+x_5=1$ |

In any set the first four equations are linearly independent. No linear combination of these involves less than three $x$'s; those with 3 or 4 $x$'s being reproduced above. For this fraction, it turns out that the interactions $A_{15}$, $A_{26}$, $A_{37}$ and $A_{48}$ are orthogonally estimable. The remaining effects could be divided into the two sets ($A_1$, $A_6$, $A_7$, $A_8$; $A_{12}$, $A_{56}$, $A_{13}$, $A_{57}$, $A_{14}$, $A_{58}$, $A_{27}$, $A_{36}$, $A_{28}$, $A_{46}$, $A_{38}$, $A_{47}$) and ($A_2$, $A_3$, $A_4$, $A_5$; $A_{16}$, $A_{25}$, $A_{17}$, $A_{35}$, $A_{18}$, $A_{45}$, $A_{23}$, $A_{67}$, $A_{24}$, $A_{68}$, $A_{34}$, $A_{78}$). The estimates of each of the two sets have the same variance matrix $M$ reproduced below, and between the two sets, the estimates are uncorrelated. For ease of presentation, write

$$(16) \quad M=\begin{bmatrix} M_1 & M_2 \\ M_2' & M_3 \end{bmatrix}$$

where $M_1(4\times 4)$ and $M_3(12\times 12)$ correspond respectively to the main effects and interactions in any set. Then

$$M_1=\frac{1}{3}I_4+\frac{1}{6}J_{44},$$

$$M_3=\left[\begin{array}{cc|cc|cc|cc|cc|cc}
a & b & c & c & c & c & c & c & c & c & d & d \\
b & a & c & c & c & c & c & c & c & c & d & d \\
\hline
 & & a & b & c & c & c & c & d & d & c & c \\
 & & b & a & c & c & c & c & d & d & c & c \\
\hline
 & & & & a & b & d & d & c & c & c & c \\
 & & & & b & a & d & d & c & c & c & c \\
\hline
 & & & & & & a & b & c & c & c & c \\
 & & & & & & b & a & c & c & c & c \\
\hline
 & & \text{Sym.} & & & & & & a & b & c & c \\
 & & & & & & & & b & a & c & c \\
\hline
 & & & & & & & & & & a & b \\
 & & & & & & & & & & b & a
\end{array}\right]$$

$$a=\frac{1}{3},\quad b=\frac{1}{12},\quad c=\frac{1}{16},\quad d=\frac{1}{24},$$

$$M_2=\begin{bmatrix}
x & x & y & y & y & y & x & x & x & x & y & y \\
y & y & x & x & y & y & x & x & y & y & x & x \\
y & y & y & y & x & x & y & y & x & x & x & x \\
x & x & x & x & x & x & y & y & y & y & y & y
\end{bmatrix},\quad x=-\frac{1}{12},\quad y=-\frac{1}{6}.$$

That the fraction has a partially balanced structure is evident from an inspection of $M$. Since $n_2(3, 2)=7$, a minimum block size $2^4$ is required to preserve strength 2 in each block. Also, since 3 such blocks (i.e., $n=48$) would provide too few error d.f., at least four seem necessary.

The figure $n=56$ appears to be a good compromise, but since 56 is not a multiple of 16, one would have to use blocks of size 8. Since each such block will be an orthogonal array of strength 1 only, the analysis will become more complicated due to the need of adjustment for block effects.

A $2^8$ fraction (64 assemblies) can also be obtained by considering one factor in the earlier $2^9$ fraction as either dummy or a block factor.

## 6. $2^7$ fraction ($\nu=29$, $n=56$ or $48$, $b=7$ or $6$)

The first block $B_1$ is obtained by using the equations

$$x_1+x_3+x_5=1 \qquad x_2+x_3+x_4=1$$
$$x_1+x_2+x_6=0 \qquad x_3+x_6+x_7=1$$

which are all linearly independent. These give rise to the following further set of equations (involving 3 or 4 $x$'s):

$$x_4+x_5+x_6=0 \qquad x_1+x_2+x_3+x_7=1$$
$$x_1+x_4+x_7=0 \qquad x_2+x_4+x_6+x_7=0$$
$$x_2+x_5+x_7=0 \qquad x_1+x_5+x_6+x_7=0$$
$$x_2+x_3+x_5+x_6=1 \qquad x_1+x_2+x_4+x_5=0$$
$$x_1+x_3+x_4+x_6=1 \qquad x_3+x_4+x_5+x_7=1$$

The other seven blocks are then $B_1T$, $B_1T^2$, $B_1T^3$, $B_1T^4$, $B_1T^5$, $B_1T^6$, where $T$ is the translation permutation on 7 symbols. As for the $2^{10}$ design, $T$ imparts a partial balance to the fraction.

The matrix $(EE')^{-1}$ splits up into 7 diagonal submatrices corresponding to the 7 sets $ST^i$ $(i=0, 1, \cdots, 6)$ where $S_1=(A_1, A_{26}, A_{34}, A_{57})$ and the other sets are obtained by using $T$ on the suffixes in $S_1$. The covariance matrices for the different sets are identical, being given by $M=\left(\frac{1}{8}\right)[4I_4+J_{44}]$.

The above design allows for 20 d.f. for error. In case fewer assemblies are desirable, another fraction obtained by omitting the last block $(BT^6)$ in the above may be used. In this case, $(EE')^{-1}$ splits into two matrices $M_1$ and $M_2$ corresponding respectively to $\{S_1; S_1T^4; S_1T^6\}$ and $\{S_1T; S_1T^2; S_1T^3; S_1T^5\}$, where

$$(8)M_1=\begin{bmatrix} P_1 & P_2 & P_3 \\ & P_1 & P_4 \\ \text{Sym.} & & P_1 \end{bmatrix}, \quad P_1=\begin{bmatrix} a & b & b & c \\ b & a & c & b \\ b & c & a & b \\ c & b & b & a \end{bmatrix}, \quad P_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ g & 0 & -g & 0 \\ -g & 0 & g & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_3=\begin{bmatrix} 0 & 0 & g & -g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -g & g \end{bmatrix}, \quad P_4=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -g & g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g & -g & 0 & 0 \end{bmatrix},$$

$$a=0.1968, \ b=0.0476, \ c=0.0413, \ g=0.0222,$$

$$(8) M_2=\begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ & Q_1 & Q_5 & Q_6 \\ \text{Sym.} & & Q_1 & Q_7 \\ & & & Q_1 \end{bmatrix}, \quad Q_1=\begin{bmatrix} z & x & x & x \\ & z & x & x \\ \text{Sym.} & & z & x \\ & & & z \end{bmatrix}, \quad Q_2=\begin{bmatrix} 0 & y & x & 0 \\ 0 & x & y & 0 \\ -x & 0 & 0 & -y \\ -y & 0 & 0 & -x \end{bmatrix},$$

$$Q_3=\begin{bmatrix} 0 & 0 & y & x \\ -y & -x & 0 & 0 \\ 0 & 0 & x & y \\ -x & -y & 0 & 0 \end{bmatrix}, \quad Q_4=\begin{bmatrix} 0 & x & 0 & y \\ -x & 0 & -y & 0 \\ -y & 0 & -x & 0 \\ 0 & y & 0 & x \end{bmatrix}, \quad Q_5=\begin{bmatrix} 0 & -y & -x & 0 \\ 0 & -x & -y & 0 \\ y & 0 & 0 & x \\ x & 0 & 0 & y \end{bmatrix},$$

$$Q_6=\begin{bmatrix} 0 & y & x & 0 \\ -x & 0 & 0 & -y \\ 0 & x & y & 0 \\ -y & 0 & 0 & -x \end{bmatrix}, \quad Q_7=\begin{bmatrix} 0 & 0 & -y & -x \\ x & y & 0 & 0 \\ 0 & 0 & -x & -y \\ y & x & 0 & 0 \end{bmatrix},$$

$$x=0.0625, \ y=0.0208, \ z=0.2292.$$

## 7. $2^6$ design ($\nu=22$, n=40, b=5)

The five blocks are respectively obtained by using the following five sets of equations:

$$\begin{array}{lllll} x_1+x_2+x_3=0 & x_2+x_3+x_4=1 & x_3+x_4+x_5=1 & x_1+x_4+x_5=1 & x_1+x_2+x_5=1 \\ x_1+x_4+x_5=0 & x_1+x_2+x_5=0 & x_1+x_2+x_3=1 & x_2+x_3+x_4=0 & x_3+x_4+x_5=0 \\ x_2+x_4+x_6=0 & x_3+x_5+x_6=1 & x_1+x_4+x_6=1 & x_2+x_5+x_6=1 & x_1+x_3+x_6=0 \end{array}$$

Each set gives rise to four other equations having 3 or 4 $x$'s. By examining the set of all equations so generated, it can be checked that the main effects are all orthogonally estimable. The interactions can be divided into 5 sets and ordered as ($A_{12}$, $A_{13}$, $A_{23}$; $A_{46}$, $A_{45}$, $A_{56}$; $A_{36}$, $A_{24}$, $A_{15}$; $A_{34}$, $A_{25}$, $A_{16}$). The covariance matrix $(EE')^{-1}$ of their estimates can then be exhibited as

$$(8)(EE')^{-1}=\left(\frac{1}{72}\right)\begin{bmatrix} Q_1 & R_1 & R_2 & R_3 & R_3 \\ R_1 & Q_1 & -R_2 & R_4 & R_4' \\ R_2 & -R_2 & Q_2 & R_4' & R_4 \\ R_3 & R_4' & R_4 & Q_3 & -R_4' \\ R_3 & R_4 & R_4' & -R_4 & Q_3 \end{bmatrix}, \text{ where } R_4=\begin{bmatrix} b & b & -c \\ -c & b & b \\ b & -c & b \end{bmatrix},$$

$$Q_1=(a+b)I-bJ, \quad Q_2=(a-c)I+cJ, \quad Q_3=(a+d)I-dJ,$$
$$R_1=(c+d)I-cJ, \quad R_2=(d-b)I+bJ, \quad R_3=(b+c)I-bJ,$$
$$a=20, \quad b=1, \quad c=5, \quad d=4,$$

and $I$ and $J$ are of order $(3\times 3)$ each.

## 8. $2^n$ factorial : general series

Let $\phi$ denote the treatment combination in which all factors are at level 0, $a_i$ the one in which only the $i$th factor is at level 1 and the rest are at level 0, and let $\bar{a}_{ij}$ denote the one in which only the $i$th and $j$th factors are at level 0, the others being at level 1. The observed response to any treatment shall be denoted by the same symbol as the treatment itself. Further, the sums over the response of certain treatments will be written as

$$S_0=\sum_{i=1}^{n}(a_i)\,, \quad \bar{S}_{i0}=\sum_{j=1,\ j\neq i}^{n}(\bar{a}_{ij})\,,$$

$$\bar{S}_{00}=\sum_{all\ i,\ j\ (i\neq j)}(\bar{a}_{ij})\,.$$

For the $2^n$ factorial fraction, we then consider the $\nu$ treatments $\phi$, $a_i$ $(i=1, 2, \cdots, n)$ and $\bar{a}_{ij}(i<j;\ i, j=1, 2, \cdots, n)$. It can then be shown that all the $\nu$ effects are estimable, and their estimates are given by

$$\hat{\mu}=\frac{1}{4(n-3)(n-2)}\left[(-n^3+10n^2-29n+24)\phi+(n^2-7n+10)S_0+2\bar{S}_{00}\right],$$

$$(16)\ \hat{A}_i=\frac{1}{4(n-3)}\left[(n-4)a_i+\left(\frac{2}{n-2}\right)\bar{S}_{00}-S_0-\bar{S}_{i0}\right]+\frac{1}{2(n-2)}\phi$$

$$\hat{A}_{ij}=\frac{1}{4}\left[\bar{a}_{ij}-\frac{1}{(n-3)}(a_i+a_j+\bar{S}_{i0}+\bar{S}_{j0})+\left(\frac{2}{n-2}\right)\phi+\frac{2}{(n-3)(n-2)}\bar{S}_{00}\right].$$

From the symmetry in the choice of assemblies it is clear that the fraction is completely balanced. Also it could be shown that the correlation between any two effects tends to zero for large $n$. The variances and covariances between any two estimates can be easily calculated for any $n$ from (16). The corresponding expressions for general $n$ are cumbersome and would be omitted here.

This series of fractions, first described in [7], has been selected after an examination of several other such series of fractions. The fractions in this series appear to give rise to relatively less correlation between the estimates, than the fractions in the other series examined. The fraction provides no degrees of freedom for error, and can not be split into blocks. However by adding a dummy factor whose main effects and interactions with other factors could be assumed zero, one gets $1+[(n+1)(n+2)/2]$ assemblies, which provides $(n+1)$ d.f. for error.

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