ON CLASSES OF BAYES SOLUTIONS

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1. Introduction

It is well-known that, under some restrictions, the class W of Bayes solutions in the wide sense and the closure B^{α} of the class B of Bayes solutions with respect to the regular topology are complete in the relevant class of decision functions (see [1]).

LeCam [2] asserts that the intersection of B^{α} and W is also complete. In this connection, it will be of some interest to know whether $B^{\alpha}\cap W$ actually gives a smaller complete class than B^{α} or W. In the present paper sufficient conditions are given for B^{α} and W^{α}, respectively, to coincide with the class of all decision functions, say D. Also examples in which $W\subsetneqq B^{\alpha}=D$ or $B^{\alpha}\subsetneqq W=D$ are given. Further, in theorem 3, it is proved that W is dense in D under fairly general conditions. Since in many cases we have $B^{\alpha}\subsetneqq D$, this fact implies that there often exist elements of W which are not contained in B^{α}. Thus, we might say that W is not likely to give a smaller complete class than B^{α}.

2. Notations and definitions

In this paper, we deal only with the non-sequential case, which does not seem to harm essentials of the theory. A measurable space $(\mathscr{X}, \mathscr{B})$ will be called a sample space, where $\mathscr{B}$ is a σ-field of subsets of $\mathscr{X}$ and we consider a family of probability measures on $\mathscr{X}$, $\mathscr{P}=\{p_{\theta}:\theta\in\Omega\}$, where the index set Ω is called a parameter space. A topological space A will be an action space and a function $w(\theta, a)$ on $\Omega\times A$ will be called a loss function.

Throughout this paper we assume the following:

Assumption (A). A is a metrizable, locally compact, and σ-compact space.

Assumption (B). $w(\theta, a)$ is lower semicontinuous on A and $w(\theta, a)\geqq 0$ for every $\theta\in\Omega$ and $a\in A$.

Assumption (C). Every p_{θ} $(\theta\in\Omega)$ is absolutely continuous with respect to a σ-finite measure μ. We denote the density $dp_\theta/d\mu$ by $f(\cdot, \theta)$.

Let $C_0(A)$ be the totality of bounded continuous real valued functions on A vanishing outside a compact set of A with norm $\|\alpha\|=\sup_{a \in A}|\alpha(a)|$. Let $M(\mathscr{X})$ be the set of all bounded $\mathscr{B}$-measurable functions with norm $\|f\|=\text{ess.}\sup_{x \in \mathscr{X}}|f(x)|$ and $\mathscr{F}$ the set of all probability measures on A. A mapping δ of $\mathscr{X}$ to $\mathscr{F}$: $x \to \delta(\cdot : x)$ is called a decision function if $\int_A \alpha(a)\delta(da : x) \in M(\mathscr{X})$ for all $\alpha \in C_0(A)$. We denote this integral by $\delta \circ \alpha$.

Let $E(\mathscr{P})$ be the closed linear space spanned by $\{f(x, \theta) : \theta \in \Omega\}$. We write $\int_{x} s(x)f(x)d\mu(x)$ by $f \circ s$ for $f \in E(\mathscr{P})$.

We can define a topology by neighborhoods

$$V(\delta^* : \varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$$
$$=\{\delta : |f_i \circ \delta \circ \alpha_i - f_i \circ \delta^* \circ \alpha_i| < \varepsilon,\ i=1, 2, \cdots, n\},$$

where α_i runs through $C_0(A)$ and f_i through $E(\mathscr{P})$ (LeCam [2]). We call it the regular topology. By assumption (B), for every fixed θ, the loss function $w(\theta, a)$ is the limit of a monotone increasing sequence $\{\alpha_n\}$ ($\in C_0(A)$, $n=1, 2, \cdots$) and so $\int_A w(\theta, a)\delta(da : x)$ is also a $\mathscr{B}$-measurable function of x. Hence the risk function

$$r(\theta, \delta)=\int_{x}\int_A w(\theta, a)\delta(da : x)f(x, \theta)d\mu(x)$$

can be defined for every $\theta(\in \Omega)$ and $\delta(\in D)$.

We call a probability measure on Ω an a priori distribution on Ω and denote by $\mathscr{M}$ the set of all a priori distributions on Ω having as its carrier a finite subset of Ω. Hereafter we denote the integral $\int_\Omega r(\theta, \delta)d\xi(\theta)$ by $r(\xi, \delta)$ for an a priori distribution ξ.

For a given class $\mathcal{E}$ of a priori distributions, we denote by $B_{\mathcal{E}}$ the set of all Bayes solutions with respect to some ξ in $\mathcal{E}$. If there exists a sequence of $\xi_1, \xi_2, \cdots$ such that $\lim_{n\to\infty}(r(\xi_n, \delta^*)-\inf_\delta r(\xi_n, \delta))=0$, δ^* is called a Bayes solution in the wide sense with respect to $\xi_1, \xi_2, \cdots$. For a given class $\mathcal{E}$ of a priori distributions, we denote by $W_{\mathcal{E}}$ the set of all Bayes solutions in the wide sense with respect to $\xi_1, \xi_2, \cdots$, where all ξ_n are included in $\mathcal{E}$.

Throughout this paper we denote by S^a the closure of $S(\subset D)$ for the regular topology.

3. Closures of B and W

A family of probability measures $\{p_\theta : \theta \in \Omega\}$ on $(\mathscr{X}, \mathscr{B})$ will be called to have the B-property if there exist mutually disjoint subsets E_1, $E_2, \cdots (\in \mathscr{B})$ and $\theta_1, \theta_2, \cdots (\in \Omega)$ such that $\bigcup_{i=1}^{\infty} E_i = \mathscr{X}$ and $p_{\theta_i}(E_i)=1$ for all i. As an example, a family of uniform distributions with range $[\theta, \theta+1]$ has this property.

THEOREM 1. *Suppose that*

(1.a) $\mathscr{P} = \{p_\theta : \theta \in \Omega\}$ *has the B-property,*

(1.b) *there exists a_θ such that $w(\theta, a_\theta) = \inf_a w(\theta, a)$ for all θ.*
Then we have $B^a_{\mathscr{M}} = D$.

Proof. From the B-property of $\mathscr{P}$, we can choose mutually disjoint $E_1, E_2, \cdots$ and $\theta_1, \theta_2, \cdots$ satisfying $\bigcup_{i=1}^{\infty} E_i = \mathscr{X}$ and $p_{\theta_i}(E_i)=1$ for all i. For a given δ_0 and a neighborhood $V(\delta_0 : \varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$, we choose n_0 such that

$$\int_{(\bigcup_{i=1}^{n_0} E_i)^c} |f_i(x)|\, d\mu(x) < \frac{\varepsilon}{2I} \qquad \text{for all } i=1, 2, \cdots, n,$$

where $I = \max_i \|\alpha_i\|$. Define a decision function δ^* as follows:

$$\delta^*(\cdot : x) = \delta_0(\cdot : x) \qquad \text{for} \quad x \in \bigcup_{i=1}^{n_0} E_i$$

$$\delta^*(\{a_{\theta_m}\} : x) = 1 \qquad \text{for} \quad x \in E_m (m > n_0)\,.$$

We have for $m > n_0$

$$\begin{aligned} r(\theta_m, \delta^*) &= \int_{E_m}\int_A w(\theta_m, a)\delta^*(da;\ x) f(x, \theta_m) d\mu(x) \\ &= \int_{E_m} \inf_a w(\theta_m, a) f(x, \theta_m) d\mu(x) \\ &= \inf_a w(\theta_m, a) p_{\theta_m}(E_m) = \inf_a w(\theta_m, a) \\ &= \inf_\delta r(\theta_m, \delta)\,. \end{aligned}$$

Thus, for $m > n_0$, δ^* is a Bayes solution with respect to ξ_{θ_m}, where ξ_{θ_m} is the a priori distribution satisfying $\xi_{\theta_m}(\{\theta_m\}) = 1$.

Further we have

$$|f_i \circ \delta_0 \circ \alpha_i - f_i \circ \delta^* \circ \alpha_i|$$

$$=\left|\int_{(\bigcup_{i=1}^{n_0} E_i)^c} \int_A \alpha_i(a)\delta_0(da:x)f_i(x)d\mu(x) - \int_{(\bigcup_{i=1}^{n_0} E_i)^c} \int_A \alpha_i(a)\delta^*(da:x)f_i(x)d\mu(x)\right|$$

$$\leqq 2I\int_{(\bigcup_{i=1}^{n_0} E_i)^c} |f_i(x)|\,d\mu(x)=2I\frac{\varepsilon}{2I}=\varepsilon \quad \text{for all } i.$$

This shows $\delta^* \in V(\delta_0 : \varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$ and we obtain $B^\alpha_{\mathcal{M}}=D$.

Using the above theorem, we can obtain an example in which $W_{\mathcal{M}} \subsetneqq B^\alpha_{\mathcal{M}}$. In general, such an example is not easily obtained. Let the sample space $\mathscr{X}$ be the real line $(-\infty, +\infty)$ and the density of p_θ $(\theta \in (-\infty, +\infty))$ be such that

$$f(x, \theta)=\begin{cases} 1 & \text{for } x \in \left(\theta-\frac{1}{2},\ \theta+\frac{1}{2}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Define $w(\theta, a)=(\theta-a)^2$. Consider the following two decision functions δ_1, δ_2:

$$\delta_1(\{x\} : x)=1 \qquad \text{for all } x \in \mathscr{X},$$

$$\delta_2(\{x+1\} : x)=1 \qquad \text{for all } x \in \mathscr{X}.$$

With easy calculations, we have $r(\theta, \delta_1)=1/12$ and $r(\theta, \delta_2)=13/12$ for all $\theta \in (-\infty, +\infty)$. So δ_2 can not be a Bayes solution in the wide sense since $r(\xi, \delta_2)-r(\xi, \delta_1)=1$ for all ξ. On the other hand this problem satisfies the conditions of the previous theorem and hence $B^\alpha_{\mathcal{M}}=D$. Thus we have $W_{\mathcal{M}} \subsetneqq B^\alpha_{\mathcal{M}}$. Since we can easily get an example in which $B^\alpha_{\mathcal{M}} \subsetneqq W$, the assertion that $B^\alpha_{\mathcal{M}} \cap W$ is complete is actually an improvement of the theorems about completeness of $B^\alpha_{\mathcal{M}}$ and W.

A family of probability measures $\{p_\theta : \theta \in \Omega\}$ on $(\mathscr{X}, \mathscr{B})$ will be called to have the A-property if there exist mutually disjoint subsets E_1, E_2, $\cdots$ $(\in \mathscr{B})$ and θ_1, θ_2, $\cdots$ $(\in \Omega)$ such that $\bigcup_{i=1}^{\infty} E_i=\mathscr{X}$ and $\lim_{i\to\infty} p_{\theta_i}(E_i)=1$. We meet such a family in many cases, e.g. a family of distributions with the location parameter θ, Poisson distributions $\frac{e^{-\theta}\theta^x}{x\,!}$, etc.

THEOREM 2. *Suppose that*

(2.a) $\mathscr{P}=\{p_\theta:\theta\in\Omega\}$ *has the A-property,*

(2.b) *there exists K such that* $w(\theta, a)\leqq K<\infty$ *for all* θ *and* a.

Then we have $W^{\alpha}_{\mathscr{M}}=D$.

Proof. By (2. a), there exist mutually disjoint $E_1, E_2, \cdots$ and $\theta_1, \theta_2, \cdots$ such that $\bigcup_{i=1}^{\infty} E_i=\mathscr{X}$ and $\lim_{i\to\infty} p_{\theta_i}(E_i)=1$. For a given $\delta_0\in D$ and a neighborhood $V(\delta_0:\varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$, we can choose a sufficiently large n_0 such that

$$\int_{(\bigcup_{i=1}^{n_0} E_i)^c} |f_i(x)|\,d\mu(x)<\frac{\varepsilon}{2I} \qquad \text{for all } i=1, 2, \cdots, n\,,$$

where I is already defined in the previous theorem.

We define a decision function δ^* as follows:

$$\delta^*(\cdot:x)=\delta_0(\cdot:x) \qquad \text{for} \quad x\in\bigcup_{i=1}^{n_0} E_i$$

$$\delta^*(\{d_{\theta_m}\};\ x)=1 \qquad \text{for} \quad x\in E_m \quad (m>n_0)\,,$$

where d_{θ_m} is an action satisfying $w(\theta_m, d_{\theta_m})<\inf_a w(\theta_m, a)+\frac{1}{m}$. Then we have for $m>n_0$

$$\begin{aligned} r(\theta_m, \delta^*)-\inf_\delta r(\theta_m, \delta) &= \int_{E_m}\int_A w(\theta_m, a)\delta^*(da:x)f(x, \theta_m)d\mu(x) \\ &+\int_{E_m^c}\int_A w(\theta_m, a)\delta^*(da:x)f(x, \theta_m)d\mu(x) \\ &-\inf_a w(\theta_m, a)\leqq p_{\theta_m}(E_m)\cdot\left(\inf_a w(\theta_m, a)+\frac{1}{m}\right) \\ &+K(1-p_{\theta_m}(E_m))-\inf_a w(\theta_m, a) \\ &\leqq\frac{1}{m}+K(1-p_{\theta_m}(E_m))\,. \end{aligned}$$

It follows from the A-property that the right hand side tends to 0 as $m\to\infty$. Hence δ^* above defined is a Bayes solution in the wide sense with respect to $\xi_{\theta_m}(m=1, 2, \cdots)$. The proof of $\delta^*\in V(\delta_0:\varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$ is completely similar to that of the previous theorem and is omitted. Hence we obtain $W^{\alpha}_{\mathscr{M}}=D$.

We note that assumption (A) and the lower semicontinuity of the loss function, which have been assumed throughout this paper, are not necessary in the proofs of theorems 1 and 2.

Before proceeding to the next theorem, we shall prove the following lemma.

LEMMA 1. *Let D^* be the set of all decision functions such that there exists a compact set C_δ depending on δ satisfying $\delta(C_\delta : x)=1$ for a.e.x (μ). Then we have $D^{*^\alpha}=D$.*

Proof. If the action space A is compact, this assertion is clear since $D^*=D$. So we assume that A is not compact. Take an arbitrary $\delta \in D$. For given $\alpha_1, \alpha_2, \cdots, \alpha_n \in C_0(A)$, $f_1, f_2, \cdots, f_n \in E(\mathscr{P})$ and $\varepsilon(>0)$, put $S=\bigcup_{i=1}^{n} \operatorname{car} \alpha_i$ where $\operatorname{car} \alpha$ stands for the carrier of α. We fix a point $\tilde{a} \in S^c$ arbitrarily. This is possible since S is compact and A is not compact.

We define $\hat{\delta}$ as follows:

$$\hat{\delta}(B : x)=1-\delta(S : x)+\delta(B \cap S : x) \qquad \text{for } \tilde{a} \in B$$

$$\hat{\delta}(B : x)=\delta(B \cap S : x) \qquad \text{for } \tilde{a} \notin B.$$

It is easily shown that $\hat{\delta}$ becomes a decision function and moreover $\hat{\delta} \in D^*$. $\hat{\delta}$ coincides with δ on the set S. Then we have

$$\begin{aligned} &|f_i \circ \delta \circ \alpha_i - f_i \circ \hat{\delta} \circ \alpha_i| \\ &=\left|\int_{\mathscr{X}}\int_S \alpha_i(a)\delta(da : x)f_i(x)d\mu(x)-\int_{\mathscr{X}}\int_S \alpha_i(a)\hat{\delta}(da : x)f_i(x)d\mu(x)\right| \\ &=0<\varepsilon. \end{aligned}$$

Consequently $\hat{\delta} \in V(\delta : \varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$ and therefore $D^{*^\alpha}=D$. Thus the proof of the lemma is completed.

The condition (2. b) of theorem 2 is rather restrictive. When the parameter space Ω is a topological space, the same assertion as in theorem 2 can be obtained under the different conditions from those of theorem 2.

THEOREM 3. *Suppose that*

(3.a) *Ω is σ-compact and metrizable,*

(3.b) *the family of functions of a: $\{f(x, \theta)w(\theta, a) : \theta \in \Omega\}$ with a fixed x is equicontinuous on A,*

(3.c) *there exists a sequence $F_1, F_2, \cdots (\subset \Omega)$ of compact subsets such that $\bigcup_{i=1}^{\infty} F_i=\Omega$ and $\lim_{n\to\infty} \sup_{\theta \in F_n^c} f(x, \theta)w(\theta, a)=0$ for a.e.x (μ) and for any a,* (3.d) *$r(\theta, \delta)$ is continuous on Ω for every fixed δ,*

(3.e) *$f(x, \theta)w(\theta, a)$ is continuous on Ω for every fixed a and x.*
Let $\mathcal{G}=\{\xi_1, \xi_2, \cdots\}$ be a sequence of a priori distributions on σ-field of Borel subsets of Ω such that, for a compact subset E of Ω,

$$\text{(3.1)} \qquad \lim_{n\to\infty} \xi_n(E)=0 .$$

Suppose further that $W_{\mathcal{G}}$ is not empty and there exists $\delta^* \in W_{\mathcal{G}}$ such that, for almost all x (μ), there exists a compact N_x for which $\delta^*(N_x : x)=1$. Then we have $W_{\mathcal{G}}^a=D$.

Proof. It is sufficient to prove $D^* \subset W_{\mathcal{G}}^a$ since $D=D^{*a} \subset (W_{\mathcal{G}}^a)^a = W_{\mathcal{G}}^a$ by lemma 1. Since A is locally compact and σ-compact, we can choose compact subsets $I_1, I_2, \cdots$ of A such that $\bigcup_{j=1}^{\infty} I_j=A$ and $I_j \subset I_{j+1}^0$ for all j, where I^0 means the interior of I. Then, for a given compact set C, there exists I_j such that $C \subset I_j$. So, if we put $K_j=\{x : \delta^*(I_j : x)=1\}$ we have $K_j \subset K_{j+1}$ for all j and $\bigcup_{j=1}^{\infty} K_j=\mathscr{X}$ for a.e. x (μ). It follows from (3.b) and (3.c) that we have

$$\text{(3.2)} \qquad \lim_{n\to\infty} \sup_{\theta \in F_n^c} \sup_{a \in C} w(\theta, a)f(x, \theta)=0 \text{ for any compact set } C.$$

Since Ω and A are metrizable and separable by assumption (A) and (3.a), and since $w(\theta, a)f(x, \theta)$ is lower semicontinuous on A and on Ω by assumption (B) and (3.e), $\sup_{\theta \in F_n^c} \sup_{a \in C} w(\theta, a)f(x, \theta)$ is measurable with respect to x. Hence, by (3.2) and the Egoroff's theorem, there exist $N_1, N_2, \cdots \in \mathscr{B}$ such that $N_i \subset N_{i+1}$, $\mu(N_i)<\infty$ for all i, $\bigcup_{i=1}^{\infty} N_i=\mathscr{X}$ (a.e. $x(\mu)$) and

$$\text{(3.3)} \qquad \sup_{\theta \in F_n^c} \sup_{a \in I_j} w(\theta, a)f(x, \theta) \to 0 \text{ uniformly on } N_i$$

as $n\to\infty$ for any i and j. Put $Z_i=N_i \cap K_i$. Then it is obvious $Z_i \subset Z_{i+1}$ for all i and $\bigcup_{i=1}^{\infty} Z_i=\mathscr{X}$ (a.e.x (μ)). For a given $\delta \in D^*$ and its neighborhood $V(\delta : \varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$, we first choose a sufficiently large m such that

$$\text{(3.4)} \qquad \int_{Z_m^c} |f_i(x)| \, d\mu(x) < \frac{\varepsilon}{2K} \qquad \text{for all } i,$$

where $K=\max_i \|\alpha_i\|$. We then define a decision function $\tilde{\delta}$ as follows:

$$\tilde{\delta}(\cdot : x) = \delta(\cdot : x) \qquad \text{for } x \in Z_m$$

$$\tilde{\delta}(\cdot : X) = \delta^*(\cdot : x) \qquad \text{for } x \notin Z_m .$$

As δ is taken from D^*, there exists a compact set C_δ such that $\delta(C_\delta : x) = 1$ for a.e. x (μ). Since $\tilde{\delta}$ thus defined coincides with δ on Z_m and with δ^* on Z_m^c, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \sup_{\theta \in F_n^c} | r(\theta, \tilde{\delta}) - r(\theta, \delta^*) | \\ &= \lim_{n\to\infty} \sup_{\theta \in F_n^c} \left| \int_{Z_m} \int_{C_\delta} w(\theta, a) \tilde{\delta}(da : x) f(x, \theta) d\mu(x) \right. \\ &\qquad \left. - \int_{Z_m} \int_A w(\theta, a) \delta^* (da : x) f(x, \theta) d\mu(x) \right| \\ &\leqq \lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \int_{C_\delta} w(\theta, a) \tilde{\delta}(da : x) f(x, \theta) d\mu(x) \\ &\qquad + \lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \int_A w(\theta, a) \delta^*(da : x) f(x, \theta) d\mu(x) . \end{aligned}$$

At first we consider about the first term of the right hand side of the above inequality. Since there exists I_j satisfying $C_\delta \subset I_j$, by (3.3) we have

$$\begin{aligned} (3.5) \qquad &\lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \int_{C_\delta} w(\theta, a) \tilde{\delta}(da : x) f(x, \theta) d\mu(x) \\ &= \lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \int_{I_j} w(\theta, a) \tilde{\delta}(da : x) f(x, \theta) d\mu(x) \\ &\leqq \lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \sup_{a \in I_j} w(\theta, a) f(x, \theta) d\mu(x) \\ &\leqq \lim_{n\to\infty} \int_{Z_m} \sup_{\theta \in F_n^c} \sup_{a \in I_j} w(\theta, a) f(x, \theta) d\mu(x) = 0 . \end{aligned}$$

The second term is

$$\begin{aligned} (3.6) \qquad &\lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \int_A w(\theta, a) \delta^*(da : x) f(x, \theta) d\mu(x) \\ &= \lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \int_{I_m} w(\theta, a) \delta^*(da : x) f(x, \theta) d\mu(x) \end{aligned}$$

$$\leqq \lim_{n\to\infty} \sup_{\theta \in F_n^c} \int_{Z_m} \sup_{a \in I_m} w(\theta, a) f(x, \theta) d\mu(x)$$

$$\leqq \lim_{n\to\infty} \int_{Z_m} \sup_{\theta \in F_n^c} \sup_{a \in I_m} w(\theta, a) f(x, \theta) d\mu(x) = 0 .$$

The first equality of (3.6) follows from $Z_m = N_m \cap K_m \subset K_m$ and $\delta^*(I_m : x) = 1$ for $x \in K_m$. The last inequalities of (3.5) and (3.6) follow from $Z_m = N_m \cap K_m \subset N_m$, $\mu(Z_m) < \infty$ and (3.3). Thus we have

$$\lim_{n\to\infty} \sup_{\theta \in F_n^c} | r(\theta, \tilde{\delta}) - r(\theta, \delta^*) | = 0. \tag{3.7}$$

Hence, for a given ε, there exists j such that

$$| r(\theta, \tilde{\delta}) - r(\theta, \delta^*) | < \varepsilon \qquad \text{for } \theta \in F_j^c . \tag{3.8}$$

By (3. d) we have $\sup_{\theta \in F_j} r(\theta, \delta^*) < \infty$ and $\sup_{\theta \in F_j} r(\theta, \tilde{\delta}) < \infty$. We denote them by L and T, respectively. If we take a sufficiently large n_0 such that $\xi_n(F_j) < \frac{\varepsilon}{L+T}$ for $n > n_0$, then we have, for $n > n_0$,

$$| r(\xi_n, \tilde{\delta}) - r(\xi_n, \delta^*) |$$

$$\leqq \int_{F_j} r(\theta, \tilde{\delta}) d\xi_n(\theta) + \int_{F_j} r(\theta, \delta^*) d\xi_n(\theta) + \int_{F_j^c} | r(\theta, \tilde{\delta}) - r(\theta, \delta^*) | d\xi_n(\theta)$$

$$\leqq (L+T)\xi_n(F_j) + \varepsilon \xi_n(F_j^c) < 2\varepsilon .$$

This implies that $\hat{\delta}$ is also a Bayes solution in the wide sense with respect to $\xi_1, \xi_2, \cdots$.

At last we shall show $\tilde{\delta} \in V(\delta : \varepsilon, f_i, \alpha_i, i=1, 2, \cdots, n)$. We have

$$| f_i \circ \tilde{\delta} \circ \alpha_i - f_i \circ \delta \circ \alpha_i |$$

$$= \left| \int_{Z_m^c} \int_A \alpha_i(a) \delta^*(da : x) f_i(x) d\mu(x) - \int_{Z_m^c} \int_A \alpha_i(a) \delta(da : x) f_i(x) d\mu(x) \right|$$

$$\leqq 2 \| \alpha_i \| \left(\int_{Z_m^c} | f_i(x) | d\mu(x) \right) < 2K \left(\frac{\varepsilon}{2K} \right) = \varepsilon .$$

The first equality follows from the fact that $\tilde{\delta}$ coincides with δ on Z_m by the definition of $\tilde{\delta}$. The inequality follows from (3.4). Thus we complete the proof.

As an instructive example, we cite the estimation of the location parameter θ of $N(\theta, \sigma^2)$ with quadratic loss when σ^2 is known. Conditions (3. a), (3. d) and (3. e) are evidently satisfied. We have

$$\left|(\theta-a)^2 \exp \frac{-(x-\theta)^2}{2\sigma^2}-(\theta-a')^2 \exp \frac{-(x-\theta)^2}{2\sigma^2}\right|$$

$$\leqq |a-a'| \cdot |a+a'| \exp \frac{-(x-\theta)^2}{2\sigma^2}+2|a-a'| \theta \exp \frac{-(x-\theta)^2}{2\sigma^2}.$$

It is clear that the right hand side converges to 0 uniformly with respect to θ as a' tends to a for every fixed x since the functions $\exp \frac{-(x-\theta)^2}{2\sigma^2}$ and $\theta \exp \frac{-(x-\theta)^2}{2\sigma^2}$ are both bounded on Ω for every fixed x. Hence the condition (3. b) of the above theorem is satisfied. As for (3. c), we have

$$\lim_{\theta\to\pm\infty} (\theta-a)^2 \exp \frac{-(x-\theta)^2}{2\sigma^2}=0$$

as is easily proved and it is therefore satisfied. It is well-known that a decision function δ^* satisfying $\delta^*(\{\bar{x}\}: x)=1$ for all x is the minimax invariant decision function and it is a Bayes solution in the wide sense with respect to $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ provided that ξ_n is uniformly distributed on the interval $[-n, n]$. It is obvious that δ^* and $\mathcal{G}=\{\xi_n : n=1, 2, \cdots\}$ satisfy assumptions of the above theorem. All conditions being satisfied, we have $W_{\mathcal{G}}^{\alpha}=D$.

The following theorem gives a sufficient condition for existence of a sequence $\mathcal{G}$ of a priori distributions which satisfies (3.1) and for which $W_{\mathcal{G}}$ is not empty.

THEOREM 4. *Suppose that the topology on Ω is induced by the metric $\rho(\theta, \theta')=\sup_{S\in\mathcal{B}} |p_\theta(S)-p_{\theta'}(S)|$. If a subset $\hat{\mathcal{P}}=\{p_{\theta_i} : i=1, 2, \cdots\}$ of $\mathcal{P}$ has the A-property and $w(\theta, a)$ is bounded on $\Omega\times A$, a sequence $\{\xi_{\theta_n} : n=1, 2, \cdots\}$ satisfies (3.1) and there exists a Bayes solution in the wide sense with respect to this sequence, where ξ_{θ_n} is the a priori distribution satisfying $\xi_{\theta_n}(\{\theta_n\})=1$.*

Proof. It follows from the A-property that there exist mutually disjoint $E_1, E_2, \cdots$ such that $\bigcup_{i=1}^{\infty} E_i=\mathcal{X}$ and $\lim_{i\to\infty} p_{\theta_i}(E_i)=1$. Evidently we have $\lim_{i\to\infty} p_{\theta_k}(E_i)=0$ for every fixed k. Hence we have

$$\lim_{i\to\infty} \rho(\theta_k, \theta_i)\geqq \lim_{i\to\infty} | p_{\theta_k}(E_i)-p_{\theta_i}(E_i) |$$

$$= \lim_{i\to\infty} p_{\theta_i}(E_i)=1.$$

Consequently any θ_k can not be an accumulation point of $\{\theta_i : i=1, 2, \cdots\}$. This implies that any compact set of Ω contains at most finite members of θ_i. So we have $\lim_{i\to\infty} \xi_{\theta_i}(S)=0$ for any compact set S. Let a_n be such that

$$w(\theta_n, a_n) \leqq \inf_{a \in A} w(\theta_n, a) + \frac{1}{n} .$$

We define a decision function $\tilde{\delta}$ as follows:

$$\tilde{\delta}(\{a_n\} : x)=1 \qquad \text{for } x \in E_n \qquad (n=1, 2, \cdots) .$$

The proof that $\tilde{\delta}$ above defined is a Bayes solution in the wide sense with respect to $\xi_{\theta_1}, \xi_{\theta_2}, \cdots$ is completely the same as that of theorem 2 and so is omitted.

In many cases we have $B^\alpha \subsetneqq D$. Hence theorems 2 and 3 imply that there often exist elements of W which are not contained in B^α.

In some problems complete classes based on B^α are characterized (for example, see [3] [4] [5]). However, any example of complete class based on W does not seem to be known.

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