ON THE DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITH A CONTINUOUS POISSON SPECTRUM

RYOICHI SHIMIZU

1. Introduction and summary

An infinitely divisible characteristic function $\varphi(t)$ has a unique expression,

$$\log \varphi(t)=\beta it-\gamma t^2+\int_0^\infty \left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x) \tag{1.1}$$
$$+\int_{-\infty}^0 \left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)$$

where $M(x)$ and $N(x)$ are monotone non-decreasing right-continuous functions defined on $(0,\infty)$, and $(-\infty, 0)$ respectively, and,

(i) $M(\infty)=N(-\infty)=0$

(ii) $\int_0^a x^2dM(x)+\int_{-a}^0 x^2dN(x)<\infty$ for any $a>0$,

(iii) $\gamma\geqq 0,\quad -\infty>\beta>\infty$.

We call γ the normal factor, $M(x)$ the positive and $N(x)$ the negative Poisson spectrum of $\varphi(t)$. When $\varphi_1(t)$ and $\varphi_2(t)$ are characteristic functions (abbr. ch.f.'s), so is their product $\varphi(t)=\varphi_1(t)\cdot\varphi_2(t)$. φ_1 and φ_2 are called the factors of $\varphi(t)$. A ch.f. is said to be indecomposable if it has no factor except itself and 1.[1] A ch.f. $\varphi(t)$ has no indecomposable factor if and only if every factor of $\varphi(t)$ is infinitely divisible.

Yu. V. Linnik [4] gave a necessary condition for $\varphi(t)$ to have no indecomposable factor when $\gamma>0$: $M(x)$ and $N(x)$ are step functions, and if u and v $(0<u<v$ or $v<u<0)$ are any two jump points of $M(x)$ or of $N(x)$, then v/u is a positive integer. H. Cramér [1] showed that an infinitely divisible ch.f. has an indecomposable factor if there exist posi-

1) In the sequel ch.f.'s $\varphi(t)$ and $e^{bit}\varphi(t)$ are identified $(-\infty<b<\infty)$.

tive constants k and c such that $M'(x)\geqq k$ in the interval $(0, c)$. P. Lévy [3] and D. A. Raikov[2] considered the problem when the Poisson spectrum M is a step function with finite steps and $\gamma=0$.

The purpose of this paper is to generalize H. Cramér's result stated above. We consider only the case where $M(x)$ is continuous, $\gamma=0$, and $N(x)\equiv 0$, i.e., the ch.f. of the form,

$$\varphi(t)=\exp\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x)\ . \tag{1.2}$$

2. Preliminaries

The function $\varphi(t)$ of (1.2) can be a ch.f. even if $M(x)$ does not satisfy all the conditions stated in the previous section, i.e.,

(a) monotone non-decreasing (b) right-continuous in $(0, \infty)$

(c) $M(\infty)=0$ (d) $\int_0^a x^2dM(x)<\infty$ for any $a>0$.

Let $\boldsymbol{A}$ be the class of all functions $M(x)$ which satisfy the conditions (b), (c), and (d) and for which (1.2) is a ch.f. It is known that the expression is unique. The subclass of functions of $\boldsymbol{A}$ which satisfy the further condition (a) is denoted by $\boldsymbol{M}$. An infinitely divisible ch.f. (1.2) has an indecomposable factor if there exist $M_1\in \boldsymbol{A}-\boldsymbol{M}$ and $M_2\in\boldsymbol{A}$ such that $M=M_1+M_2$. In this case we also say that M has an indecomposable factor.

Let $X=(0, \infty)$ and $\boldsymbol{S}$ be the σ-algebra of Borel subsets of X. If $N(x)$ is a monotone non-decreasing right-continuous function on X, there corresponds a unique σ-finite measure μ on $(X, \boldsymbol{S})$ such that for any interval $A=(a, b]$, $\mu(A)=N(b)-N(a+)$. We shall denote it by $\mu_N(\cdot)$.

A point x is called a point of increase of the monotone non-decreasing function $M(x)$ if $M(x+\varepsilon)-M(x-\varepsilon)>0$ for any $\varepsilon>0$. The set of all points of increase of $M(x)$ will be denoted by $P(M)$.

A statement that holds for all points except possibly on a set of μ-measure zero is said to hold a.e.μ.

Let $\mu_1, \cdots, \mu_n$ be non-zero (i.e., not identically zero) σ-finite measures on $(X, \boldsymbol{S})$. They are absolutely continuous with respect to the σ-finite measure $\mu=\mu_1+\cdots+\mu_n$, and their respective Radon-Nikodym derivatives $f_i=d\mu_i/d\mu$ satisfy

$$1\geqq f_i(x)\geqq 0 \qquad \text{a.e.}\mu \qquad i=1, \cdots, n$$

and

2) See, e.g., [4], 249-252 or [5], 176-182.

$$f_1(x)+\cdots+f_n(x)=1. \qquad \text{a.e.}\mu.$$

We say that $\mu_1, \cdots, \mu_n$ are mutually non-singular if

$$\mu(\{x; f_1(x)f_2(x)\cdots f_n(x)>0\})>0,$$

or equivalently if $\mu_i(E)=0$ implies $\mu_j(E')>0$ $(i, j=1, 2, \cdots, n)$. If, further, there exists a positive constant r such that

$$f_i(x)\geqq r>0 \qquad \text{a.e.}\mu \qquad i=1, \cdots, n,$$

then we say that $\mu_1, \cdots, \mu_n$ are similar. Now let $M\in \boldsymbol{M}$, and $a_1, \cdots, a_n \in P(M)$ and let

$$M_i(x)=\begin{cases} 0 & 0<x<a-\varepsilon \\ M(x+a_i-a)-M(a_i-\varepsilon) & a-\varepsilon\leqq x<a+\varepsilon \\ M(a_i+\varepsilon)-M(a_i-\varepsilon) & a+\varepsilon\leqq x \end{cases} \tag{2.2}$$

where

$$a=\min_i a_i>\varepsilon>0.$$

Then $M_1, \cdots, M_n$ are monotone non-decreasing right-continuous functions and the corresponding measures $\mu_i=\mu_{i,\varepsilon}$ are finite and vanish outside the interval $E=(a-\varepsilon, a+\varepsilon]$. We shall say that the points $a_1, \cdots, a_n$ of X are mutually non-singular if $\mu_{1,\varepsilon}, \cdots, \mu_{n,\varepsilon}$ are non-singular for all $\varepsilon>0$. We say that they are similar (or the similar points of M) if for sufficiently small $\varepsilon>0$, $\mu_{1,\varepsilon}, \cdots, \mu_{n,\varepsilon}$ are similar. The definition of similarity is not ambiguous, since if $\varepsilon\geqq\delta>0$ and if $\mu_{1,\varepsilon}, \cdots, \mu_{n,\varepsilon}$ are similar, so are $\mu_{1,\delta}, \cdots, \mu_{n,\delta}$.

3. Generalizations of H. Cramér's theorem

3.1 We shall first prove the following

THEOREM 1. *Let $M\in \boldsymbol{M}$ be a continuous function. If there exists a positive constant a such that a, $2a$, $4a$ and $5a$ are similar points of M, then the ch.f.*

$$\varphi(t)=\exp\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x) \tag{3.1}$$

has an indecomposable factor.

PROOF. Without any loss of generality, we assume that $a=1$. Choose $1/2>\varepsilon>0$ so small that finite measures μ_1, μ_2, μ_4 and μ_5 corresponding to the monotone functions $M_i(x)$'s defined below ((3.2)) are similar. (2.2) is reduced in this case to

$$(3.2)\qquad M_i(x)=\begin{cases} 0 & x<1-\varepsilon/3 \\ M(x+i-1)-M(i-\varepsilon/3) & 1-\varepsilon/3\leqq x<1+\varepsilon/3 \\ M(i+\varepsilon/3)-M(i-\varepsilon/3) & x\geqq 1+\varepsilon/3 \end{cases}$$

$$i=1,\ 2,\ 4,\ 5$$

and $\mu=\mu_1+\mu_2+\mu_4+\mu_5$ is a finite measure on $(X, \boldsymbol{S})$, which vanishes outside $F_1=(1-\varepsilon/3,\ 1+\varepsilon/3]$.

If $Y=(-\infty, \infty)$ and if $\boldsymbol{T}$ is the σ-algebra of Borel subsets of Y, a σ-finite measure μ_0 on $(Y, \boldsymbol{T})$ is defined by

$$\mu_0(A)=\sum_{n=-\infty}^{\infty}\mu(A_{(n)}\cap F_1)\,, \qquad A\in \boldsymbol{T}$$

where

$$A_{(n)}=\{x;\ x-n\in A\}\,.$$

Clearly we have for any $A\in \boldsymbol{S}$, $\mu(A)=\mu_0(A\cap F_1)$. Let, for any integer n,

$$F_n=(n-\varepsilon/3,\ n+\varepsilon/3]$$

$$E_n=(n-\varepsilon,\ n+\varepsilon]\,.$$

It is easily verified that the set function ν on $\boldsymbol{T}$ defined by

$$\nu(A)=\int_{F_0}\int_{F_3}\chi_A(u+t)d\mu_0(u)d\mu_0(t) \qquad A\in \boldsymbol{T}$$

is a non-zero, finite measure on $(Y, \boldsymbol{T})$ which vanishes outside E_3. Some lemmas are necessary.

LEMMA 1. *If a finite measure μ_* is defined by $\mu_*(A)=\mu_0(A\cap F_3)$, then μ_* and ν are mutually non-singular.*

PROOF. It is enough to show that $A\subseteqq F_3$ and $\mu_*(A)>0$ imply $\nu(A)>0$. Let

$$g(t)=g_A(t)=\int\chi_A(u+t)d\mu_*(u)\,.$$

Then we have

$$\nu(A)=\int_{F_0}g(t)d\mu_0(t)\,.$$

Clearly $g(t)$ is continuous, and $g(0)=\int\chi_A(u)d\mu_*(u)=\mu_*(A)>0$. Hence there exists a neighbourhood N of 0 such that $g(t)>0$ whenever $t\in N$, and $\nu(A)\geqq\int_N g(t)d\mu_0(t)>0$. (Note that $\int_N d\mu_0(t)>0$ for all $\delta>0$.) q.e.d.

LEMMA 2. *There exist a σ-finite measure ν_0 on $(Y, \boldsymbol{T})$ and an integer $n_0(>1)$ such that*

(i) $\nu_0(A \cap F_3) \leqq \nu(A)$

(ii) $\nu_0(A) = \nu_0(A_{(n)})$ *for all* $A \in \boldsymbol{T}$ *and* n

(iii) $d\mu_0/d\lambda_0 \geqq 1/n_0$ *a.e.*λ_0

where $\lambda_0 = \mu_0 + \nu_0$.

PROOF. Let $\lambda = \mu_* + \nu$, $Z_n = \left\{x;\ d\mu_*/d\lambda \geqq \frac{1}{n}\right\}$ $(n = 1, 2, \cdots)$, and $Z_0 = \{x;\ d\mu_*/d\lambda > 0\} = \bigcup_1^\infty Z_n$. Then $\mu_*(Z_0') = \int_{Z_0'} \frac{d\mu_*}{d\lambda} d\lambda = 0$, hence there exists a positive integer n_0 such that $\nu(Z_{n_0}) > 0$, since otherwise $\nu(Z_0) \leqq \sum_1^\infty \nu(Z_n) = 0$ contrary to the non-singularity of μ_* and ν. Since

$$\mu_*(Z_1) = \int_{Z_1} \frac{d\mu_*}{d\lambda} d\lambda = \lambda(Z_1) = \mu_*(Z_1) + \nu(Z_1)$$

and since μ_* is finite, we must have $\nu(Z_1) = 0$, and hence $n_0 > 1$.

If ν_* is the finite measure defined for Borel set A, by $\nu_*(A) = \nu(A \cap Z_{n_0})$, if $\lambda_* = \mu_* + \nu_*$ and if $Z_* = \left\{x;\ d\mu_*/d\lambda_* \geqq \frac{1}{n_0}\right\} (\supseteqq Z_{n_0})$, then we have

$$\nu_*(Z_*') = 0 \ . \tag{3.3}$$

On the other hand,

$$\mu_*(Z_*') = \int_{Z_*'} \frac{d\mu_*}{d\lambda_*} d\lambda_* \leqq \frac{1}{n_0} \int_{Z_*'} d\lambda_* = \frac{1}{n_0} \mu_*(Z_*') \ .$$

Since $n_0 > 1$ and since μ_* is finite, we have

$$\mu_*(Z_*') = 0 \ . \tag{3.4}$$

(3.3) and (3.4) imply $\lambda_*(Z_*') = 0$ or $d\mu_*/d\lambda_* \geqq \frac{1}{n_0}$ a.e.λ_* .

σ-finite measure defined by

$$\nu_0(A) = \sum_{n=-\infty}^{\infty} \nu_*(A_{(n)})$$

satisfies the required conditions. q.e.d.

Remark. (iii) and $n_0 > 1$ imply the non-singularity of μ_0 and ν_0 .

From lemma 2, we see that

$$1 \geqq \xi_1(x) \geqq \frac{1}{n_0} \qquad \text{a.e.}\lambda_0$$

$$1-\frac{1}{n_0} \geqq \xi_2(x) \geqq 0 \qquad \text{a.e.}\lambda_0$$

where $\xi_1 = d\mu_0/d\lambda_0$ and $\xi_2 = d\nu_0/d\lambda_0$.

Moreover the invariance property (under the transformation $x \to x+n$) of μ_0 and ν_0 implies

$$\xi_k(x) = \xi_k(x+n) \qquad (k=1,\, 2) \qquad \text{a.e.}\lambda_0\,. \tag{3.5}$$

LEMMA 3. *ν_0 and hence λ_0 is absolutely continuous with respect to μ_0. Especially if $\mu_{0i}(A) = \sum_{-\infty}^{\infty} \mu_i(A_{(n)})$, then*

$$d\mu_{0i}/d\mu_0 \geqq r > 0 \qquad \textit{a.e.}\lambda_0\,.$$

PROOF. Suppose $\mu_0(A) = \sum \mu(A_{(n)}) = 0$, then

$$\mu(A_{(n)}) = \mu_0(A_{(n)} \cap F_1) = 0\,, \qquad (n = \cdots -1,\, 0,\, 1,\, 2,\, \cdots)$$

and

$$\nu_0(A_{(n)} \cap F_1) = \int_{A_{(n)} \cap F_1} d\nu_0 = \int_{A_{(n)} \cap F_1} \xi_2(x) d\lambda_0(x)$$
$$\leqq \left(1-\frac{1}{n_0}\right)\lambda_0(A_{(n)} \cap F_1) = \left(1-\frac{1}{n_0}\right)\nu_0(A_{(n)} \cap F_1)\,.$$

Since $\nu_0(A_{(n)} \cap F_1) < \infty$, we have $\nu_0(A_{(n)} \cap F_1) = 0$ $(n = \cdots -1,\, 0,\, 1,\, 2,\, \cdots)$. Hence

$$\nu_0(A) = \sum \nu(A_{(n)}) = \sum \nu_0(A_{(n)} \cap F_1) = 0\,. \qquad \text{q.e.d.}$$

Let

$$f(x) = \sum_{n=1}^{5} h_n(x) \chi_n(x)$$

$$\tilde{f}(x) = r\xi_1(x) \sum_{0}^{5} \chi_n(x)$$

where

$$h_n(x) = \begin{cases} r\xi_1(x) & (n=1,\, 2,\, 4,\, 5) \\ -\delta r \xi_1(x) \xi_2(x) & (n=3)\,. \end{cases}$$

$\chi_n(x)$ is the indicator function of F_n and $\delta > 0$ is chosen so small that

$$\begin{aligned} &1-12\delta > 0 \\ &(1-2\delta)r^2 - 2r\delta > 0\,. \end{aligned} \tag{3.6}$$

The given Poisson spectrum $M(x)$ can be docomposed as

$$M(x)=M_0(x)+M_1(x)$$

where

(3.7)
$$M_0(x)=-\int_{(x,\,\infty)} f(t)d\lambda_0(t)$$

$$M_1(x)=\int_{(x,\,\infty)}\Big[h_3(t)-\sum_{\substack{n=1\\ n\neq 3}}^{5}\Big(\frac{d\mu_{0,\,n}}{d\lambda_0}-r\Big)\chi_n(t)\Big]\,d\lambda_0(t)$$
$$-\int_{(x,\,\infty)}\prod_{n=1}^{5}(1-\chi_n(t))d\mu_M(t)\,.$$

It is not so difficult to verify that $M_1\in \boldsymbol{M}$, and $M_0\notin \boldsymbol{M}$. Hence the theorem is proved if we can show that

(3.8) $$\varphi(t)=\exp\int_0^\infty\Big(e^{itx}-1-\frac{itx}{1+x^2}\Big)dM_0(x)$$

is a ch.f.

LEMMA 4. *Suppose that $M_0(x)=N_1(x)-N_2(x)$ (N_1, $N_2\in \boldsymbol{M}$) is a bounded, continuous function which vanishes outside some finite interval $(0,\,c)$. Define a sequence of continuous functions $B_n(x)$ on X recursively by*

$$B_1(x)=M_0(x)-M_0(+0)$$
$$B_n(x)=\int_0^x B_{n-1}(x-t)dB_1(t)\,.$$

The function $\varphi(t)$ given by (3.8) is a ch.f. if $B_2(x)$, $B_2(x)+2B_1(x)$ and $B_3(x)$ are monotone non-decreasing.

PROOF. From the definition of $B(x)$'s, we have for some γ_0,

$$|B_1(x)|\leqq\gamma_0<\infty$$
$$|B_n(x)|\leqq\gamma_0^n\,.$$

Hence the series

$$B(x)=\sum_{n=1}^{\infty}\frac{1}{n\,!}B_n(x)$$

is absolutely and uniformly convergent in $(0,\,\infty)$.

Observing that

$$\int_0^\infty e^{itx}\, dB_n(x) = \left(\int_0^\infty e^{itx} dB_1(x)\right)^n = \left(\int_0^c e^{itx} dM_0(x)\right)^n$$

we get

$$\exp \int_0^\infty e^{itx} dM_0(x) = 1 + \sum_1^\infty \frac{1}{n!}\left(\int_0^c e^{itx} dM_0(x)\right)^n$$

$$= 1 + \sum_1^\infty \frac{1}{n!} \int_0^\infty e^{itx} dB_n(x)$$

$$= 1 + \int_0^\infty e^{itx} dB(x)\ .$$

Hence, putting

$$\eta = \int_0^\infty \frac{x}{1+x^2} dM_0(x)\ ,$$

we have

$$\varphi(t) = \exp \int_0^c \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM_0(x)$$

$$= e^{-\gamma_0} \cdot e^{-it\eta}\left(1 + \int_0^\infty e^{itx} dB(x)\right)$$

$$= \int_{-\infty}^\infty e^{-itx} dH(x)$$

where

$$H(x) = \begin{cases} e^{-\gamma_0}(\epsilon(x+\eta) + B(x+\eta)) & x > -\eta \\ 0 & x \leqq -\eta\ . \end{cases}$$

As is easily seen, $H(-\infty)=0$, $H(\infty)=\varphi(0)=1$. Hence $\varphi(t)$ is a ch.f. if $H(x)$ is monotone non-decreasing.

It is easy to verify (using induction) that

$$B_n(x) = \int_0^x B_{n-2}(x-t) dB_2(t)$$

is monotone non-decreasing in $(0, \infty)$ whenever $n \geqq 4$. Hence

$$B(x) = \sum_3^\infty \frac{1}{n!} B_n(x) + \frac{1}{2} B_2(x) + B_1(x)$$

is monotone non-decreasing in $(0, \infty)$ and $H(x)$ is also monotone non-decreasing in $(-\infty, \infty)$. q.e.d.

$M_0(x)$ defined by (3.7) is a bounded, continuous function which vanishes outside the finite interval $(0, 5+\varepsilon)$. Hence for the proof of the theorem 2 it is enough to show that $B_2(x)$, $B_2(x)+2B_1(x)$ and $B_3(x)$ are monotone non-decreasing, where

$$B_n(x)=\int_0^x B_{n-1}(x-t)dB_1(t)=\int_{(0,\,x]} B_{n-1}(x-t)f(\lambda)d\lambda_0(t) \qquad n\geqq 1\,.$$

Lemmas 5 and 6 below are easily verified, lemma 6 being the consequence of the periodicity of $\xi(x)$ (c.f. (3.5)) and the invariance property of λ_0.

LEMMA 5. *For $y<x$, denote by $U_t(y,\,x)$ the interval $(y-t,\,x-t]$. If $x\notin\bigcup_{n=2}^{10} E_n$ (resp. $x\notin\bigcup_{n=3}^{15} E_n$), there exists an open interval U containing x such that $B_2(x)$ (resp. $B_3(x)$) is constant on U.*

Moreover, if $n-\frac{4}{3}\varepsilon<x<n+\frac{4}{3}\varepsilon$,

$$B_2(x)=\int_{(0,\,x]} B_1(x-t)f(t)d\lambda_0(t)$$

$$=\sum_{i=1}^{n-1}\int_{F_i}\int_{F_{n-i}\cap U_t(0,\,x)} f(u)f(t)d\lambda_0(u)d\lambda_0(t) \tag{3.10}$$

$$B_3(x)=\sum_{i=1}^{n-2}\int_{F_i} B_2(x-t)f(t)d\lambda_0(t)\,.$$

LEMMA 6. *Let*

$$n-\frac{4}{3}\varepsilon\leqq y<x\leqq n+\frac{4}{3}\varepsilon,\quad i+j=n\,,$$

and let

$$J_{i,\,j}=J_{i,\,j}(y,\,x)=\int_{F_i}\int_{F_j\cap U_t(y,\,x)} \tilde{f}(u)\tilde{f}(t)d\lambda_0(u)d\lambda_0(t)\,.$$

Then we have

(i) $J_{i,\,j}=J_{0,\,n}$,

(ii) *if* $i+j+k=n$,

$$\int_{F_k} J_{i,\,j}(y-t,\,x-t)\tilde{f}(t)d\lambda_0(t)=\int_{F_0} J_{0,\,n}(y-t,\,x-t)\tilde{f}(t)d\lambda_0(t)\,.$$

LEMMA 7. *$B_2(x)$ is monotone non-decreasing.*

PROOF. It is enough, because of lemma 5, to show that $B_2(x)\geqq B_2(y)$ whenever

$$y<x, \quad y, x \in E_n \ (n=2, 3, \cdots, 10)$$

and hence whenever

$$n-\frac{4}{3}\varepsilon<y<x<n+\frac{4}{3}\varepsilon .$$

But then we have, using (3.10),

$$B_2(x)-B_2(y)=\sum_1^{n-1}\int_{F_i}\int_{F_{n-i}\cap U_t(y,x)} f(u)f(t)d\lambda_0(u)d\lambda_0(t) .$$

It is easily seen that there exists an integer i_0 such that $1\leqq i_0\leqq n-1$, $i_0\neq 3$ and $n-i_0\neq 3$. Suppose first that $n\neq 2, 3, 6$. Then

$$B_2(x)-B_2(y)=I_1+I_2+I_3 \tag{3.11}$$

where

$$\begin{aligned} I_1 &= \sum_{\substack{i\neq 3\\ n-i\neq 3}}\int_{F_i}\int_{F_{n-i}\cap U_t(y,x)} f(u)f(t)d\lambda_0(u)d\lambda_0(t) \\ &\geqq \int_{F_{i_0}}\int_{F_{n-i_0}\cap U_t(y,x)} \tilde{f}(u)\tilde{f}(t)d\lambda_0(u)d\lambda_0(t) \qquad (3.12)\\ &= J_{0,n}(y, x) \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{F_3}\int_{F_{n-3}\cap U_t(y,x)} f(u)f(t)d\lambda_0(u)d\lambda_0(t) \qquad (3.13)\\ &\geqq -\delta J_{0,n}(y, x) \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_{F_{n-3}}\int_{F_3\cap U_t(y,x)} f(u)f(t)d\lambda_0(u)d\lambda_0(t) \qquad (3.14)\\ &\geqq -\delta J_{0,n}(y, x) . \end{aligned}$$

From (3.12), (3.13) and (3.14), we have

$$B_2(x)-B_2(y)\geqq(1-2\delta)J_{0,n}(y, x)\geqq 0 . \tag{3.15}$$

When $n=2$, 3 or 6, (3.11) and I_1 are the same as before, while I_2 and I_3 are replaced by

$$I_2=I_3=\int_{F_3}\int_{F_3\cap U_t(y,x)} f(u)f(t)d\lambda_0(u)d\lambda_0(t)\geqq 0$$

if $n=6$, and by $I_2=I_3=0$ if $n=2$ or 3. In any case (3.15) holds true. q.e.d.

LEMMA 8. *$B_2(x)+2B_1(x)$ is monotone non-decreasing.*

PROOF. It is enough to show that $B_2(x)+2B_1(x)\geqq B_2(y)+2B_1(y)$ when $y<x$, $y, x\in F_3$. Then, since $F_3\subset E_3$, we have from (3.15)

$$B_2(x)-B_2(y)\geqq(1-2\delta)J_{0,3}=(1-2\delta)\int_{F_0}\int_{F_3\cap U_t(y,x)}\tilde{f}(u)\tilde{f}(t)d\lambda_0(u)d\lambda_0(t)$$

$$=(1-2\delta)r^2\int_{F_0}\int_{F_3\cap U_t(y,x)}\xi_1(u)\xi_1(t)d\lambda_0(u)d\lambda_0(t) \tag{3.16}$$

$$=(1-2\delta)r^2\int_{F_0}\int_{F_3\cap U_t(y,x)}d\mu_0(u)d\mu_0(t)$$

$$=(1-2\delta)r^2\nu_0(U_0(y,x))\,.$$

On the other hand,

$$B_1(x)-B_1(y)=\int_{F_3\cap U_0(y,x)}f(t)d\lambda_0(t)$$

$$\geqq -r\delta\int_{F_3\cap U_0(y,x)}\xi_2(t)d\lambda_0(t) \tag{3.17}$$

$$=-r\delta\nu_0(U_0(y,x))\,.$$

The result follows from (3.16), (3.17) and (3.6). q.e.d.

LEMMA 9. *$B_3(x)$ is monotone non-decreasing.*

PROOF. If $n-\frac{4}{3}\varepsilon\leqq\eta<\xi\leqq n+\frac{4}{3}\varepsilon$, $\quad(n=2,\cdots,10)$,

we have from (3.15)

$$B_2(\xi)-B_2(\eta)\geqq(1-2\delta)J_{0,n}(\eta,\xi)\,. \tag{3.18}$$

On the other hand,

$$B_3(\xi)-B_3(\eta)=\sum_{i=1}^{n-1}\int_{F_i}\int_{F_{n-i}\cap U_t(\eta,\xi)}f(u)f(t)d\lambda_0(u)d\lambda_0(t) \tag{3.19}$$

$$\leqq\sum_{1}^{n-1}\int_{F_i}\int_{F_{n-i}\cap U_t(\eta,\xi)}\tilde{f}(u)\tilde{f}(t)d\lambda_0(u)d\lambda_0(t)$$

$$\leqq 10J_{0,n}(\eta,\xi)\,.$$

Now we shall show that $B_3(x)\geqq B_3(y)$ if $x>y$ and if $y, x\in E_n$ $(n=3,\cdots,15)$,

$$B_3(x)-B_3(y)$$
$$=\sum_1^{n-2}\int_{F_i}(B_2(x-t)-B_2(y-t))f(t)d\lambda_0(t)$$
$$\geqq\int_{F_1}(B_2(x-t)-B_2(y-t))f(t)d\lambda_0(t)$$
$$+\int_{F_3}(B_2(x-t)-B_2(y-t))f(t)d\lambda_0(t)$$

using (3.18), (3.19) with $y-t$ and $x-t$ in place of η and ξ,

$$\geqq(1-2\delta)\int_{F_1}J_{0,\,n-1}(y-t,\ x-t)\tilde{f}(t)d\lambda_0(t)$$
$$-10\delta\int_{F_3}J_{0,\,n-3}(y-t,\ x-t)\tilde{f}(t)d\lambda_0(t)$$
$$=(1-12\delta)\int_{F_0}J_{0,\,n}(y-t,\ x-t)\tilde{f}(t)d\lambda_0(t)\geqq 0\,. \qquad \text{q.e.d.}$$

PROOF OF THE THEOREM. The theorem follows immediately from lemmas 4, 7, 8 and 9. q.e.d.

Remark. Finite measure ν is in general not absolutely continuous with respect to μ_0. If it is, we need not introduce either ν_0 or λ_0 and the proof of the theorem will be much simpler.

This is the case if, in particular, finite measure μ is similar to the Lebesgue measure on F_1.

COROLLARY (H. Cramér). *If there exist positive constants k and c such that $M(\in \boldsymbol{M})$ has first derivative which is not less than k in the interval $(0, c)$, then M has an indecomposable factor.*

PROOF. Without any loss of generality we assume,

$$M(x)=\begin{cases} k\cdot(x-c) & 0<x\leqq c \\ 0 & x>c\,. \end{cases} \tag{3.20}$$

If $0<a<c/6$, then a, $2a$, $4a$ and $5a$ are similar points of M with $r=1/4$. q.e.d.

The theorem 1 can be extended to more general form: "If M is a continuous positive Poisson spectrum, and if there exist a positive constant a and vector $(n_1, n_2, \cdots, n_r)\in V$ such that $n_1a, n_2a, \cdots, n_ra$ are similar points of M, then M has an indecomposable factor," where V is a set of finite dimensional vectors which satisfy some "suitable conditions." Theorem 1 states that the vector $(1, 2, 4, 5)$ is in V. The similar argument as the proof of the theorem 1 can apply to show that V contains the vector (2, 3, 5, 6). It can be shown that (0, 1, 3, 4) is also in V (with suitable modification on the definition of similarity), if M satisfies the further assumption

$$M(a+\varepsilon)-M(a)>0 \quad \text{for all} \quad \varepsilon>0 .$$

Characterization of V is one of the problem to solve. A more general problem is, of course, the characterization of $\boldsymbol{A}$.

The similar problem of characterization of V was solved by P. Lévy [3], when M is a step function which has its jump points on the finite positive integers. Note that all the points of increase are similar in this case. According to Yu. V. Linnik [4], V is trivial when the normal factor γ is positive, since then M has always an indecomposable factor. This means that V can be any set (provided of course that M is continuous).

3.2 The following theorem is also a generalization of H. Cramér's theorem.

THEOREM 2. *Let $M \in \boldsymbol{M}$ be continuous and suppose for any $\varepsilon_1>0$, there exist points of increase a_1 and a_2 of M which are mutually non-singular and $\varepsilon_1>a_2>a_2/3>a_1>0$.*

If there exist a finite interval (b, c) and a positive number k such that

$$M(x)-M(y)\geqq k\cdot(x-y)$$

whenever

$$0<b\leqq y<x\leqq c<\infty ,$$

then M has an indecomposable factor.

PROOF. Let

$$0<\varepsilon_1<b/5 , \quad (c-b)/5 ,$$
$$0<a_1<3a_1<a_2<a_1+a_2<\varepsilon_1 ,$$
$$0<\varepsilon<a_1/10 ,$$

where a_1 and a_2 are points of increase of M which are mutually non-singular. Let ν_1 and ν_2 be finite measures on $(X, \boldsymbol{S})$ corresponding to the non-decreasing, continuous functions

$$M_i(x)=\begin{cases} 0 & 0<x<a_1-\varepsilon \\ M(x+a_i-a_1)-M(a_i-\varepsilon) & a_1-\varepsilon\leqq x<a_1+\varepsilon \\ M(a_i+\varepsilon)-M(a_i-\varepsilon) & a_1+\varepsilon\leqq x \end{cases}$$
$$i=1, 2 .$$

Let $\xi_1(x)$ and $\xi_2(x)$ be the Radon-Nikodym derivatives of ν_1 and ν_2 with respect to the finite measure $\nu_0=\nu_1+\nu_2$ and let

$$\xi(x)=\min(\xi_1(x), \xi_2(x))$$
$$E_n=(a_n-\varepsilon, \; a_n+\varepsilon] \qquad (n=1, 2)$$
$$E_0=(b, c)$$
$$E=\{x; \; \xi(x)>0\}\cap E_1 .$$

Non-sigularity of ν_1 and ν_2 implies $0<\nu_0(E)=\nu_0(E\cap P(M))$, and we assume without any loss of generality that $a_1, \; a_2\in E\cap P(M)$. Then the function

$$g(x)=\frac{\xi(x)}{\xi_1(x)}\chi_{E_1}(x)+\frac{\xi(x-\alpha)}{\xi_1(x-\alpha)}\chi_{E_2}(x)+\chi_{E_0}(x)$$

is defined a.e.ν_0 and satisfies

$$1\geqq g(x)\geqq 0 \qquad \text{a.e.}\nu_0 .$$

The function defined by

$$M_*(x)=\begin{cases} -\int_{(x,\,\infty)} g(t)d\mu_M(t)+k\cdot(b-c) & x<b \\ k\cdot(x-c) & b\leqq x<c \\ 0 & x\geqq c \end{cases}$$

belongs to $\boldsymbol{M}$ with $M-M_*$. a_1 and a_2 are similar points of M_* with $r=1/2$, i.e., if ν_{*1} and ν_{*2} are the measures corresponding to the monotone functions defined by

$$M_{*k}(x)=\begin{cases} 0 & x<a_1-\varepsilon \\ M_*(x+a_k-a_1)-M_*(a_k-\varepsilon) & a_1-\varepsilon\leqq x<a_1+\varepsilon \\ M_*(a_k+\varepsilon)-M_*(a_k-\varepsilon) , & a_1+\varepsilon\leqq x \end{cases}$$
$$k=1, 2 .$$

then we have

$$\nu_{*1}=\nu_{*2}\,(=\nu_*) . \tag{3.21}$$

We shall show that M_* has an indecomposable factor.
Let

$$0<a_1<a_2<5a_2<b+\varepsilon<a_3<b+2\varepsilon$$
$$\alpha=a_2-a_1 , \qquad \beta=a_1 \qquad (\alpha>2\beta)$$
$$a_4=a_3+\alpha ,$$
$$a_5=a_4+\alpha=a_3+2\alpha$$

$$a_6=a_5+(\alpha-\beta)=a_3+3\alpha-\beta$$
$$a_7=a_6+\beta=a_3+3\alpha$$
$$a_8=a_7+(\alpha-\beta)=a_3+4\alpha-\beta$$
$$a_9=a_8+\beta=a_3+4\alpha$$
$$a_{10}=a_9+\beta=a_3+4\alpha+\beta$$
$$a_{11}=a_{10}+\alpha=a_3+5\alpha+\beta \qquad (<c-\varepsilon)$$
$$E_n=(a_n-\varepsilon,\ a_n+\varepsilon]$$
$$F_n=(a_n-\varepsilon/2,\ a_n+\varepsilon/2]\,.$$

Let μ be the Lebesgue measure on (X, S) and ν be the non-zero finite measure defined by

(3.22) $$\nu(A)=\nu_*(A)+\nu_*(A_{(\alpha)})$$

where $$A_{(\alpha)}=\{x;\ x-\alpha\in A\}\,,$$

and λ be the σ-finite measure $\lambda=\mu+\nu$. Let

$$f(x)=\begin{cases}\dfrac{d\nu}{d\lambda} & \text{if } n=1, 2\\[2ex] \dfrac{d\mu}{d\lambda} & \text{if } n\geqq 3,\ n\neq 7\\[2ex] -\delta\dfrac{d\mu}{d\lambda}\chi_{0,\,7}(x) & \text{if } n=7\end{cases}$$

where $\chi_{0,\,7}(x)$ is the indicator function of F_7 and δ is a positive number such that

(3.23) $$(1-\delta)\nu(F_1)-2\delta>0$$
$$1-4\delta>0\,.$$

Let

$$f(x)=\sum_{n=1}^{11} f_n(x)\chi_n(x)\,,$$

$$\tilde{f}(x)=\frac{d\nu}{d\lambda}\sum_1^2\chi_n(x)+\frac{d\mu}{d\lambda}\sum_3^{11}\chi_n(x)$$

where $\chi_n(x)$ is the indicator function of E_n $(n=1, 2, \cdots, 11)$. Note that we have

$$1\geqq\tilde{f}(x)\geqq 0 \qquad \text{a.e.}\lambda$$

$$f(x)=\begin{cases}\tilde{f}(x) & x \notin F_7 \\ -\delta\tilde{f}(x)=-\delta k & x \in F_7\end{cases}.$$

$$f(x)=\begin{cases}f(x+\alpha) & x \in E_1 \\ f(x-\alpha) & x \in E_2\end{cases}$$

(c.f. (3.21) and (3.22)).

M_* can now be written as

$$M_*(x)=M_0(x)+M_1(x)$$

where

$$M_0(x)=-\int_{(x,\infty)} f(t)d\lambda(t)$$

(3.24)

$$M_1(x)=-\int_{(x,\infty)} \left(\frac{d\mu}{d\lambda}\chi_E(t)-f_7(t)\chi_7(t)\right)d\lambda(t)$$

and $\chi_E(x)$ is the indicator function of $E=E_0-\bigcup_1^{11} E_n$. It is not difficult to verify that $M_1 \in \boldsymbol{M}$ and that $M_0 \notin \boldsymbol{M}$. We shall prove that $\varphi(t)$ of (3.8) is a ch.f. by showing that $B_2(x)$, $B_2(x)+2B_1(x)$ and $B_3(x)$ are monotone non-decreasing, where B_n's are defined as in the lemma 4. Let $Q_1=\{1, 2, \cdots, 11\}$, $Q_2=Q_1\times Q_1$ and $Q_3=Q_2\times Q_1$. When $(i, j) \in Q_2$(resp. $(i, j, k,) \in Q_3$), write $I_{i,j}(I_{i,j,k})$ to denote the set of all elements $(i', j') \in Q_2$ $((i', j', k') \in Q_3)$ such that

$$a_i+a_j=a_{i'}+a_{j'}$$

$$(a_i+a_j+a_k=a_{i'}+a_{j'}+a_{k'}).$$

$I_{i,j}$ consists of at most 3 elements.

Write $\quad E_{i,j}=(a_i+a_j-2\varepsilon\,,\ a_i+a_j+2\varepsilon)$

$$(E_{i,j,k}=(a_i+a_j+a_k-3\varepsilon\,,\ a_i+a_j+a_k+3\varepsilon))$$

where $\quad (i, j) \in Q_2 \quad ((i, j, k) \in Q_3)$.

Clearly $\quad E_{i,j}=E_{i',j'} \quad (E_{i,j,k}=E_{i',j',k'})$

if and only if $\quad (i', j') \in I_{i,j} \quad ((i', j', k') \in I_{i,j,k})$.

Verification of the lemmas 10 and 11 below is not so difficult and is omitted here.

LEMMA 10. *If $x \notin \bigcup_{(i,j)\in Q_2} E_{i,j}$ (resp. $x \notin \bigcup_{(i,j,k)\in Q_3} E_{i,j,k}$), there exists an open interval U containing x such that $B_2(x)$ (resp. $B_3(x)$) remains constant on U. Moreover, if $a_i+a_j-4\varepsilon \leqq y < x \leqq a_i+a_j+4\varepsilon$, then we have*

$$B_2(x)-B_2(y)=\sum_k \int_{E_k}(B_1(x-t)-B_1(y-t))f(t)d\lambda(t) \tag{3.25}$$

where the summation extends over all $k\in Q_1$ such that for some $l\in Q_1$, $(l, k)\in I_{i,j}$. If $y<x$, $y, x\in E_{i,j,k}$, then

$$B_3(x)-B_3(y)=\sum_{k'} \int_{E_{k'}}(B_2(x-t)-B_2(y-t))f(t)d\lambda(t) \tag{3.26}$$

where the summation extends over all $k'\in Q_1$ such that for some $(i', j')\in Q_2$, $(i', j', k')\in I_{i,j,k}$.

LEMMA 11. *For $(i, j)\in Q_2$ and $y, x\in E_{i,j}$, write*

$$J_{i,j}=J_{i,j}(y, x)=\int_{E_i}\int_{E_j\cap U_t(y,x)}\tilde{f}(u)\tilde{f}(t)d\lambda(u)d\lambda(t)\,. \tag{3.27}$$

Then we have

(i) *if $(i', j')\in I_{i,j}$ and if $i'\leqq j'$, $i\leqq j$ (or $i'>j'$, $i>j$), then $J_{i,j}=J_{i',j'}$ and $J_{j,i}=J_{j',i'}$,*

(ii)
$$\int_{E_i}\int_{E_j\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)$$
$$=\begin{cases} J_{i,j} & \text{if } i\neq 7,\ j\neq 7\\ -\delta J_{i,j} & \text{if } i=7,\ j\neq 7 \text{ or } i\neq 7,\ j=7\\ \delta^2 J_{i,j} & \text{if } i=j=7\,,\end{cases}$$

(iii) *suppose $(i, j, k)\in Q_3$, $(i', j', k')\in I_{i,j,k}$, $y, x\in E_{i,j,k}$, $i\leqq j$, $i'\leqq j'$ (or $i>j$, $i'>j'$) and $k\geqq 3$, $k'\geqq 3$ (or $k<3$, $k'<3$), then we have*

$$\int_{E_k} J_{i,j}(y-t,\ x-t)\tilde{f}(t)d\lambda(t)=\int_{E_{k'}} J_{i',j'}(y-t,\ x-t)\tilde{f}(t)d\lambda(t)\,.$$

LEMMA 12. *$B_2(x)\geqq B_2(y)$ if $x>y$.*

PROOF. According to lemma 10, it is enough to prove only for $y, x \in E_{i,j}((i, j)\in Q_2)$, and hence for

$$a_i+a_j-4\varepsilon \leqq y < x \leqq a_i+a_j+4\varepsilon\,.$$

Now we can find an element (i_0, j_0) of $I_{i,j}$ such that $i_0<j_0$, $i_0\neq 7$, and

$j_0 \neq 7$. Suppose first $(7, r) \in I_{i,j}$ for some $r \in Q_1$ $(r \neq 7)$, then using (3.25) we have

$$B_2(x)-B_2(y)=\sum_{\substack{(p,q)\in I_{i,j}\\ p\leq q}}\int_{E_p}\int_{E_q\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t) \tag{3.28}$$

$$+\sum_{\substack{(p,q)\in I_{i,j}\\ p>q}}\int_{E_p}\int_{E_q\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)$$

$$\geqq I_1+I_2+I_3+I_4 \tag{3.29}$$

where

$$I_1=\int_{E_{i_0}}\int_{E_{j_0}\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)=J_{i_0,j_0}$$

$$I_2=\int_{E_{j_0}}\int_{E_{i_0}\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)=J_{j_0,i_0}$$

$$I_3=\int_{E_r}\int_{E_7\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)\geqq -\delta J_{r,7}$$

$$I_4=\int_{E_7}\int_{E_r\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)\geqq -\delta J_{7,r}\,.$$

Hence

$$B_2(x)-B_2(y)\geqq(1-\delta)(J_{i,j}+J_{j,i})\geqq 0\,. \tag{3.30}$$

If for any $r \in Q_1$, $(r, 7) \notin I_{i,j}$, then (3.29) I_1 and I_2 are the same as before while $I_3=I_4=0$.

If $(7, 7) \in I_{i,j}$, then I_3 and I_4 are replaced by

$$I_3=I_4=\int_{E_7}\int_{E_7\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t)=\delta^2 J_{7,7}\geqq 0\,.$$

In any case inequality (3.30) holds true. q.e.d.

LEMMA 13. *$B_2(x)+2B_1(x)\geqq B_2(y)+2B_1(y)$, if $x>y$.*

PROOF. It is enough to consider the case $y, x \in F_7$. Since $F_7\subseteqq E_7\subseteqq E_{1,6}$, we have by (3.30)

$$B_2(x)-B_2(y)\geqq(1-\delta)(J_{1,6}+J_{6,1})\geqq(1-\delta)J_{1,6}$$

$$=(1-\delta)\int_{E_1}\int_{E_6\cap U_t(y,x)} f(u)f(t)d\lambda(u)d\lambda(t) \tag{3.31}$$

$$\geqq(1-\delta)\int_{F_1}\int_{E_6\cap U_t(y,x)} kf(t)d\mu(u)d\lambda(t)$$

$$=(1-\delta)k\int_{F_1}\int_{y-t}^{x-t} dud\lambda(t)=(1-\delta)k\cdot(x-y)\cdot\lambda(F_1)$$

$$\geqq(1-\delta)k\cdot(x-y)\cdot\nu(F_1)\,.$$

On the other hand,

$$B_1(x)-B_1(y)=\int_{U_0(y,\,x)} f(t)d\lambda(t)=-\delta k\int_y^x dt=-\delta k\cdot(x-y)\,. \tag{3.32}$$

The result follows from (3.23), (3.31) and (3.32). q.e.d.

LEMMA 14. $B_3(x)\geqq B_3(y)$ *if* $x>y$.

PROOF. If

$$a_i+a_j-4\varepsilon\leqq\eta<\xi\leqq a_i+a_j+4\varepsilon\,,$$

we have from (3.30)

$$B_2(\xi)-B_2(\eta)\geqq(1-\delta)(J_{i,\,j}(\eta,\,\xi)+J_{j,\,i}(\eta,\,\xi))\,. \tag{3.33}$$

On the other hand,

$$\begin{aligned}B_2(\xi)-B_2(\eta)&=\sum_k\int_{E_k}(B_1(\xi-t)-B_1(\eta-t))f(t)d\lambda(t)\\&=\sum_{\substack{(\alpha,\,\beta)\in I_{i,\,j}\\ \alpha\leqq\beta}}\int_{E_\alpha}\int_{E_\beta\cap U_t(\eta,\,\xi)} f(u)f(t)d\lambda(u)d\lambda(t)\\&\qquad+\sum_{\substack{(\alpha,\,\beta)\in I_{i,\,j}\\ \alpha>\beta}}\int_{E_\alpha}\int_{E_\beta\cap U_t(\eta,\,\xi)} f(u)f(t)d\lambda(u)d\lambda(t)\\&\leqq3(J_{i,\,j}(\eta,\,\xi)+J_{j,\,i}(\eta,\,\xi))\,.\end{aligned} \tag{3.34}$$

Now let $y,\,x\in E_{i,\,j,\,k}$, $y<x$. It is enough to consider the case $(\alpha,\,\beta,\,7)\in I_{i,\,j,\,k}$ for some $(\alpha,\,\beta)\in Q_2$, since otherwise $B_3(x)\geqq B_3(y)$ is clear. Then we can find an element $(i_0,\,j_0,\,k_0)$ of $I_{i,\,j,\,k}$ such that $k_0\geqq3$ and $k_0\neq7$. Using lemmas 10, 11 and 12 as well as the inequalities (3.33) and (3.34) (replacing η and ξ by $y-t$ and $x-t$, respectively), we get

$$\begin{aligned}B_3(x)-B_3(y)&\geqq\int_{E_{k_0}}(B_2(x-t)-B_2(y-t))f(t)d\lambda(t)\\&\qquad+\int_{E_1}((B_2(x-t)-B_2(y-t))f(t)d\lambda(t)\\&\geqq(1-\delta)\int_{E_{k_0}}(J_{i_0,\,j_0}(y-t,\,x-t)+J_{j_0,\,i_0}(y-t,\,x-t))\tilde{f}(t)d\lambda(t)\end{aligned}$$

$$-3\delta\int_{E_7}(J_{\alpha,\beta}(y-t,\ x-t)+J_{\beta,\alpha}(y-t,\ x-t)\tilde{f}(t)d\lambda(t)$$

$$=(1-4\delta)\int_{E_k}(J_{i,j}+J_{j,i})\tilde{f}(t)d\lambda(t)\geqq 0\ . \quad \text{q.e.d.}$$

PROOF OF THE THEOREM. The theorem follows from lemmas 4, 12, 13 and 14. q.e.d.

COROLLARY 2. *If there exists a positive number c such that $M(x)\in$ **M** has a positive and continuous first derivative in the interval $(0, c)$, then M has an indecomposable factor.*

3.3 The following theorem is another generalization of H. Cramér's result.

THEOREM 3. *Let $M\in \boldsymbol{M}$. If there exist three constants b, c and k such that $0\leqq b<2b<c<\infty$, $k>0$, and $M(x)-M(y)\geqq k\cdot(x-y)$ whenever $b<y<x<c$, then M has an indecomposable factor.*

PROOF. Without loss of generality, we assume

$$M(x)=\begin{cases} -k\cdot(c-b) & \text{if } \ 0<x<b \\ -k\cdot(c-x) & \text{if } \ b\leqq x<c \\ 0 & \text{if } \ x\geqq c\ . \end{cases}$$

Let

$$c-2b\geqq 2\alpha+3\varepsilon>0$$

$$a_1>\alpha>0$$

$$\varepsilon<\alpha/6, \quad \varepsilon<(a_1-\alpha)/6$$

$$a_1=b+\varepsilon$$

$$a_2=a_1+\alpha$$

$$a_3=2(a_1-\alpha)$$

$$a_4=2a_1-\alpha$$

$$a_5=2a_1$$

$$a_6=a_5+\alpha$$

$$a_7=a_6+\alpha\ .$$

Let

$$f(x)=k(\sum_1^4 \chi_n(x)-\delta\chi_5(x)+\sum_6^7 \chi_n(x))$$

$$\tilde{f}(x)=k\sum_1^7 \chi_n(x)$$

$$g(x)=k\chi_0(x)+\delta k\chi_5(x)$$

where $\chi_n(x)$ is the indicator function of the interval $E_n=(a_n-\varepsilon/3,\ a_n+\varepsilon/3)$ if $n\geqq 1$ and of the set $(b, c)-\bigcup_{n\neq 5} E_n$ if $n=0$.

The Poisson spectrum $M(x)$ is decomposed as

$$M=M_0+M_1$$

where

$$M_0(x)=-\int_x^\infty f(t)dt \quad \notin \boldsymbol{M}$$

$$M_1(x)=-\int_x^\infty g(t)dt \quad \in \boldsymbol{M}\ .$$

The result follows from the essentially same argument as before.

The process is much simpler in this case and the further detail is omitted.

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