ON THE GLIVENKO-CANTELLI THEOREM

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Summary

In this note, it is shown that for any probability measure μ on the Borel sets of k dimensional Euclidian space E_k, the Glivenko-Cantelli theorem can be generalized (section 3, theorem 1). Furthermore, it is shown that for every μ satisfying some conditions, the similar result holds with the supremum taken over the class of all Borel subsets of E_k (theorem 6).

1. Introduction

Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $\boldsymbol{\xi}=(\xi^1, \xi^2, \cdots, \xi^k)$ a k dimensional random vector. Let μ be the induced probability measure on $\boldsymbol{S}=\boldsymbol{S}(E_k)$ by $\boldsymbol{\xi}$;

$$\mu(S)=P\{\boldsymbol{\xi} \in S\}, \quad S \in \boldsymbol{S}.$$

Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots$ be a sequence of independent random vectors which have the same distribution as $\boldsymbol{\xi}$. For each $S \in \boldsymbol{S}$, we shall define a sequence of random variables, which we call an empirical distribution of $\boldsymbol{\xi}$, by

$$\mu_n(S)=\frac{1}{n}\sum_{i=1}^{n} \eta_i(S), \qquad n=1, 2, \cdots,$$

where

$$\eta_i(S)=\eta_i(\omega \mid S)=\begin{cases} 1, & \text{if } \boldsymbol{\xi}_i(\omega) \in S \\ 0, & \text{if } \boldsymbol{\xi}_i(\omega) \notin S . \end{cases}$$

When a family $\boldsymbol{F}$ of subsets of E_k and a class $\mathfrak{C}$ of the induced probability measures on $\boldsymbol{S}$ satisfy the relation

$$\inf_{\mu \in \mathfrak{C}} P\{\lim_{n\to\infty} \sup_{S \in \boldsymbol{F}} |\mu(S)-\mu_n(S)| =0\} =1 ,$$

we shall say that the pair $(\boldsymbol{F}, \mathfrak{C})$ belongs to the Glivenko-Cantelli class of E_k and conventionally denote it by

$$(\boldsymbol{F}, \mathfrak{C}) \in GC(E_k)\,.$$

Then, the classical theorem of Glivenko-Cantelli can be stated as $(\boldsymbol{R}_0, \mathfrak{M}) \in GC(E_1)$, where $\boldsymbol{R}_0 = \boldsymbol{R}_0(E_1)$ denotes the family of all open half intervals and $\mathfrak{M} = \mathfrak{M}(E_1)$ the class of all probability measures on $\boldsymbol{S} = S(E_1)$.

Let $\boldsymbol{A} = \boldsymbol{A}(E_k)$ be the collection of all open half-spaces in E_k. Let $\boldsymbol{B}_1$ be the class of sets $B \subset E_k$ each of which has the following property: if $\boldsymbol{x} = (x^1, \cdots, x^k) \in B$ and $\boldsymbol{y} = (y^1, \cdots, y^k)$ is such that $y^i < x^i$ for $i = 1, \cdots, k$, then $\boldsymbol{y} \in B$. Let $\boldsymbol{B}_j$, $j = 2, 3, \cdots, 2^k$, be the $2^k - 1$ classes of sets which can be obtained by reversing, one at a time, the k inequalities occurring in the definition of $\boldsymbol{B}_1$. Then we put $\boldsymbol{B} = \boldsymbol{B}(E_k) = \bigcup_{j=1}^{2^k} \boldsymbol{B}_j$. Let $\boldsymbol{C} = \boldsymbol{C}(E_k)$ be the class of all measurable convex subsets of E_k.

Let $\mathfrak{M} = \mathfrak{M}(E_k)$ be the class of all probability measures on $\boldsymbol{S} = \boldsymbol{S}(E_k)$, $\mathfrak{N} = \mathfrak{N}(E_k)$ the class of probability measure $\mu \in \mathfrak{M}$ such that every convex set has a μ-null boundary, and $\mathfrak{L} = \mathfrak{L}(E_k)$ the class of the measure $\mu \in \mathfrak{M}$ which is absolutely continuous with respect to Lebesgue measure. Using the above notations, we can state some earlier main results concerning the same problem as follows:

(i) Fortet-Mourier [5]: $(\boldsymbol{A}, \mathfrak{L}) \in GC(E_k)$,

(ii) Wolfowitz [13, 14]: $(\boldsymbol{A}, \mathfrak{M}) \in GC(E_k)$,

(iii) Elum [2]: $(\boldsymbol{B}, \mathfrak{L}) \in GC(E_k)$,

(iv) Ahmad [1] and Rao [9]: $(\boldsymbol{C}, \mathfrak{L}) \in GC(E_k)$,

(v) Rao [10]: $(\boldsymbol{C}, \mathfrak{N}) \in GC(E_k)$,

(vi) Sazonov [11]: $(\boldsymbol{A}, \mathfrak{M}) \notin GC(E_\infty)$.

The pourpose of this article is to show:

$$(\boldsymbol{R}_0, \mathfrak{M}) \in GC(E_k),\ (\boldsymbol{R}, \mathfrak{L}) \notin GC(E_k),\ (\boldsymbol{S}, \mathfrak{S}) \in GC(E_k),$$

the notations being defined below.

2. Preliminalies

We shall use the following notations.

E_k: k dimensional Euclidian space,

$\boldsymbol{S} = \boldsymbol{S}(E_k)$: the class of all Borel subsets of E_k(σ-ring generated by the class of all compact subsets of E_k),

$$\boldsymbol{R}_0=\boldsymbol{R}_0(E_k)=\{(-\infty, \boldsymbol{x})=\mathop{\times}_{i=1}^{k}(-\infty, x^i);\ \boldsymbol{x}=(x^1, \cdots, x^k)\in E_k\},$$

$$\boldsymbol{R}_1=\boldsymbol{R}_1(E_k)=\{[\boldsymbol{x}, \boldsymbol{y})=\mathop{\times}_{i=1}^{k}[x^i, y^i);\ \boldsymbol{x}, \boldsymbol{y}\in E_k,\ x^i\leqq y^i\quad (i=1, 2, \cdots, k)\},$$

$\boldsymbol{R}_m=\boldsymbol{R}_m(E_k)$: the class of all elements which are the union of at most m elements of $\boldsymbol{R}_1$,

$$\boldsymbol{R}=\boldsymbol{R}(E_k)=\bigcup_{m=1}^{\infty}\boldsymbol{R}_m.$$

LEMMA 1. *$\boldsymbol{R}$ is a ring and $\boldsymbol{S}=S(\boldsymbol{R})$, where $S(\boldsymbol{R})$ denotes the σ-ring generated by the class $\boldsymbol{R}$ of sets.*

This is easily shown by the k-dimensional analogue of the 15.B of [7]. The following lemma is a direct consequence of the 6.B of [7] and lemma 1.

LEMMA 2. *Let $\boldsymbol{M}=M(\boldsymbol{R})$ be a monotone class generated by $\boldsymbol{R}$. Then $\boldsymbol{M}=\boldsymbol{S}$.*

LEMMA 3. *The class $\boldsymbol{M}=M(\boldsymbol{R})$ is given by*

$$\boldsymbol{C}=\{C=\bigcup_{j=1}^{\infty}S_j;\ S_j\in\boldsymbol{R},\ S_1\subset S_2\subset\cdots\}.$$

PROOF. Let $C_1\subsetneqq C_2\subsetneqq\cdots$, $C_i\in\boldsymbol{C}$. Then there exist increasing sequences of sets $\{S_{1j}\}$, $\{S_{2j}\}$, $\cdots$ such that

$$C_i=\bigcup_{j=1}^{\infty}S_{ij},\qquad S_{ij}\in\boldsymbol{R}\quad (j=1, 2, \cdots),$$

for all $i=1, 2, \cdots$. Since

$$S_{11}\subset C_1\subsetneqq C_2=\bigcup_{j=1}^{\infty}S_{2j},$$

there exists a finite positive integer j_2 for which

$$S_{11}\subset C_1\subset S_{2j_2}.$$

Similarly, we can select a sequence of positive integers $\{j_i\}$ such that

$$S_{i, j_i}\subset C_i\subset S_{i+1, j_{i+1}}\qquad (i=2, 3, \cdots).$$

Put

$$S_1=S_{11},\ S_i=S_{ij_i}\qquad (i=2, 3, \cdots).$$

Then for every positive integer N,

$$\bigcup_{i=1}^{N} S_i \subset \bigcup_{i=1}^{N} C_i \subset \bigcup_{i=2}^{N+1} S_i \subset \bigcup_{i=1}^{N+1} S_i .$$

Since $S_1 \subset S_2 \subset \cdots$ and $S_i \in \boldsymbol{R}$, we have

$$\bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} S_i \in \boldsymbol{C} .$$

This shows that $\boldsymbol{C}$ is a monotone class containing $\boldsymbol{R}$.

To prove that $\boldsymbol{C}$ is the smallest one, take any other monotone class $\tilde{\boldsymbol{C}} \supset \boldsymbol{R}$. Then, since every element of $\boldsymbol{C}$ is the limit of a monotone sequence of sets in $\boldsymbol{R} \subset \tilde{\boldsymbol{C}}$, we have $\boldsymbol{C} \subset \tilde{\boldsymbol{C}}$.

LEMMA 4. *Each element of $\boldsymbol{R}$ is the difference of two sums of 2^{k-1} elements of $\boldsymbol{R}_0$. Every element of $\boldsymbol{R}_m$ can be expressed by the disjoint sum and the difference of at most 2^m elements of $\boldsymbol{R}_1$.*

3. Results

First we shall state three lemmas.

LEMMA 5. *For any $\mu \in \mathfrak{M}$ and for each $\varepsilon > 0$, there exists a bounded measurable subset $\tilde{K} = \tilde{K}_\varepsilon$ such that $\mu(\tilde{K}) > 1 - \varepsilon$ and $\tilde{K}$ includes a set $Q = Q_\varepsilon$ of at most finite number of points having positive μ-measure.*

PROOF. It can be easily shown that for every $\mu \in \mathfrak{M}$ and for each $\varepsilon > 0$, there exists a bounded measurable subset $K = K_{\varepsilon/2} \subset E_k$ for which $\mu(K) > 1 - \frac{\varepsilon}{2}$. For each $i = 1, 2, \cdots, k$, let $\{h_1^{(i)}, h_2^{(i)}, \cdots\}$ be the family of hyperplanes which are perpendicular to the $x^{(i)}$-axes and have positive μ-measure. Since $\sum_{j=1}^{\infty} \mu(h_j^{(i)}) \leqq 1$, there exists a positive integer m_i such that $\sum_{j=m_i+1}^{\infty} \mu(h_j^{(i)}) \leqq \frac{\varepsilon}{2k}$ for each $i = 1, 2, \cdots, k$. Put $\tilde{K} = \tilde{K}_\varepsilon = K_{\varepsilon/2} \cap [\bigcup_{i=1}^{k} \bigcup_{j=m_i+1}^{\infty} h_j^{(i)}]^c$. Then it is easily shown that $\mu(\tilde{K}) > 1 - \varepsilon$ and that there is no point in $\tilde{K}$ which has positive μ-measure, but the points $P_{j_1, \dots, j_k}$ of $\bigcap_{i=1}^{k} h_{j_i}^{(i)}$. Therefore the set Q of such points has at most $m_1 m_2 \cdots m_k$ elements.

LEMMA 6. *Let $\boldsymbol{F}$ be a class of subsets of E_k. For any $\mu \in \mathfrak{M}$ and $\varepsilon > 0$, we put*

$$\boldsymbol{F}_\varepsilon = \{F \cap \tilde{K} \cap Q^c;\ \mu(F \cap \tilde{K} \cap Q^c) > 0,\ F \in \boldsymbol{F}\}, \tag{3.1}$$

where $\tilde{K}$ and Q are subsets of E_k as stated in the last lemma. If we as-

sume that

$$\lim_{l\to\infty} P\{\bigcup_{n\geq l}[\sup_{S\in \boldsymbol{F}_\varepsilon}|\mu(S)-\mu_n(S)|\geqq\varepsilon]\}=0\,, \tag{3.2}$$

then

$$(\boldsymbol{F},\ \mathfrak{M})\in GC(E_k)\,. \tag{3.3}$$

PROOF. For every $\mu\in\mathfrak{M}$ and $F\in\boldsymbol{F}$, the relation

$$\begin{aligned}|\mu(F)-\mu_n(F)| &\leqq |\mu(F\cap\tilde{K}\cap Q^c)-\mu_n(F\cap\tilde{K}\cap Q^c)|\\ &+|\mu(F\cap\tilde{K}\cap Q)-\mu_n(F\cap\tilde{K}\cap Q)|+|\mu(F\cap\tilde{K}^c)-\mu_n(F\cap\tilde{K}^c)|\\ &\leqq \sup_{S\in\boldsymbol{F}_\varepsilon}|\mu(S)-\mu_n(S)|+\max_{S\in\boldsymbol{Q}}|\mu(S)-\mu_n(S)|+|\mu(\tilde{K}^c)-\mu_n(\tilde{K}^c)|+\varepsilon\end{aligned} \tag{3.4}$$

holds with probability one, where $\boldsymbol{Q}$ is a finite family of subsets of Q. Put

$$A_n=[\sup_{F\in\boldsymbol{F}}|\mu(F)-\mu_n(F)|\geqq 4\varepsilon]\,,$$

$$B_n=[\sup_{S\in\boldsymbol{F}_\varepsilon}|\mu(S)-\mu_n(S)|\geqq\varepsilon]\,,$$

$$C_n=[\max_{S\in\boldsymbol{Q}}|\mu(S)-\mu_n(S)|\geqq\varepsilon]\,,$$

$$D_n=[\,|\mu(\tilde{K}^c)-\mu_n(\tilde{K}^c)|\geqq\varepsilon]\,.$$

Then, by the relation (3.4), for each n, we have

$$A_n\subset B_n\cup C_n\cup D_n\,.$$

Therefore, for every l

$$P\{\bigcup_{n\geq l}A_n\}\leqq P\{\bigcup_{n\geq l}B_n\}+P\{\bigcup_{n\geq l}C_n\}+\{\bigcup_{n\geq l}D_n\}\,. \tag{3.5}$$

When $l\to\infty$ the first term of the right-hand side of the inequality (3.5) tends to zero by the assumption (3.2) and the second and the third term also tend to zero according to the strong law of large numbers and Loève's definition of a.s. convergence [8, p. 151]. This completes the proof of lemma 6.

LEMMA 7. (Blum [2]'s lemma 1) *Let $\boldsymbol{F}$ be a family of subsets of E_k such that for each $\varepsilon>0$ there exists a finite class of sets $\boldsymbol{F}(\varepsilon)$ and for every $F\in\boldsymbol{F}$ there exist F_1 and F_2 in $\boldsymbol{F}(\varepsilon)$, satisfying*

$$F_1\subset F\subset F_2\,,\qquad \textit{with}\qquad \mu(F_2)-\mu(F_1)\leqq\varepsilon\,.$$

Then

$$P\{\lim_{n\to\infty}\sup_{F\in \boldsymbol{F}}|\mu(F)-\mu_n(F)|=0\}=1\,.$$

The proof is immediately obtained by using the same method as in the proof of the classical theorem of Glivenko-Cantelli (cf. for example, [6, p. 391]).

THEOREM 1. $(\boldsymbol{R}_0, \mathfrak{M})\in GC(E_k)\,.$

PROOF. By lemma 6 it is sufficient to show that

$$\lim_{l\to\infty} P\{\bigcup_{n\geqq l}[\sup_{R_0\in \boldsymbol{R}_\varepsilon}|\mu(R_0)-\mu_n(R_0)|\geqq\varepsilon]\}=0\,, \tag{3.6}$$

where

$$\boldsymbol{R}_\varepsilon=\{R_0=R\cap\tilde{K}\cap Q^c;\ \mu(R_0)>0,\ R\in \boldsymbol{R}_0\}\,.$$

With no loss of generality we can assume that $k=2$.

For any $\mu\in\mathfrak{M}$ and $\varepsilon>0$, we can select the $K_{\varepsilon/2}$ stated in the proof of lemma 5 as an element of $\boldsymbol{R}_1$. Put $m=[2\delta/\varepsilon]+1$, where $\delta=\mu(\tilde{K}\cap Q^c)$ and $[x]$ denotes the largest integer not larger than x. Let $D_0^{(1)}=D_0^{(2)}=\phi$ (empty set) and $D_m^{(1)}=D_m^{(2)}=\tilde{K}\cap Q^c$. Let $D_j^{(1)}$ be the intersection of $\tilde{K}\cap Q^c$ and the open left half-space of such line $l_j^{(1)}$ as paralleling $x^{(2)}$-axes $(j=1,\cdots,m-1)$, $D_j^{(2)}$ the intersection of $\tilde{K}\cap P^c$ and the open under half-space of $l_j^{(2)}$ paralleling $x^{(1)}$-axes $(j=1, 2,\cdots,m-1)$ such that it holds:

$$\mu(D_j^{(i)}-D_{j-1}^{(i)})=\frac{\delta}{m}\,,\qquad j=1, 2,\cdots,m,\quad i=1, 2.$$

We put

$$\boldsymbol{F}(\varepsilon)=\{D_{ij}=D_i^{(1)}\cap D_j^{(2)};\ i, j=1,\cdots,m\}\,.$$

Then for any $R_0\in \boldsymbol{R}_\varepsilon$ we can find $D_{i,j}$ and $D_{i+1,j+1}$ for which

$$D_{i,j}\subset R_0\subset D_{i+1,j+1}$$

$$\mu(D_{i+1,j+1})-\mu(D_{ij})\leqq\mu(D_{i+1}^{(1)},\ D_i^{(1)})+\mu(D_{j+1}^{(2)}-D_j^{(2)})\leqq\varepsilon\,.$$

Hence by lemma 7 the relation (3.6) holds.

Combining theorem 1 with lemma 4, we obtain

THEOREM 2. *For each fixed integer $m>0$,*

$$(\boldsymbol{R}_m, \mathfrak{M})\in GC(E_k)\,.$$

But the result can not be extended to the class $\boldsymbol{R}$. In fact,

THEOREM 3. $$(\boldsymbol{R}, \mathfrak{L}) \notin GC(E_k).$$

PROOF. Let $\mu \in \mathfrak{L}$. For each $\boldsymbol{\omega} \in \Omega^\infty$ and every positive integer n, we can construct $S_n = S_n(\boldsymbol{\omega}) \in \boldsymbol{R}$ such that $\mu(S_n) > 1-\varepsilon$ and $\mu_n(S_n)=0$, where $\varepsilon>0$ is an arbitrarily chosen number. In fact, for every $i=1, 2, \cdots, n$, we can find the rectangle $R_i = R_i(\boldsymbol{\omega})$ which includes the point $\xi_i(\boldsymbol{\omega})$ and the μ-measure of which is smaller than $\varepsilon\, 2^{-i}$, since μ is absolutely continuous with respect to Lebesgue measure. Put $S_n = E_k - \bigcup_{i=1}^{n} R_i$. Then, for every n,

$$\Pr\{\sup_{S \in \boldsymbol{R}} |\mu(S) - \mu_n(S)| > 1-\varepsilon\} = 1.$$

This completes the proof.

In the following, we shall say that a family $\boldsymbol{F}$ of sets has the property (P_m) for a probability measure $\mu \in \mathfrak{M}$, where m is a positive integer, if for each $R \in \boldsymbol{F}$ there exists at least one subset $R_0 \subset R$ such that $R_0 \in \boldsymbol{R}_m$ and $\mu(R) = \mu(R_0)$.

Let $\tilde{\boldsymbol{R}}$ be a subfamily of $\boldsymbol{R}$ such that for every $\mu \in \mathfrak{M}$ and each $\varepsilon > 0$ we can select a positive integer $m = m_\varepsilon$ for which

$$\tilde{\boldsymbol{R}}_\varepsilon = \{R \cap K;\ \mu(R \cap K) > 0,\ R \in \tilde{\boldsymbol{R}}\}$$

has the property (P_m), where $K = K_\varepsilon$ is the bounded measurable subsets of E_k stated in the proof of lemma 5.

From theorem 2 and lemma 6 we obtain

THEOREM 4. $$(\tilde{\boldsymbol{R}}, \mathfrak{M}) \in GC(E_k).$$

Let $\mathfrak{F}$ be the class of probability measures $\mu \in \mathfrak{M}$ for which for every $\varepsilon > 0$ there exists a positive integer $m = m_\varepsilon$ such that

$$\boldsymbol{R}_\varepsilon = \{R \cap K;\ \mu(R \cap K) > 0,\ R \in \boldsymbol{R}\}$$

has the property (P_m). Then, according to the same reasoning as above, we obtain

THEOREM 5. $$(\boldsymbol{R}, \mathfrak{F}) \in GC(E_k).$$

LEMMA 9. *Let $\mathfrak{C}$ be a class of probability measures on E_k. If $(\boldsymbol{R}, \mathfrak{C}) \in GC(E_k)$, then $(\boldsymbol{S}, \mathfrak{C}) \in GC(E_k)$.*

PROOF. By lemmas 1 and 2 it is sufficient to show that $(\boldsymbol{M}, \mathfrak{C}) \in GC(E_k)$. From lemma 3, for any $M \in \boldsymbol{M}$, there exist $S_i \in \boldsymbol{R}$ such that $S_1 \subset$

$S_2 \subset \cdots$ and $M = \bigcup_{i=1}^{\infty} S_i$. Since for every n and for each m the relation

$$|\mu(\bigcup_{i=1}^{m} S_i) - \mu_n(\bigcup_{i=1}^{m} S_i)| = |\mu(S_m) - \mu_n(S_m)| = \sup_{S \in \boldsymbol{R}} |\mu(S) - \mu_n(S)|$$

holds with probability one, for every n we have

$$|\mu(M) - \mu_n(M)| \leqq \sup_{S \in \boldsymbol{R}} |\mu(S) - \mu_n(S)|$$

with probability one. Hence, for every n

$$\Pr\{\sup_{M \in \boldsymbol{M}} |\mu(M) - \mu_n(M)| = \sup_{S \in \boldsymbol{R}} |\mu(S) - \mu_n(S)|\} = 1$$

which completes the proof of lemma 9.

Combining the above lemma with theorem 5, we have

THEOREM 6. $(\boldsymbol{S}, \mathfrak{G}) \in GC(E_k)$.

It is easily seen that the class $\mathfrak{G}$ contains the class $\mathfrak{D}$ of all discrete probability measures on E_k. Thus we obtain

COROLLARY 1. $(\boldsymbol{S}, \mathfrak{D}) \in GC(E_k)$.

4. Remarks

(i) For a strictly stationary sequence of random vectors, similar results can be deduced, for example Tucker [12], Rao [10], Burke [3], etc. Furthermore, another generalization of the Glivenko-Cantelli theorem is developed by regarding $F_n(t)$ as a non-decreasing stochastic process, for example Fisz [4] and Burke [3]. We did not consider such cases in this note.

(ii) Since $\mathfrak{N} \subset \mathfrak{M}$ but $\boldsymbol{R}_0 \subset \boldsymbol{C}$, our theorems 1 and 2 cannot be reduced from Rao's result [10], which is not a corollary of our results either.

(iii) Sazonov [11] obtained a subfamily $\boldsymbol{A}_0 \subset \boldsymbol{A}$ for which $(\boldsymbol{A}_0, \mathfrak{M}_0) \in GC(E_\infty)$, where $\mathfrak{M}_0 \subset \mathfrak{M}$ is a family of distributions on E_∞. But he did not consider the class $\boldsymbol{R}_0$. Does a subfamily $\mathfrak{C} \subset \mathfrak{M}$ exist for which $(\boldsymbol{R}_0, \mathfrak{C})$ belongs to the Glivenko-Cantelli class of the infinite dimensional Euclidian space E_∞ ?

(iv) Furthermore we could not answer the following questions:

Q_1: Does the pair $(\boldsymbol{C}, \mathfrak{M})$ belong to the class $GC(E_k)$? The affirmative answer to this question is Rao [10]'s extension.

Q_2: Is there the largest class $\mathfrak{C}\subset\mathfrak{M}$ such that $(S, \mathfrak{C}) \in GC(E_k)$? If exist, what is that?

Q_3: Is the class $\mathfrak{D}$ the proper sub-class of $\mathfrak{C}$?

(v) We are preparing another paper in which we shall show some results of the compound decision problem using theorem 4.

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CORRECTIONS TO
"ON THE GLIVENKO-CANTELLI THEOREM"

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In the above titled article (this Annals 18 (1966), 29–37) the following corrections should be made.

(i) On page 30, line 7, replace

"$\boldsymbol{B}_1$ be the class of sets . . ."

by

"$\boldsymbol{B}_1$ be the class of measurable sets . . .".

(ii) The part from page 31, line 9 (The following lemma is . . .) to page 32, line 7 (. . . , we have $C \subset \tilde{C}$) should be deleted.

(iii) The part from page 35, line 4 from bottom to page 36, line 9 should be deleted.

(iv) On page 36, line 12, replace

$(\boldsymbol{S}, \mathfrak{D}) \in GC(E_k)$

by

$(\boldsymbol{R}, \mathfrak{D}) \in GC(E_k)$.

The author is indebted to Dr. F. Topsoe, who pointed out that lemma 3 does not hold. Accordingly, lemma 9, theorem 6 and corollary 1 are also false. Therefore lemmas 1 and 2 are not necessary. Thus these should be deleted.