# THE SWEEPING-OUT OF ADDITIVE FUNCTIONALS AND PROCESSES ON THE BOUNDARY

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## 1. Introduction

In this paper we shall consider the sweeping-out of additive functionals in Markov processes, and its application to processes on the boundary (U-processes). In sections 3 and 4 we define the sweeping-out of additive functionals and investigate their properties. In section 5, we consider a special additive functional (i.e. a time additive functional) and its inverse function. In section 6, using the additive functional defined in section 5, we transform the original process by the time change, and obtain a certain form of the process on the boundary (U-process) introduced by T. Ueno [9]. We show that this process is sufficiently regular if the original one is so. This paper is an introductory part of the investigation of U-processes. More detailed arguments of U-processes and their application to the boundary value problems are treated in [8]. The author wishes to express his gratitude to Mr. K. Sato and Mr. T. Ueno for their helpful advices and encouragements.

## 2. Definitions and notations

Let $S$ be a locally compact separable Hausdorff space and $S^*=S\cup\{\partial\}$ its one point compactification. (If $S$ is compact we consider $\{\partial\}$ an isolated point.) Let $\boldsymbol{B}$ be the topological Borel field generated by the open sets in $S^*$ and $M(S)$ the set of all bounded non-negative measures $\mu$ on $\boldsymbol{B}$. We define $\boldsymbol{F}=\bigcap_{\mu\in M(S)}\boldsymbol{B}^{\mu}$ where $\boldsymbol{B}^{\mu}$ is the $\mu$-completion of $\boldsymbol{B}$. Let $C(S)$, $B(S)$ and $F(S)$ be the sets of all bounded continuous functions, bounded $\boldsymbol{B}$-measurable functions and bounded $\boldsymbol{F}$-measurable functions on $S$, respectively. We always extend any function $f$ defined on $S$ to the one defined on $S^*$ by setting $f(\partial)=0$ unless particularly mentioned. Path space $W$ is the set of all mappings $w$ from $[0, \infty]$ into $S^*$ which satisfies the following properties:

($W$. 1) $w(t)$ is right continuous and has a left limit in $[0, \infty)$

($W$. 2) $w(\infty)=\partial$

(*W.* 3) There exists $\zeta=\zeta(w)$ such that

$$w(t)\in S \quad \text{if} \quad t<\zeta,$$
$$w(t)=\partial \quad \text{if} \quad t\geqq\zeta.$$

We shall write $w(t)=x_t(w)$ or simply $x_t$. For any $w\in W$ and $t\in[0,\infty]$, we define $w_t^+$ such that $x_s(w_t^+)=x_{s+t}$ $(0\leqq s\leqq\infty)$. Let $\mathfrak{B}_t$ be a Borel field generated by the cylinder sets $\{w: x_s(w)\in A\}$ $(s\leqq t, A\in \boldsymbol{B})$ and $\mathfrak{B}=\mathfrak{B}_\infty$.

Let $\boldsymbol{M}=\{P_x\}_{x\in S^*}$ be a system of probability measures on $\mathfrak{B}$ which satisfies:

(*P.* 1) $P_x(x_0=x)=1$ for any $x\in S^*$

(*P.* 2) $P_x(\mathfrak{A})$ is $\boldsymbol{B}$-measurable function in $x$ $(x\in S^*)$ for any $\mathfrak{A}\in\mathfrak{B}$. For any $\mu\in \boldsymbol{M}(S)$, we write

$$P_\mu(\mathfrak{A})=\int P_x(\mathfrak{A})\mu(dx)$$

$$E_x(F(w))=\int F(w)dP_x$$

$$E_\mu(F(w))=\int F(w)dP_\mu$$

where $\mathfrak{A}\in\mathfrak{B}$ and $F(w)$ is any (bounded) $\mathfrak{B}$-measurable function. Let $\mathfrak{F}_t{}^*=\bigcap_{\mu\in \boldsymbol{M}(S)}\mathfrak{B}_t^\mu$ where $\mathfrak{B}_t^\mu$ is the $P_\mu$-completion of $\mathfrak{B}_t$ and $\mathfrak{F}=\mathfrak{F}_\infty^*=\bigcap_{\mu\in \boldsymbol{M}(S)}\mathfrak{B}^\mu$. We say an event $\mathfrak{A}$ in $\mathfrak{F}$ occurs almost everywhere $P_x$ (a.e. $P_x$) if and only if $P_x(\mathfrak{A})=1$ and almost everywhere (a.e.) if and only if $P_x(\mathfrak{A})=1$ for any $x$ in $S^*$.

A $[0,\infty]$-valued function $\sigma=\sigma(w)$ on $W$ is called Markov time if $\{\sigma<t\}\in\mathfrak{F}_t{}^*$[1], and we set

$$\mathfrak{F}_\sigma=\{\mathfrak{A}: \mathfrak{A}\in\mathfrak{F} \quad \mathfrak{A}\cap\{\sigma<t\}\in\mathfrak{F}_t{}^* \text{ for any } t\}.$$ [2]

For any (nearly) Borel set $A$ in $S$ (or $S^*$), set

$$\sigma_A=\begin{cases}\inf t: & t>0,\ x_t\in A\\ \infty & \text{if there is not such } t.\end{cases}$$

Then $\sigma_A$ is a Markov time which is called a first passage time to $A$. For any $f$ in $F(S)$, we set

$$H_\sigma^\alpha f(x)=E_x(e^{-\alpha\sigma}f(x_\sigma))$$

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1) Without using (*P.* 3), (*P.* 4) and (*P.* 5) below, we see that if $\{\sigma_n\}$ be a monotone sequence of Markov times and $\sigma=\lim\sigma_n$, then $\sigma$ is also Markov time (c.f. section 6).

2) In general, $\mathfrak{F}_t{}^*\subset\mathfrak{F}_t$.

$$G_\alpha f(x)=E_x\left(\int_0^\infty e^{-\alpha t}f(x_t)dt\right)=\int_0^\infty e^{-\alpha t}H_t^0 f(x)dt\ ,$$

where $\alpha\geqq 0$.

For the system $\boldsymbol{M}$, we further assume the following:

$(P.\ 3)$ (Strong Markov property) Let $\sigma$ be any Markov time, for any $\mathfrak{A}\in\mathfrak{F}$ and $x\in S^*$,

$$P_x(w_\sigma^+\in\mathfrak{A}\mid\mathfrak{F}_\sigma)=P_{x_\sigma}(w\in\mathfrak{A})\quad\text{a.e. } P_x\ .$$

$(P.\ 4)$ (Property of quasi-left continuity) Let $\sigma_n$ be an increasing sequence of Markov times and $\sigma=\lim_{n\to\infty}\sigma_n$. Then for $w$ such as $\sigma(w)<\infty$,

$$x_\sigma=\lim x_{\sigma_n}\quad\text{a.e.}$$

That is,

$$P_x(x_\sigma=\lim x_{\sigma_n}\ ,\ \sigma<\infty)=P_x(\sigma<\infty)$$

for any $x$ in $S^*$.

$(P.\ 5)$ (Existence of reference measure) There exists an $\nu_0$ in $M(s)$ which has the following property: For any $A\in\boldsymbol{B}$, $P_{\nu_0}(\sigma_A<\infty)=0$ implies $P_x(\sigma_A<\infty)=0$ for any $x$ in $S^*$.

We call $\nu_0$ a reference measure of $\boldsymbol{M}$. Under $(P.\ 4)$, $\mathfrak{F}_t{}^*=\mathfrak{F}_t$. Under $(P.\ 5)$, Green operator $G_\alpha$ can be written in the form ([7]):

$$G_\alpha f(x)=\int g_\alpha(x,\ y)f(y)\nu_1(dy)$$

where $\nu_1(A)=\int G_{\alpha_0}(x,\ A)\nu_0(dx)$, and $\nu_0$ is a reference measure of $\boldsymbol{M}$ ($\alpha_0$ is a fixed positive number). From this we can easily see:

PROPOSITION 2.1. *For any* $f\in F(S)$, $G_\alpha f$ *is in* $B(S)$.

PROPOSITION 2.2. *Let* $f$ *be in* $F(S)$, *and if* $\lim_{t\downarrow 0}H_t^\alpha f=f$, *then* $f$ *is in* $B(S)$.

PROOF. Since $f=\lim_{\beta\to\infty}\beta\int_0^\infty e^{-\beta t}H_t^\alpha f dt=\lim_{\beta\to\infty}\beta G_{\alpha+\beta}f$ and $G_{\alpha+\beta}f$ is in $B(S)$ by (2.1), $f$ is in $B(S)$.

The non-negative $\boldsymbol{F}$-measurable function $u$ $(u(\partial)=0)$ is called $\alpha$-excessive if and only if

$(E_\alpha.\ 1)$ $H_t^\alpha u(x)\leqq u(x)$ for any $x$ in $S$ and $t$,

$(E_\alpha.\ 2)$ $\lim_{t\downarrow 0}H_t^\alpha u(x)=u(x)$.

A function $u$ is called $\alpha$-potential in class $D$ if and only if $u$ is $\alpha$-excessive and

$(E_\alpha . D_p)$ for any increasing sequence $\{\sigma_n\}$ of Markov times such as $\lim_{n\to\infty} \sigma_n \geqq \zeta$,

$$\lim_{n\to\infty} H^\alpha_{\sigma_n} u(x) = 0 .$$

A function $u$ is called regular $\alpha$-potential if and only if $u$ is $\alpha$-excessive and

$(E_\alpha . R)$ for any increasing sequence $\{\sigma_n\}$ of Markov times, if $\sigma = \lim \sigma_n$, then

$$\lim_{n\to\infty} H^\alpha_{\sigma_n} u(x) = H^\alpha_\sigma u(x)^{3)}.$$

From (2.2) and $(E_\alpha . 2)$, $\alpha$-excessive function is $\boldsymbol{B}$-measurable.

A $[0, \infty]$-valued function $A(t, w)$ on $[0, \infty] \times W$ is called (non-negative right continuous) additive functional, when it satisfies the following properties:

$(A. 1)$ $\quad 0 \leqq A(t, w) \leqq \infty$

$(A. 2)$ $\quad A(t, w)$ is right continuous in $t$

$(A. 3)$ $\quad A(t, w)$ is continuous at $t = \zeta$ and $A(t) = A(\zeta)$ for $t \geqq \zeta$

$(A. 4)$ $\quad A(t, w)$ is $\mathfrak{F}_t$-measurable in $w$ for any fixed $t$

$(A. 5)$ $\quad A(t, w) + A(s, w_t^+) = A(t+s, w)$

$(A. 6)$ $\quad A(t, w) < \infty$ for all $t < \infty$ a.e.

Two additive functionals $A$ and $B$ are equivalent if and only if

$$A(t, w) = B(t, w) \quad \text{for all } t \quad \text{a.e.},$$

and in this case we use the notation $A \sim B$. We also write $A \ll B$ if $A(t, w) \leqq B(t, w)$ for all $t$ a.e. For a non-negative $\mathfrak{F}$-measurable function $f$, we define

$$f \cdot A(t, w) = \int_{(0,\, t]} f(x_s(w)) dA(s, w) .$$

Then, for a suitable version, $f \cdot A$ is an additive functional if $f \cdot A$ satisfies condition $(A. 6)$. Let $\mathfrak{U}$ (unicity class) be the set of all additive functionals whose discontinuity points are continuity points of $x_t(w)$, and $\mathfrak{C}$ be the set of all additive functionals which are continuous (in $t$). Let $\mathfrak{U}_\alpha$ be a set of all additive functionals which are in $\mathfrak{U}$ and

$$u_\alpha(x) \equiv u_{\alpha A}(x) = E_x\left(\int_0^\infty e^{-\alpha t} dA(t, w)\right)$$

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3) As to these definitions, see Meyer [6]. The regular $\alpha$-potential in class $D$ in [6] is simply called regular $\alpha$-potential in this paper.

is finite. We define $\mathfrak{C}_\alpha=\mathfrak{C}\wedge\mathfrak{U}_\alpha$, $\bar{\mathfrak{U}}=\bigwedge_{\alpha>0}\mathfrak{U}_\alpha$ and $\bar{\mathfrak{C}}=\bigwedge_{\alpha>0}\bar{\mathfrak{C}}_\alpha$. It is easily seen that $u_{\alpha,A}$ satisfies the equation similar to the resolvent equation:

PROPOSITION 2.3. *If $A\in\bar{\mathfrak{C}}$, then for any $\alpha$, $\beta>0$*

$$u_{\alpha,A}-u_{\beta,A}+(\alpha-\beta)G_\alpha u_\beta=0^{4)}.$$

We define the time additive functional $T$ as follows:

$$T(t,\,w)=\begin{cases} t & \text{if} \quad t<\zeta(w)\\ \zeta & \text{if} \quad t\geqq\zeta(w)\,.\end{cases}$$

Then $T$ is in $\bar{\mathfrak{C}}$ and in $\mathfrak{C}_0$ if and only if $G_0 1$ is finite. We set

$$g_\alpha(x)\equiv u_{\alpha,T}(x)=G_\alpha 1(x)\,.$$

If $A\in\mathfrak{U}_\alpha$, $u_{\alpha,A}$ is an $\alpha$-potential in class $D$, and if $A\in\mathfrak{C}_\alpha$, $u_{\alpha,A}$ is a regular $\alpha$-potential. On the other hand, for any $\alpha$-potential $u$ in class $D$, there corresponds $A$ in $\mathfrak{U}_\alpha$ such as $u=u_{\alpha,A}$. If $u$ is a regular $\alpha$-potential, then the above $A$ is in $\mathfrak{C}_\alpha$ [5]. $A$ is determined uniquely by $u$ (except equivalence) ([6]).

## 3. Sweeping-out of the additive functional

Throughout this paper we shall fix a Borel set $V$ in $S$. We set

$$\sigma\equiv\sigma_V,\ H^\alpha\equiv H^\alpha_{\sigma_V}\,,$$

$$G^0_\alpha f(x)=E_x\left(\int_0^\sigma e^{-\alpha t}f(x_t)dt\right)\qquad(\alpha\geqq 0)\,,$$

where $f$ is in $F(S)$, and for $A$ in $\bar{\mathfrak{U}}$

$$u^0_\alpha=u^0_{\alpha,A}(x)=E_x\left(\int_0^\sigma e^{-\alpha t}dA\right)$$

$$g^0_\alpha(x)=E_x\left(\int_0^\sigma e^{-\alpha t}dT\right)=G^0_\alpha 1(x)\qquad(\alpha>0)\,.$$

4) For $A\in\mathfrak{C}$, see Dynkin [2]. But in this special case, the proof is almost the same as that of resolvent equation even in the discontinuous case.

5) It is noticed that, if $u$ is a regular $\alpha$-potential, we can choose $A$ in $\mathfrak{C}_\alpha$ to be $\mathfrak{B}$-measurable. If $u$ is uniformly $\alpha$-excessive, noting $u$ is $\boldsymbol{B}$-measurable and following the construction of Volkonskii and Tanaka ([10], [5]), we see $A$ can be chosen $\mathfrak{B}$-measurable. In general case, since $u(x)$ is a limit of strongly increasing sequence of uniformly $\alpha$-excessive functions, $A$ is a (almost everywhere) uniform limit of increasing sequence of $\mathfrak{B}$-measurable functionals, and so we can choose $A$ to be $\mathfrak{B}$-measurable. The above fact also shows that, *if $A\in\mathfrak{C}_\alpha$, there exists an equivalent functional which is $\mathfrak{B}$-measurable.* We shall use this remark.

For the later use, we state the following equalities, which are easily proved. For positive $\alpha$ and $\beta$,

$$H^{\alpha}-H^{\beta}+(\alpha-\beta)G_{\alpha}^{0}H^{\beta}=0\,, \tag{3.1}$$

$$u_{\alpha,A}=u_{\alpha,A}^{0}+H^{\alpha}u_{\alpha,A}\,,\text{[6]} \tag{3.2}$$

$$G_{\alpha}=G_{\alpha}^{0}+H^{\alpha}G_{\alpha}\,. \tag{3.3}$$

Since, for any $A$ in $\bar{\mathfrak{U}}$ and $\alpha>0$, $H^{\alpha}u_{\alpha,A}$ is $\alpha$-excessive ([3]) and dominated by $u_{\alpha,A}$ which is $\alpha$-potential in class $D$, $H^{\alpha}u_{\alpha,A}$ is also a potential in class $D$. Therefore, there exists a unique additive functional $\tilde{A}_{(\alpha)}$ in $\mathfrak{U}_{\alpha}$ such that

$$u_{\alpha,\tilde{A}_{(\alpha)}}=H^{\alpha}u_{\alpha,A}$$

$$E_x\left(\int_0^{\infty}e^{-\alpha t}d\tilde{A}_{(\alpha)}\right)=E_x\left(\int_{\sigma}^{\infty}e^{-\alpha t}dA\right)$$

for any $x$ in $S$. We shall call $\tilde{A}_{(\alpha)}$ the $\alpha$th order sweeping-out of $A$[7]. In general, $\tilde{A}_{(\alpha)}$ depends on $\alpha$.

PROPOSITION 3.4. *For $0<\alpha<\beta$*

$$\tilde{A}_{(\alpha)}\sim\tilde{A}_{(\beta)}+(\beta-\alpha)\widetilde{(u_{\alpha}^{0}\cdot T)}_{(\beta)}\,.$$

PROOF.
$$\begin{aligned}
E_x\left(\int_0^{\infty}e^{-\beta t}d\tilde{A}_{(\alpha)}\right)&=u_{\beta,\tilde{A}_{(\alpha)}}\\
&=u_{\alpha,A_{(\alpha)}}+(\alpha-\beta)G_{\beta}u_{\alpha,A_{(\alpha)}} && \text{(by (2.3))}\\
&=H^{\alpha}u_{\alpha}+(\alpha-\beta)G_{\beta}H^{\alpha}u_{\alpha}\\
&=H^{\beta}u_{\alpha}+(\alpha-\beta)(G_{\beta}-G_{\beta}^{0})H^{\alpha}u_{\alpha} && \text{(by (3.1))}\\
&=H^{\beta}u_{\alpha}+(\alpha-\beta)H^{\beta}G_{\beta}H^{\alpha}u_{\alpha} && \text{(by (3.3))}\\
&=H^{\beta}u_{\beta}+(\beta-\alpha)(H^{\beta}G_{\beta}u_{\alpha}-H^{\beta}G_{\beta}H^{\alpha}u_{\alpha}) && \text{(by (2.3))}\\
&=H^{\beta}u_{\beta}+(\beta-\alpha)H^{\beta}G_{\beta}u_{\alpha}^{0}(x) && \text{(by (3.2))}\\
&=u_{\beta,\tilde{A}_{(\beta)}}+(\beta-\alpha)u_{\beta,\widetilde{(u_{\alpha}^{0}\cdot T)}_{(\beta)}}\\
&=E_x\left(\int_0^{\infty}e^{-\beta t}(d\tilde{A}_{(\beta)}+(\beta-\alpha)d\widetilde{(u_{\alpha}^{0}\cdot T)}_{(\beta)})\right)
\end{aligned}$$

which proves the proposition.

From this proposition we can easily see

[6] c.f. Meyer [6].

[7] It is natural to define the 0th order sweeping-out $\tilde{A}_{(0)}$ of $A$ as

$$\tilde{A}_{(0)}(t)=\lim_{\alpha\downarrow 0}\tilde{A}_{(\alpha)}(t)$$

if the limit in the right hand side is finite for finite $t$ a.e. For example, this is the case when $u_0^0(x)$ is bounded. ($u_0(x)$ may be infinite.)

COROLLARY 3.5. *$\tilde{A}_{(\alpha)}$ increases as $\alpha$ decreases, and $\tilde{A}_{(\alpha)}$ is in $\bar{\mathfrak{U}}$ ($\alpha>0$).*

PROPOSITION 3.6. *If $A, B \in \bar{\mathfrak{U}}$ and $A \ll B$, then*

$$\tilde{A}_\alpha \ll \tilde{B}_\alpha \quad \text{for any} \quad \alpha>0 .$$

PROOF. By the assumption $u_{\alpha,B}-u_{\alpha,A}$ is $\alpha$-excessive, so $H^\alpha(u_{\alpha,B}-u_{\alpha,A})=u_{\alpha,\tilde{B}(\alpha)}-u_{\alpha,\tilde{A}(\alpha)}$ is also $\alpha$-excessive and dominated by $u_\alpha$. Thus, $\tilde{B}_{(\alpha)}(t)-\tilde{A}_{(\alpha)}(t)$ is equal to a certain (non-negative) additive functional a.e.

PROPOSISION 3.7. *If $A, B \in \bar{\mathfrak{U}}$ and $k$ is a non-negative constant, then*

$$\widetilde{(A+B)}_{(\alpha)}=\tilde{A}_{(\alpha)}+\tilde{B}_{(\alpha)}$$
$$\widetilde{(kA)}_{(\alpha)}=k\tilde{A}_{(\alpha)} .$$

PROOF. The above relations are only versions of the relations

$$H^\alpha(u_{\alpha,A}+u_{\alpha,B})=H^\alpha u_{\alpha,A}+H^\alpha u_{\alpha,B}$$
$$H^\alpha(ku_{\alpha,A})=kH^\alpha u_{\alpha,A} .$$

In general, we can not say $\tilde{A}_{(\alpha)} \in \bar{\mathfrak{C}}$, even if $A \in \bar{\mathfrak{C}}$. For example, let us consider the uniform motion on the real line and $P$ is a fixed point on the real line and $V=\{P\}$. Then $\tilde{T}_{(\alpha)}$ has a jump of height $1/\alpha$ at $6P$, where $T$ is the time additive functional. In the following we shall consider the condition for $\tilde{A}_{(\alpha)} \in \bar{\mathfrak{C}}$.

LEMMA 3.8. *For any $t \geqq 0$, $\varphi(x)=P_x(\sigma \leqq t)$[8] is B-measurable.*

PROOF. Since $P_x(\sigma=0)=\lim_{t \downarrow 0} P_x(\sigma \leqq t)$, it is sufficient to prove the lemma when $t>0$. Because

$$\{\sigma(w_u^+) \leqq t\} \subset \{\sigma \leqq t+s\}$$

for any $s \geqq u$, we have

$$\overline{\lim_{u \downarrow 0}}\, \{\sigma(w_u^+) \leqq t\} \subset \bigcap_{s>0} \{\sigma \leqq t+s\}=\{\sigma \leqq t\}.$$

On the other hand, if $\sigma(w)<t$ ($t>0$), there exists $s_0$ such as $0<s_0<t$ and $x_{s_0} \in V$, and so we have $\sigma(w_s^+) \leqq t$ for all $s<s_0$. If $\sigma(w)=t>0$, then $\sigma(w_s^+) \leqq t$ for all $s<t$. Therefore, we have

$$\underline{\lim}_{u \downarrow 0}\, \{\sigma(w_u^+) \leqq t\} \supset \{\sigma \leqq t\} .$$

That is,

8) $\sigma=\sigma_V$.

$$\lim_{u \downarrow 0} \{\sigma(w_u^+) \leqq t\} \text{ exists} = \{\sigma \leqq t\} .$$

Therefore,

$$\begin{aligned}\lim_{u \downarrow 0} H_u\varphi(x) &= \lim E_x(P_{x_u}(\sigma \leqq t)) \\ &= \lim P_x(\sigma(w_u^+) \leqq t) \\ &= P_x(\sigma \leqq t) = \varphi(x) .\end{aligned}$$

By this and (2.2) $\varphi(x)$ is in $B(S)$.

We set

$$\begin{aligned}&V_0 = \{x : x \in \bar{V} \text{ and } P_x(\sigma > 0) = 1\} \\ &V_r = \{x : P_x(\sigma = 0) = 1\} \\ &V' = V \cup V_r\end{aligned}$$

where $\bar{V}$ is the closure of $V$. Then these sets are $\boldsymbol{B}$-measurable and $\bar{V} - V_0 = V_r$ since $P_x(\sigma = 0) = 0$ or 1. Now we set the following assumption.

*Assumption R.* $\sigma_{V_0} = \infty$ a.e.[9]

THEOREM 3.9. *Under the assumption R, $A \in \bar{\mathfrak{C}}$ implies that $\tilde{A}_{(\alpha)} \in \bar{\mathfrak{C}}$ for any $\alpha > 0$.*

PROOF. It is sufficient to prove that $H^\alpha u$ is a regular $\alpha$-potential where $u = u_{\alpha, A}$. Let $\{\tau_n\}$ be any increasing sequence of Markov times and $\tau = \lim \tau_n$, and $\hat{\tau}_n = \tau_n + \sigma(w_{\tau_n}^+)$, $\hat{\tau} = \tau + \sigma(w_\tau^+)$.

1) If $\tau = \infty$, then $\lim \hat{\tau}_n \geqq \lim \tau_n = \tau = \infty$ and $\hat{\tau} \geqq \tau = \infty$. Therefore, $\lim \hat{\tau}_n = \hat{\tau} = \infty$.

2) If $\hat{\tau}_{n_0} > \tau$ for some $n_0$, then $\tau_{n_0} \leqq \tau_n \leqq \tau < \hat{\tau}_{n_0}$ for all $n \geqq n_0$. Therefore, $\hat{\tau}_{n_0} = \hat{\tau}_n = \hat{\tau}$ for all $n \geqq n_0$ which shows $\hat{\tau} = \lim \hat{\tau}_n$.

3) If $0 = \tau$ and $\hat{\tau}_n \leqq \tau$ for all $n$, then $\tau_n = \tau = 0$ and $\hat{\tau} = \hat{\tau}_n$ for all $n$. Hence, we have $\lim \hat{\tau}_n = \hat{\tau} = 0$.

4) Finally if $0 < \tau < \infty$ and $\hat{\tau}_n \leqq \tau$ for all $n$, then $\tau \geqq \lim \hat{\tau}_n \geqq \lim \tau_n = \tau$ or $\lim \hat{\tau}_n = \tau$. Noting $x_{\hat{\tau}_n} \in \bar{V}$ and $x_\tau = \lim x_{\hat{\tau}_n}$ a.e. by $(P. 4)$, we have $x_\tau \in \bar{V}$ a.e. Moreover, by the assumption $R$, $x_\tau \notin V_0$ a.e. (since $\tau > 0$). Therefore for any $x$ in $S$, we have

$$\begin{aligned}&P_x(\hat{\tau} = \lim \hat{\tau}_n,\ 0 < \tau < \infty,\ \hat{\tau}_n \leqq \tau \text{ for all } n) \\ &\quad = P_x(\tau = \hat{\tau},\ 0 < \tau < \infty,\ \hat{\tau}_n \leqq \tau \text{ for all } n) \\ &\quad = P_x(\tau = \hat{\tau},\ 0 < \tau < \infty,\ x_\tau \in \bar{V} - V_0,\ \hat{\tau}_n \leqq \tau \text{ for all } n) \\ &\quad = E_x(P_{x_\tau}(\sigma = 0) : 0 < \tau < \infty,\ x_\tau \in \bar{V} - V_0,\ \hat{\tau}_n \leqq \tau \text{ for all } n)\end{aligned}$$

9) If $V$ is closed, the assumption $R$ follows from the condition $H$ of Hunt [4]. For, $V_0 = V - V_r$ has any regular point, which is negligible.

$$=P_x(0<\tau<\infty,\ x_\tau\in\bar{V}-V_0,\ \hat{\tau}_n\leqq\tau \text{ for all } n)$$
$$=P_x(0<\tau<\infty,\ \hat{\tau}_n\leqq\tau \text{ for all } n)$$

since $P_x(\sigma=0)=1$ if $x\in\bar{V}-V_0$. The above equality shows that, if $0<\tau<\infty$ and $\hat{\tau}_n\leqq\tau$ for all $n$, then $\lim\hat{\tau}_n=\hat{\tau}$ a.e. From (1), (2), (3) and (4) we conclude that $\lim\hat{\tau}_n=\hat{\tau}$ a.e. Noting $u$ itself is a regular $\alpha$-potential, we can see that

$$H^\alpha_{\tau_n}H^\alpha u=H^\alpha_{\hat{\tau}_n}u \text{ tends to } H^\alpha_\tau H^\alpha u=H^\alpha_{\hat{\tau}}u,$$

which proves the theorem.

In the remainder of this section, we shall consider the condition for $A\sim\tilde{A}_{(\alpha)}$ when $A\in\mathfrak{C}$. Let

$$U_\delta=\left\{x: g^0_\alpha(x)=E_x\left(\int_0^\sigma e^{-\alpha t}dT\right)\geqq\delta\right\}$$

for any $\alpha$ and $\delta>0$, and $\rho_\delta=\sigma_{U_\delta}$.

LEMMA 3.10. *Let* $\rho_0=0$

$$\sigma_n=\rho_{n-1}+\sigma(w^+_{\rho_{n-1}}),\ \rho_n=\sigma_n+\rho_\delta(w^+_{\sigma_n}),\ n=1,2,\cdots.$$

*Then*

$$\lim_{n\to\infty}\rho_n=\lim_{n\to\infty}\sigma_n=\infty\ .$$

PROOF. Since $g^0_\alpha(x)=g_\alpha(x)-H^\alpha g_\alpha(x)$ is the difference of two $\alpha$-excessive functions, $g^0_\alpha(x_t,w)$ is right continuous in $t\in[0,\infty)$ (c.f. Hunt [3]). Hence, $g^0_\alpha(x_{\rho_n})\geqq\delta$. Let $\bar{\rho}=\lim_{n\to\infty}\rho_n$. Then

$$\lim_{n\to\infty}E_x\left(\int_0^{\rho_n}e^{-\alpha t}dT\right)=E_x\left(\int_0^{\bar{\rho}}e^{-\alpha t}dT\right)\leqq\frac{1}{\alpha},$$

since $T(t)$ is continuous in $t$. Noting $\rho_n\leqq\sigma_{n+1}\leqq\bar{\rho}$, we have $\lim_{n\to\infty}E_x\left(\int_{\rho^n}^{\sigma_{n+1}}e^{-\alpha t}dT\right)=0$. However,

$$E_x\left(\int_{\rho_n}^{\sigma_{n+1}}e^{-\alpha t}dT\right)=E_x\left(e^{-\alpha\rho_n}E_{x_{\rho_n}}\left(\int_0^\sigma e^{-\alpha t}dT\right)\right)$$
$$=E_x(e^{-\alpha\rho_n}g^0_\alpha(x_{\rho_n}))\geqq\delta E_x(e^{-\alpha\rho_n})\ .$$

Thus

$$E_x(e^{-\alpha\bar{\rho}})=\lim_{n\to\infty}E_x(e^{-\alpha\rho_n})\leqq\frac{1}{\delta}\lim_{n\to\infty}E_x\left(\int_{\rho^n}^{\sigma_{n+1}}e^{-\alpha t}dT\right)=0,$$

which proves the lemma.

We can now prove the following theorem:

THEOREM 3.11. *If* $A \in \bar{\mathfrak{C}}$, *then the following four conditions are equivalent for any* $\alpha > 0$.

(1) $A \sim \tilde{A}_{(\alpha)}$

(2) $A \sim \chi_{V_r} \cdot A$

(3) $A \sim \chi_{V'} \cdot A$

(4) $A(\sigma) = 0$ *a.e.*

PROOF.

(i) $A(\sigma)=0$ if and only if

$$E_x\left(\int_0^\infty e^{-\alpha t} d\tilde{A}_{(\alpha)}\right) = H^\alpha u_{\alpha, A}(x) = E_x\left(\int_\sigma^\infty e^{-\alpha t} dA\right)$$
$$= E_x\left(\int_0^\infty e^{-\alpha t} dA\right) .$$

Thus (1) and (4) are equivalent.

(ii) If $A \sim \chi_{V_r} \cdot A$, then $\chi_{V'} A \sim \chi_{V'} \chi_{V_r} \cdot A \sim \chi_{V_r} \cdot A \sim A$, which shows that (2) implies (3).

(iii) Noting $\sigma = \sigma_{V'}$ a.e. (c.f. Dynkin [1]) and $A(t)$ is continuous, if $A \sim \chi_{V'} \cdot A$, we have

$$A(\sigma) = \int_0^\sigma \chi_{V'} \cdot dA = \int_{(0, \sigma_{V'})} \chi_{V'} \cdot dA = 0 \quad \text{a.e.},$$

which shows that (3) implies (4).

(iv) Finally, if $A(\sigma)=0$ a.e., let $\rho_n$ and $\sigma_n$ be as in lemma 4.2, and we have

$$E_x\left(\int_0^\infty e^{-\alpha t} \chi_{U_\delta} dA\right)$$
$$= E_x\left(\sum \int_{\rho_n}^{\sigma_{n+1}} \chi_{U_\delta} \cdot dA\right) + E_x\left(\sum \int_{\sigma_n}^{\rho_n} \chi_{U_\delta} \cdot dA\right)$$
$$= E_x\left(\sum \int_{\rho_n}^{\sigma_{n+1}} e^{-\alpha t} \chi_{U_\delta} \cdot dA\right) \quad (\text{since } x_t \notin U_\delta \quad \text{if} \quad t \in (\sigma_n, \rho_n))$$
$$\leqq E_x\left(\sum e^{-\alpha \rho_n} E_{x_{\rho_n}}\left(\int_0^\sigma e^{-\alpha t} dA\right)\right) = 0 .$$

Thus $\chi_{U_\delta} \cdot A \sim 0$. Noting $\chi_{U_\delta} \uparrow \chi_{S-V_r}$ as $\delta \to 0$, we have

$$\chi_{S-V_r} \cdot A \sim 0 \qquad \text{or} \qquad \chi_{V_r} \cdot A \sim A ,$$

which shows that (4) implies (2).

LEMMA 3.12. *Under the assumption R*, $\sigma < \infty$ implies $x_\sigma \in V_r$ *a.e.*

PROOF. If $x \in V_r$, then $\sigma = 0$ a.e. $P_x$ and therefore $x_\sigma = x_0 \in V_r$ a.e. $P_x$.

If $x \notin V_r$, then $\sigma>0$ a.e. $P_x$ and by the assumption $R$, $x_\sigma \in \bar{V}-V_0=V_r$ a.e.

THEOREM 3.13. *Under the assumption R, if $A \in \bar{\mathfrak{C}}$, then $\tilde{A}_{(\alpha)}$ satisfies the conditions (1), (2), (3) and (4) of theorem 3.11 for any $\alpha>0$.*

PROOF. Let $\hat{\sigma}=\sigma+\sigma(w_\sigma^+)$. Then by lemma 3.12 $x_\sigma \in V_r$ a.e. and therefore $\sigma(w_\sigma^+)=0$ a.e., that is, $\sigma=\hat{\sigma}$ a.e. Therefore, $H^\alpha H^\alpha=H^\alpha$ and $H^\alpha u_{\alpha,\tilde{A}_{(\alpha)}}=u_{\alpha,\tilde{A}_{(\alpha)}}$ which shows $\tilde{A}_{(\alpha)}=(\tilde{\tilde{A}}_{(\alpha)})_{(\alpha)}$. Since $\tilde{A}_{(\alpha)} \in \bar{\mathfrak{C}}$ by theorem 3.9, theorem 3.13 follows from theorem 3.11.

## 4. The approximation theorem

Throughout this section we shall assume the assumption $R$ without referring.

LEMMA 4.1. *Let $\tau$ be any Markov time and $A \in \bar{\mathfrak{U}}$. If $x_\tau \in \bar{V}$ a.e., then*

$$E_\xi\left(\int_0^\tau e^{-\alpha t}dA\right)=E_\xi\left(\int_0^\tau e^{-\alpha t}d\tilde{A}_{(\alpha)}\right)$$

*for any $\xi$ in $V_r$ $(\alpha>0)$.*

PROOF. Noting $P_\xi(\tau=0)=0$ or 1, we have $x_\tau=\xi \in V_r$ if $\tau=0$ a.e. $P_\xi$ and by the assumption $R$, $x_\tau \in V_r=\bar{V}-V_0$ a.e. $P_\xi$ if $\tau>0$ a.e. $P_\xi$. Therefore

$$\begin{aligned} H_\tau^\alpha H^\alpha u(\xi) &= E_\xi(e^{-\alpha\tau}H^\alpha u(x_\tau)) \\ &= E_\xi(e^{-\alpha\tau}u(x_\tau)) \\ &= H_\tau^\alpha u(\xi) , \end{aligned}$$

and $H^\alpha u(\xi)=u(\xi)$ since $\xi \in V_r$, where $u=u_{\alpha,A}$. We have

$$\begin{aligned} E_\xi\left(\int_0^\tau e^{-\alpha t}dA\right) &= u(\xi)-H_\tau^\alpha u(\xi) \\ &= H^\alpha u(\xi)-H_\tau^\alpha H^\alpha u(\xi) \\ &= E_\xi\left(\int_0^\tau e^{-\alpha t}d\tilde{A}_{(\alpha)}\right) . \end{aligned}$$

LEMMA 4.2. *Let $\rho$ be any Markov time, $A \in \bar{\mathfrak{C}}$, and $\tilde{\rho}=\rho+\sigma(w_\rho^+)$. Then for any $x$ in $S$*

$$E_x\left(\int_0^\rho e^{-\alpha t}d\tilde{A}_{(\alpha)}\right)=E_x\left(\int_0^{\tilde{\rho}} e^{-\alpha t}d\tilde{A}_{(\alpha)}\right) , \qquad (\alpha>0) .$$

PROOF. By the assumption $R$ and theorem 3.13, we have $\tilde{A}_{(\alpha)}(\sigma)=0$ a.e. Therefore

$$E_x\left(\int_\rho^{\tilde{\rho}} e^{-\alpha t} d\tilde{A}_{(\alpha)}\right)=E_x\left(E_{x_\rho}\left(\int_0^\sigma e^{-\alpha t} d\tilde{A}_{(\alpha)}\right)\right)=0\ ,$$

which proves the lemma.

In the remainder of this section we shall assume $V$ is closed.

Let $D=S-B$ and $\{D_k\}$ be a sequence of open sets in $D$ such as $\bar{D}_k \subset D_{k+1}$ and $D=\lim_{k\to\infty} D_k$. Since $D$ is open, such $\{D_k\}$ exists. Let $\{\rho(k)\}$ be a decreasing sequence of Markov times which satisfies the condition:

$$(*) \qquad \rho(k) \leqq \frac{1}{k} \wedge \sigma_{D_k}\ .$$

And we set

$$\rho_0(k)=0\ ,$$

$$\sigma_n(k)=\rho_{n-1}(k)+\sigma(w^+_{\rho_{n-1}(k)})\ ,$$

$$\rho_n(k)=\sigma_n(k)+\rho(k)(w^+_{\sigma_n(k)})\ ,$$

for $n=1, 2, \cdots$. Omitting the suffix $k$, we shall often write $\rho$ for $\rho(k)$, $\rho_n$ for $\rho_n(k)$ and $\sigma_n$ for $\sigma_n(k)$.

LEMMA 4.3. *For any $\alpha$, $\beta>0$ and $A \in \bar{\mathfrak{C}}$,*

$$E_x\left(\sum_{n=1}^{\infty} e^{-\alpha\sigma_n} u^0_\beta(x_{\rho_n})\right) \leqq e^{(\alpha+\beta)/k} v_{\alpha,\beta}(x)$$

*for any $x$ in $S$, where*

$$u^0_\beta(x)=u^0_{\beta,A}(x)=E_x\left(\int_0^\sigma e^{-\beta t} dA\right)$$

$$v_{\alpha,\beta}(x)=u_{\alpha,\tilde{A}_{(\beta)}}(x)=E_x\left(\int_0^\infty e^{-\alpha t} d\tilde{A}_{(\beta)}\right)\ .$$

PROOF. If $\xi \in V_r$, $\rho_0=\sigma_1=0$, $\rho_1=\rho$ and $\sigma_2=\rho+\sigma(w^+_\rho)$ a.e. $P_\xi$. By lemma 3.12 if $\sigma_n<\infty$ $(n=1, 2, \cdots)$ a.e. $x_{\sigma_n} \in V_r$. Noting these facts, we have

$$\begin{aligned}
&E_x\left(\sum e^{-\alpha\sigma_n} u^0_\beta(x_{\rho_n})\right) \\
&=E_x\left(\sum e^{-\alpha\sigma_n} E_{x_{\rho_n}}\left(\int_0^\sigma e^{-\beta t} dA\right)\right) \\
&=E_x\left(\sum e^{-\alpha\sigma_n} E_{x_{\sigma_n}}\left(e^{\beta\rho}\int_\rho^{\sigma_2} e^{-\beta t} dA\right)\right) \\
&\leqq e^{\beta/k} E_x\left(\sum e^{-\alpha\sigma_n} E_{x_{\sigma_n}}\left(\int_0^{\sigma_2} e^{-\beta t} dA\right)\right)
\end{aligned}$$

$$=e^{\beta/k}E_x\left(\sum e^{-\alpha\sigma_n}E_{x_{\sigma_n}}\left(\int_0^{\sigma_2} e^{-\beta t}d\tilde{A}_{(\beta)}\right)\right) \qquad \text{(by (4.1))}$$

$$=e^{\beta/k}E_x\left(\sum e^{-\alpha\sigma_n}E_{x_{\sigma_n}}\left(\int_0^{\rho} e^{-\beta t}d\tilde{A}_{(\beta)}\right)\right) \qquad \text{(by (4.2))}$$

$$\leqq e^{\beta/k}E_x\left(\sum e^{-\alpha\sigma_n}E_{x_{\sigma_n}}\left(e^{\alpha\rho}\int_0^{\rho} e^{-\alpha t}d\tilde{A}_{(\beta)}\right)\right)$$

$$\leqq e^{(\alpha+\beta)/k}E_x\left(\sum e^{-\alpha\sigma_n}E_{x_{\sigma_n}}\left(\int_0^{\rho} e^{-\alpha t}d\tilde{A}_{(\beta)}\right)\right)$$

$$\leqq e^{(\alpha+\beta)/k}E_x\left(\int_0^{\infty} e^{-\alpha t}d\tilde{A}_{(\beta)}\right)$$

which proves the lemma.

THEOREM 4.4. *The assumption $R$ and $V=\bar{V}$ are assumed. Let $A$ be in $\mathfrak{E}$ and $\chi_{V_r}\cdot A\sim 0$. Let $\{\rho(k)\}$ be a sequence of Markov times satisfying the condition $(*)$ and $\lim_{n\to\infty}\rho_n(k)=\lim_{n\to\infty}\sigma_n(k)=\infty$ a.e. Then, for any $f$ in $B(S)$ such that $f(x_{t-})=\lim_{s\uparrow t}f(x_s)$ exists for all $t<\infty$ a.e. and that $|f(x_t)-f(x_0)|\leqq 1/k$ if $0\leqq t<\rho(k)$*[10]*, we have*

$$\lim_{k\to\infty} E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}f(x_{\rho_n(k)-})u_\beta^0(x_{\rho_n(k)})\Big)$$
$$=E_x\left(\int_0^\infty e^{-\alpha t}f(x_t)d\tilde{A}_{(\beta)}\right), \qquad \textit{for any } \alpha,\ \beta>0\,.$$

PROOF.

(i) Noting $\rho_n-\sigma_n\leqq 1/k$ and $|f(x_{\rho_n-})-f(x_{\sigma_n})|\leqq 1/k$, we have

$$\Big|E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n}f(x_{\rho_n-})u_\beta^0(x_{\rho_n})\Big)-E_x\Big(\sum e^{-\alpha\sigma_n}f(x_{\sigma_n})u_{\beta_0}(x_{\rho_n})\Big)\Big|$$
$$\leqq E_x\Big(\sum|e^{-\alpha\rho_n}-e^{-\alpha\sigma_n}||f(x_{\rho_n-})(u_\beta^0(x_{\rho_n})\Big)$$
$$+E_x\Big(\sum e^{-\alpha\sigma_n}|f(x_{\rho_n-})-f(x_{\sigma_n})|u_\beta^0(x_{\rho_n})\Big)$$
$$\leqq\{(1-e^{-\alpha/k})\|f\|+1/k\}E_x\Big(\sum e^{-\alpha\sigma_n}u_\beta^0(x_{\rho_n})\Big)$$
$$\leqq\{(1-e^{-\alpha/k})\|f\|+1/k\}e^{(\alpha+\beta)/k}v_{\alpha,\beta}(x)$$

where $\|f\|=\sup_x|f(x)|$ and $v_{\alpha,\beta}$ is the same as defined in lemma 4.3.

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10) If $f$ is a bounded continuous or bounded excessive function, set

$$\rho'(k)=\begin{cases}\inf t:\ x_t\in D_k \text{ or } |f(x_t)-f(x_0)|\geqq 1/k\\ \infty \qquad \text{if there is no such } k\end{cases}$$

and $\rho(k)=\text{Min}\,(\rho'(k),\ 1/k)$. Then, these $f$ and $\{\rho(k)\}$ satisfy the conditions of the theorem.

The right hand side of this inequality tends to zero as $k$ tends to infinity.

(ii) Since

$$E_x(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) e^{-\beta(\rho_n-\sigma_n)} u_\beta^0(x_{\rho_n}))$$
$$= E_x\Big(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) E_{x_{\sigma_n}}\Big(\int_\rho^{\sigma_2} e^{-\beta t} dA\Big)\Big)$$

we have

$$(**) \qquad \Big|E_x\Big(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) u_\beta^0(x_{\rho_n})\Big) - E_x\Big(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) E_{x_{\sigma_n}}\Big(\int_0^{\sigma_2} e^{-\beta t} dA\Big)\Big)\Big|$$
$$\leqq (1-e^{-\beta/k}) \,\|f\|\, E_x(\sum e^{-\alpha\sigma_n} u_\beta^0(x_{\rho_n}))$$
$$+ \|f\|\, E_x\Big(\sum e^{-\alpha\sigma_n} E_{x_{\sigma_n}}\Big(\int_0^\rho e^{-\beta t} dA\Big)\Big) .$$

Since $x_t \notin D_k$ for $0 \leqq t < \rho$,

$$E_x\Big(\sum e^{-\alpha\sigma_n} E_{x_{\sigma_n}}\Big(\int_0^\rho e^{-\beta t} dA\Big)\Big)$$
$$= E_x\Big(\sum e^{-\alpha\sigma_n} E_{x_{\sigma_n}}\Big(\int_0^\rho e^{-\beta t} \chi_{D_k^c}(x_t) dA\Big)\Big)$$
$$\leqq E_x\Big(\int_0^\infty e^{-\alpha t} d(\chi_{D_k^c}\cdot A)\Big) ,$$
$$\chi_{D_k^c}\cdot A \ll A \quad\text{and}\quad \chi_{D_k^c}\cdot A(t) \downarrow \chi_V\cdot A(t) \quad\text{as}\quad k\to\infty ,$$

and since $P_x(x_t \in V - V_r,\ t>0) = 0$ by the assumption $R$, $\chi_V\cdot A \sim \chi_{V_r}\cdot A \sim 0$ by the assumption of the theorem. Therefore, $\lim_{k\to\infty} E_x\Big(\int_0^\infty e^{-\alpha t} d(\chi_{D_k^c}\cdot A)\Big) = 0$. Thus, the right hand side of the inequality (**) is dominated by

$$(1-e^{\beta/k}) \,\|f\|\, e^{(\alpha+\beta)/k} v_{\alpha,\beta}(x) + \|f\| E_x\Big(\int_0^\infty e^{-\alpha t} d(\chi_{D_k^c}\cdot A)\Big)$$

which tends to zero as $k$ tends to infinity.

(iii) Since $x_{\sigma_n} \in V_r$ a.e. (by lemma 3.11), using (4.1) and (4.2) we have

$$E_x\Big(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) E_{x_{\sigma_n}}\Big(\int_0^{\sigma_2} e^{-\beta t} dA\Big)\Big)$$
$$= E_x\Big(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) E_{x_{\sigma_n}}\Big(\int_0^{\sigma_2} e^{-\beta t} d\tilde{A}_{(\beta)}\Big)\Big)$$
$$= E_x\Big(\sum e^{-\alpha\sigma_n} f(x_{\sigma_n}) E_{x_{\sigma_n}}\Big(\int_0^\rho e^{-\beta t} d\tilde{A}_{(\beta)}\Big)\Big) .$$

And

$$\Big|E_x\Big(\sum e^{-\alpha\sigma_n}f(x_{\sigma_n})E_{x_{\sigma_n}}\Big(\int_0^{\sigma} e^{-\beta t}d\tilde{A}_{(\beta)}\Big)\Big)$$
$$-E_x\Big(\sum e^{-\alpha\sigma_n}f(x_{\sigma_n})E_{x_{\sigma_n}}\Big(\int_0^{\rho} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)\Big)\Big|$$
$$\leqq (e^{(\alpha+\beta)/k}-1)\,||f||\,E_x\Big(\sum e^{-\alpha\sigma_n}E_{x_{\sigma_n}}\Big(\int_0^{\rho} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)\Big)$$
$$=(e^{(\alpha+\beta)/k}-1)\,||f||\,E_x\Big(\sum_{n=1}^{\infty}\int_{\sigma_n}^{\rho_n} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)$$
$$\leqq (e^{(\alpha+\beta)/k}-1)\,||f||e^{(\alpha+\beta)/k}\,v_{\alpha,\beta}(x)\,,$$

which tends to zero as $k$ tends to infinity.

(iv) Now,

$$E_x\Big(\sum e^{-\alpha\sigma_n}f(x_{\sigma_n})E_{x_{\sigma_n}}\Big(\int_0^{\rho} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)\Big)$$
$$=E_x\Big(\sum f(x_{\sigma_n})\int_{\sigma_n}^{\rho_n} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)\,.$$

Noting (since $\tilde{A}_{(\beta)}(\sigma)=0$)

$$\Big|E_x\Big(\sum_{n=0}^{\infty}\int_{\rho_n}^{\sigma_{n+1}} e^{-\alpha t}f(x_t)d\tilde{A}_{(\beta)}\Big)\Big|$$
$$\leqq ||f||\,E_x\Big(\sum e^{-\alpha\rho_n}E_{x_{\rho_n}}\Big(\int_0^{\sigma} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)\Big)=0\,,$$

we have (since $\lim \rho_n=\lim \sigma_n=\infty$)

$$E_x\Big(\int_0^{\infty} e^{-\alpha t}f(x_t)d\tilde{A}_{(\beta)}\Big)=E_x\Big(\sum\int_{\sigma_n}^{\rho_n} e^{-\alpha t}f(x_t)d\tilde{A}_{(\beta)}\Big)\,.$$

On the other hand, (since $|f(x_t)-f(x_{\sigma_n})|\leqq 1/k$ for $\sigma_n\leqq t<\rho_n$), we have

$$\Big|E_x\Big(\sum f(x_{\sigma_n})\int_{\sigma_n}^{\rho_n} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)-E_x\Big(\sum_{n=1}^{\infty}\int_{\sigma_n}^{\rho_n} e^{-\alpha t}f(x_t)d\tilde{A}_{(\beta)}\Big)\Big|$$
$$\leqq\frac{1}{k}E_x\Big(\sum\int_{\sigma_n}^{\sigma_{n+1}} e^{-\alpha t}d\tilde{A}_{(\beta)}\Big)\leqq\frac{1}{k}v_{\alpha,\beta}(x)\,,$$

which tends to zero as $k$ tends to infinity.

(i), (ii), (iii) and (iv) prove the theorem.

This theorem shows $\sum e^{-\alpha\rho_n}u_\beta^0(x_{\rho_n})$ approximates $\int e^{-\alpha t}d\tilde{A}_{(\beta)}$ in a certain sense. The approximation in other forms may be given[11]. The

[11] For example, under sufficiently regular conditions for $\boldsymbol{M}$ and $A$, we can prove the convergence is in $L^2(P_x)$-sense and almost everywhere if we choose a suitable subsequence.

theorem is useful for the investigation of the process on the boundary ([8]).

## 5. Properties of $\Phi_\alpha$ and $\mu_\alpha$

In this section we shall also assume the assumption $R$ and that $V$ is closed, and only consider the functional $T$ and its sweeping-out $\Phi_\alpha$: $\Phi_\alpha = \tilde{T}_{(\alpha)}$. By theorem 3.9 $\Phi_\alpha \in \bar{\mathfrak{C}}$ for any $\alpha > 0$.

Let

$$W_1^* = \{w: \text{ there exists such } t \text{ as } x_t(w) \in V\},$$

and

$$W_1 = \{w: \ w \in W_1^* \text{ and } \sup\{t: x_t \in V\} = \infty\}.$$

Let

$$\rho = 1 + \sigma(w_1^+),$$

$$\rho_0 = 0, \qquad \rho_{n+1} = \rho_n + \rho(w_{\rho_n}^+).$$

Then

$$W_1 = \{\rho_n < \infty \text{ for all } n\},$$

for $\rho_{n+1} \geqq \rho_n + 1$ and $x_{\rho_n} \in V$ (since $V$ is closed). Therefore, $W_1$ is $\mathfrak{F}_\infty$-measurable. In the first, we note,

PROPOSITION 5.1. *$w \in W_1$, then $\zeta = \infty$.*

THEOREM 5.2. *If $w \in W_1$, then $\Phi_\alpha(\infty, w) = \infty$ a.e. for any $\alpha > 0$, that is,*

$$P_x(\Phi_\alpha(\infty) = \infty, \ W_1) = P_x(W_1) \quad \textit{for any } x \textit{ in } S.$$

PROOF. Let $\beta > \text{Max}(1, \alpha)$ and $X = \int_0^\rho e^{-\beta t} d\Phi_\beta$.

(i) If $\xi \in V_r$ and $P_\xi(\zeta > \rho) \geqq 1/2$, then noting $x_\rho \in V$, we have

$$\begin{aligned} E_\xi(X) &= E_\xi\left(\int_0^\rho e^{-\beta t} d\Phi_\beta\right) \\ &= E_\xi\left(\int_0^\rho e^{-\beta t} dT\right) && \text{(by (4.1))} \\ &\geqq E_\xi\left(\int_0^1 e^{-\beta t} dT\right) \\ &\geqq E_\xi\left(\int_0^1 e^{-\beta t} dt; \ \zeta > \rho \geqq 1\right) \end{aligned}$$

$$\geqq \frac{1}{\beta}(1-e^{-\beta})P_\xi(\zeta>\rho)$$

$$\geqq \frac{1}{2\beta}(1-e^{-\beta})=q$$

where $q$ is a positive number independent of $\xi$. Also we have

$$\begin{aligned} E_\xi(X^2) &= 2E_\xi\left(\int_0^\rho e^{-\beta t}d\Phi_\beta(t)\int_t^\rho e^{-\beta s}d\Phi_\beta(s)\right) \\ &\leqq 2E_\xi\left(\int_0^\rho e^{-\beta t}H^\beta g_\beta(x_t)dt\right) \\ &\leqq \frac{2}{\beta}E_\xi(X)\,, \end{aligned}$$

since $H_\beta g_\beta(\xi)\leqq g_\beta(\xi)\leqq 1/\beta$. Therefore,

$$\begin{aligned} E_\xi(e^{-X};\ \zeta>\rho) &\leqq 1-E_\xi(X)+\frac{1}{2}E_\xi(X^2) \\ &\leqq 1-\left(1-\frac{1}{\beta}\right)E_\xi(X) \\ &\leqq 1-\left(1-\frac{1}{\beta}\right)q\,. \end{aligned}$$

(ii) If $P_x(\zeta>\rho)<1/2$, it is easily seen that

$$E_x(e^{-X};\ \zeta>\rho)<\frac{1}{2}\,.$$

Therefore, for any $\xi\in V_r$, we have

$$E_\xi(e^{-X};\ \zeta>\rho)\leqq p<1,$$

where

$$p=\text{Max}\left(\frac{1}{2},\ 1-\left(1-\frac{1}{\beta}\right)q\right)\,.$$

(iii) Since $x_{\rho_n}\in V_r$ a.e. by (3.12) and $\Phi_\beta(\rho)\geqq X$

$$\begin{aligned} E_x(e^{-\Phi_\beta(\infty)}:\ W_1) &= E_x(e^{-\Phi_\beta(\infty)}:\ W_1,\ \zeta=\infty) \qquad \text{(by (5.1))} \\ &\leqq E_x(e^{-\Phi_\beta(\rho_n)}:\ \zeta>\rho_n) \\ &= E_x(e^{-\Phi_\beta(\rho_{n-1})}E_{x_{\rho_{n-1}}}(e^{-\Phi_\beta(\rho)}:\ \zeta>\rho):\ \zeta>\rho_{n-1}) \\ &\leqq pE_x(e^{-\Phi_\beta(\rho_{n-1})}:\ \zeta>\rho_{n-1}) \quad (n=2,\ 3,\ 4,\ \cdots)\,. \end{aligned}$$

Therefore, by induction,

$$E_x(e^{-\Phi_\beta(\infty)}: W_1) \leqq p^{n-1} \qquad (n=2, 3, \cdots)$$

which shows that

$$E_x(e^{-\Phi_\beta(\infty)}: W_1)=0 \quad \text{or} \quad P_x(\Phi_\beta(\infty)<\infty, W_1)=0 .$$

Noting $\Phi_\alpha(\infty) \geqq \Phi_\beta(\infty)$ (since $\alpha<\beta$), we have the theorem.

COROLLARY 5.3. *If* $M$ *is conservative (that is,* $\zeta=\infty$ *a.e.) and* $\sigma<\infty$ *a.e. then* $\Phi_\alpha(\infty)=\infty$ *a.e.*

We define the inverse function $\tau_\alpha(s, w)$ of $\Phi_\alpha(t, w)$ as follows:

$$\tau_\alpha(s, w)=\sup t: \Phi_\alpha(t) \leqq s .$$

Then, it is easily seen ([10]).

PROPOSITION 5.4.

(1) $\tau_\alpha(s, w)$ *is a Markov time for a fixed* $s$.

(2) $\tau_\alpha(s, w)$ *is a right continuous increasing function in* $s$.

(3) $\tau_\alpha(s+t, w)=\tau_\alpha(s, w)+\tau_\alpha(t, w^+_{\tau_\alpha(s)})$ .

For the later use, we shall prove the following lemmas.

LEMMA 5.5. $P_\xi(\tau_\alpha(0)=0)=1$ *for any* $\xi$ *in* $V_r$ .

PROOF. Since $\tau_\alpha(s)$ is a Markov time, $g_\alpha \geqq H^\alpha g_\alpha$ and $H^\alpha g_\alpha(\xi)=g_\alpha(\xi)$,

$$\begin{aligned} 0=E_\xi\left(\int_0^{\tau_\alpha(0)} e^{-\alpha t} d\Phi_\alpha\right) &= H^\alpha g_\alpha(\xi)-H^\alpha_{\tau_\alpha(0)} H^\alpha g_\alpha \\ &= g_\alpha(\xi)-H^\alpha_{\tau_\alpha(0)} g_\alpha(\xi) \\ &= E_\xi\left(\int_0^{\tau_\alpha(0)} e^{-\alpha t} dT\right) \geqq 0 \end{aligned}$$

where $g_\alpha(x)=E_x\left(\int_0^\infty e^{-\alpha t} dT\right)$. Thus, $T(\tau_\alpha(0))=\text{Min}\,(\tau_\alpha(0), \zeta)=0$ a.e. $P_\xi$, namely $\tau_\alpha(0)=0$ a.e. $P_\xi$ (since $P_\xi(\zeta>0)=1$).

LEMMA 5.6. $\tau_\alpha(0)=\sigma$ *a.e.*

PROOF. By theorem 3.13 $\Phi_\alpha(\sigma)=0$ a.e. and therefore $\tau_\alpha(0) \geqq \sigma$ a.e. On the other hand, noting $x_\sigma \in V_r$ a.e. by (3.12), we have

$$\begin{aligned} P_x(\tau_\alpha(0)>\sigma) &= P_x(\tau_\alpha(0, w^+_\sigma)>0, \Phi_\alpha(\sigma)=0, \sigma<\infty) \\ &= E_x(P_{x_\sigma}(\tau_\alpha(0)>0): \Phi_\alpha(\sigma)=0, \sigma<\infty) \\ &= 0 \qquad \text{(by (5.5))}. \end{aligned}$$

Thus, $\tau_\alpha(0)=\sigma$ a.e.

THEOREM 5.7. *$x_{\tau_\alpha(s)} \in V$ for any $s$ such as $0 \leqq \tau_\alpha(s) < \infty$ a.e. In other words, $P_x(x_{\tau_\alpha(s)} \notin D$ for all $s)=1$ for any $x$, where $D=S-V$*[12].

PROOF. Since $V$ is closed, $V=V'$ and by theorem 3.13 $\Phi_\alpha \sim \chi_V \cdot \Phi_\alpha$. We consider the only $w$ which satisfies $\Phi_\alpha(t)=\int_0^t \chi_V(x_s)d\Phi_\alpha$ for all $t$, which occurs for almost every $w$. If $x_u \in D$, then

$$\begin{aligned}\Phi_\alpha(u+\sigma(w_u^+)) &= \Phi_\alpha(u)+\int_u^{u+\sigma(w_u^+)} \chi_V(x_t)d\Phi_\alpha \\ &= \Phi_\alpha(u)\end{aligned}$$

and $\sigma(w_u^+)>0$ since $V$ is closed. Therefore, $u$ can not be equal to $\tau(s, w)$ for any $s$. The theorem is proved.

Let $G$ be any open set in $S$. For any positive $\varepsilon$ and $k$, we define

$$\rho_{\varepsilon,k}(G)=\begin{cases}\inf t:\ t \geqq \varepsilon, \quad x_t \in G, \quad \Phi_\alpha(t)-\Phi_\alpha(t-\varepsilon)<k \\ \infty \qquad \text{if there is no such } t.\end{cases}$$

Then, from the right continuity of $x_t$ and $\Phi_\alpha(t)$ in $t$, we have

$$\{\rho_{\varepsilon,k}(G)<t\}=\bigcup_{\substack{\varepsilon \leqq r<t \\ r:\ \text{rational}}}\{x_r \in G,\ \Phi_\alpha(r)-\Phi_\alpha(r-\varepsilon)<k\}$$

for any $t \in [0, \infty)$. Therefore, $\rho_{\varepsilon,k}(G)$ is a Markov time. Let $\{D_n\}$ be a decreasing sequence of open sets which contains $V$ and $\cap \bar{D}_n=V$[13]. Set

$$\rho_{\varepsilon,n}=\rho_{\varepsilon,\,1/n}(D_n)$$

$$\rho_\varepsilon=\begin{cases}\inf t:\quad t \geqq \varepsilon, \quad x_t \in V, \quad \Phi_\alpha(t)-\Phi_\alpha(t-\varepsilon)\ <k \\ \infty \qquad \text{if there is not such } t\,.\end{cases}$$

The sequence $\{\rho_{\varepsilon,n}\}$ is an increasing sequence of Markov times and $\rho_{\varepsilon,n} \leqq \rho_\varepsilon$. Let $\bar{\rho}=\lim_{n\to\infty} \rho_{\varepsilon,n}$. Then $x_{\bar{\rho}}=\lim_{n\to\infty} x_{\rho_{\varepsilon,n}}$ a.e. on $\bar{\rho}<\infty$ (by (*P*. 4)) and $\bar{\rho} \leqq \rho_\varepsilon$. On the other hand, since $x_{\rho_n} \in \bar{D}_n$ and $\Phi_\alpha(\rho_{\varepsilon,n})-\Phi_\alpha(\rho_{\varepsilon,n}-\varepsilon) \leqq 1/n$, we have $x_{\bar{\rho}} \in V$ if $\bar{\rho}<\infty$ a.e. $\Phi_\alpha(\bar{\rho})-\Phi_\alpha(\bar{\rho}-\varepsilon)=0$. Thus $\bar{\rho} \geqq \rho_\varepsilon$ a.e., and so $\rho_\varepsilon=\bar{\rho}$ a.e. is a Markov time. Moreover, since $x_{\rho_\varepsilon} \in V$ if $\rho_\varepsilon<\infty$ and $\rho_\varepsilon \geqq \varepsilon$, $x_{\rho_\varepsilon} \in V_r$ if $\rho_\varepsilon<\infty$ a.e. by the assumption $R$. Let

$$\rho_0=0, \qquad \rho_{n+1}=\rho_n+\rho_\varepsilon(w_{\rho_n}^+)\,.$$

Then $x_{\rho_n} \in V_r$ if $\rho_n<\infty$ a.e. and $\Phi_\alpha(\rho_n)=\Phi_\alpha(\rho_n-\varepsilon)$. Also we have

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[12] If $\tau_\alpha(s)<\infty$, then $\tau_\alpha(s)<\zeta$. For, if $\tau_\alpha(s) \geqq \zeta$, then $\Phi_\alpha(\zeta)=\Phi_\alpha(\tau_\alpha(s))=\Phi_\alpha(\infty)=s$ and $\tau_\alpha(s)=\infty$.

[13] Such $\{D_n\}$ exists since $V$ is closed.

$$P_x(\tau_\alpha(0,\ w^+_{\rho_n})>0,\ \rho_n<\infty)=E_x(P_{x_{\rho_n}}(\tau_\alpha(0)=\sigma>0)\colon\ \rho_n<\infty)$$
$$=0 \qquad \text{(by (5.6))}.$$

Hence, $\Phi_\alpha(s)>\Phi_\alpha(\rho_n)$ for any $s>\rho_n\,(n=1,\ 2,\ \cdots)$ a.e. Therefore, for any $x$ in $S$, except the subset of $W$ of $P_x$-measure 0, the following assertion holds: If $\rho_{n-1}<t\leqq\rho_{n+1}$, $\varepsilon\leqq t<\infty$, $x_t\in V$ and $\Phi_\alpha(t)=\Phi_\alpha(t-\varepsilon)$, then $t-\varepsilon\geqq\rho_{n-1}\,(n=1,\ 2,\ \cdots)$[14] and $t=\rho_n$ or $\Phi_\alpha(s)>\Phi_\alpha(t)=\Phi_\alpha(t-\varepsilon)$ for any $s>t$. Noting $\lim\rho_n=\infty$ (since $\rho_{n+1}\geqq\rho_n+\varepsilon$), we have proved the following lemma.

LEMMA 5.8. *Let*

$$\mathfrak{A}_\varepsilon=\left\{\begin{array}{ll} w: & \textit{there exist } t \textit{ and } s \textit{ such as } s>t\,, \\ & x_t\in V \textit{ and } \Phi_\alpha(s)=\Phi_\alpha(t-\varepsilon)\,. \end{array}\right\}$$

*Then*

$$P_x(\mathfrak{A}_\varepsilon)=0 \qquad \textit{for any } x \textit{ in } S\,.$$

THEOREM 5.9. *Let*

$$\mathfrak{A}=\left\{\begin{array}{ll} w: & \textit{there exists } t,\ s \textit{ and } u \textit{ such as } s>t>u\,, \\ & x_t\in V \textit{ and } \Phi_\alpha(s)=\Phi_\alpha(u)\,. \end{array}\right\}$$

*Then*

$$P_x(\mathfrak{A})=0 \qquad \textit{for any } x \textit{ in } S.$$

PROOF. Since $\mathfrak{A}=\bigcup_n\mathfrak{A}_{1/n}$, the theorem follows from the lemma 5.8. Roughly speaking, $\Phi_\alpha(t)$ can not be constant near $t$ where $x_t\in V$ a.e.

THEOREM 5.10. *Let*

$$\bar{\zeta}=\left\{\begin{array}{ll} \sup t: & x_t\in V \\ 0 & \textit{if there is no such } t\,. \end{array}\right.$$

*Then*

(1) $\bar{\zeta}=\inf t\colon\ \Phi_\alpha(t)=\Phi_\alpha(\infty) \qquad (t\leqq\infty)$ *a.e.*

(2) $\bar{\zeta}=\infty$ *if and only if* $\Phi_\alpha(\infty)=\infty$ *a.e.*

PROOF. We confine our attention to the event that $x_{\tau_\alpha(s)}\in V$ for any $s$ such as $\tau_\alpha(s)<\infty$ and $x_t\notin V$ if there exist $s$ and $u$ such as $s>t>u$ and $\Phi_\alpha(s)=\Phi_\alpha(u)$. This event occurs almost everywhere by theorems (5.7) and (5.9). Let $\rho=\inf t\colon\ \Phi_\alpha(t)=\Phi_\alpha(\infty)\ (t\leqq\infty)$. If $\Phi_\alpha(\bar{\zeta})<\Phi_\alpha(\zeta)$, then

[14] If $n=1$, $t-\varepsilon\geqq0$ is obvious, and if $n\geqq2$, then $\Phi_\alpha(t)=\Phi_\alpha(t-\varepsilon)>\Phi_\alpha(\rho_{n-1})$ and $t-\varepsilon>\rho_{n-1}$.

for any $s$ in $\Phi_\alpha(\bar{\zeta})<s<\Phi_\alpha(\infty)$, we have $\bar{\zeta}<\tau_\alpha(s)<\infty$ and $x_{\tau_\alpha(s)}\in V$ which contradicts the definition of $\bar{\zeta}$. Therefore, $\Phi_\alpha(\bar{\zeta})=\Phi_\alpha(\infty)$ or $\bar{\zeta}\geqq\rho$. On the other hand, since $\Phi_\alpha(\rho)=\Phi_\alpha(\infty)$, $x_t\notin V$ if $t$ is in $(\rho, \infty]$. Therefore $\bar{\zeta}\leqq\rho$. The first part of the theorem is proved. If $\bar{\zeta}=\infty$, then $\Phi_\alpha(\infty)=\infty$ a.e. by theorem 5.2. If we note that $\rho=\bar{\zeta}$ a.e., the converse is obvious by the definition of $\rho$.

## 6. Properties of U-process

Changing the time scale of the process $\boldsymbol{M}$ by $\Phi_\alpha(t)$, we shall get a certain form of the process on the boundary given by T. Ueno [9] (that is, U-process). We shall show this process is a Markov process on $V$ in the sense of section 2. Throughout this section, we shall assume

*Assumption* $R^*$. $V$ is closed and any point in $V$ is regular to $V$[15], that is, $V=\bar{V}=V_r$.[16]

Noting $\Phi_\alpha(t, w)$[17] can be chosen so as it is $\mathfrak{B}$-measurable (c.f. remark in section 1), we can consider

$$\tau_\alpha(s, w)=\sup\{r: r \text{ is rational and } \Phi_\alpha(r)\leqq s\}$$

is also $\mathfrak{B}$-measurable for fixed $t$. Therefore, we have

PROPOSITION 6.1. *$x_{\tau_\alpha(s, w)}$ is $\mathfrak{B}$-measurable for fixed $t$.*

It is obvious that

PROPOSITION 6.2. *$x_{\tau_\alpha(s, w)}$ is right continuous in $t$, and $x_{\tau_\alpha(s)}=\partial$ if $s$ is in $[\Phi_\alpha(\infty), \infty]$.*

Now, we set

$$\mathfrak{A}_1=\{w:\ x_{\tau_\alpha(s)}\notin D, \text{ for any } s\},$$

$$\mathfrak{A}_2=\{w:\ \Phi_\alpha(\infty)=\infty, \text{ or there exists such } \bar{\zeta}(w) \text{ as } \bar{\zeta}<\infty \text{ and } \Phi_\alpha(t)=\Phi_\alpha(\infty) \text{ for } t\geqq\bar{\zeta}\},$$

and $\mathfrak{A}_0=\mathfrak{A}_1\wedge\mathfrak{A}_2$. Then, by theorems (5.7) and (5.10), $P_x(\mathfrak{A}_0)=1$ for any $x$ in $S$, and

$$\mathfrak{A}_1=\{w:\ x_{\tau_\alpha(r)}\notin D \text{ for any rational } r\}^{18)}$$

$$\mathfrak{A}_2=\{\Phi_\alpha(\infty)=\infty\}\vee\{\Phi_\alpha(r)=\Phi_\alpha(\infty) \quad \text{for some rational } r\}$$

15) $x$ is regular to $V$ if and only if $P_x(\sigma_V=0)=1$.
16) Obviously the assumption $R^*$ implies the assumption $R$.
17) In this section a positive number $\alpha>0$ is fixed (except the last remark of this section).
18) For, $V\vee\{\partial\}$ is closed in $S^*$ and $x_{\tau_\alpha(s)}$ is right continuous in $S$.

are $\mathfrak{B}$-measurable. Moreover, for any $w \in \mathfrak{A}_2$, let $\{s_n\}$ be an increasing sequence of non-negative numbers such as $s=\lim s_n<\infty$. Then (1) $\tau_\alpha(s_n)\leqq \tau_\alpha(s)<\infty$ if $s<\Phi_\alpha(\infty)$, (2) $\tau_\alpha(s_n)<\bar{\zeta}<\infty$ for all $n$ if $s=\Phi_\alpha(\infty)$ and $s_n<\Phi_\alpha(\infty)$ for all $n$, and (3) $\tau_\alpha(s_n)=\infty$ for $n\geqq n_0$ if $s\geqq\Phi_\alpha(\infty)$ and $s_{n_0}\geqq\Phi_\alpha(\infty)$ for some $n_0$. Therefore $\lim_{n\to\infty} x_{\tau_\alpha(s_n)}$ exists in any case.

PROPOSITION 6.3. *If* $w \in \mathfrak{A}_2$, $x_{\tau_\alpha(s,w)}(w)$ *has a left limit in s which is in* $[0, \infty)$.

We shall define the fields of sets, the sets of functions and the path space etc. on $V$ in the same way as in section 2 in which we replace $S$ by $V$. To distinguish the new notations from the original notations, we add the wave marks, that is, $\tilde{\boldsymbol{B}}$ is a topological Borel fields on $V$ and $\tilde{W}$ is the space of all paths on $V^*$[19] which satisfies ($W$. 1), ($W$. 2) and ($W$. 3), etc.

We define a mapping $\pi_\alpha$ of $W$ into $\tilde{W}$ as follows:

$$\xi_s(\pi_\alpha(w))=\begin{cases} x_{\tau_\alpha(s,w)}(w) & \text{if } w \in \mathfrak{A}_0 \\ \partial & \text{if } w \notin \mathfrak{A}_0 \end{cases}$$

where $\xi_s(\tilde{w})=\tilde{w}_s$[20]. We shall often write $\tilde{w}_\alpha$ for $\pi_\alpha(w)$. We can assure $\tilde{w}_\alpha \in \tilde{W}$ by definition of $\mathfrak{A}_0$, (6.2) and by (6.3) if $w \in \mathfrak{A}_0$, and by the definition if $w \notin \mathfrak{A}_0$. Moreover,

PROPOSITION 6.4.

(1) $P_x(\xi_s(\tilde{w}_\alpha)=x_{\tau_\alpha(s)}$ *for all* $s)=1$ *for any* $x$ *in* $S$.

(2) $\pi_\alpha^{-1}(\tilde{\mathfrak{A}}) \in \mathfrak{B}$ *for any* $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{B}}$.

PROOF. The first assertion follows from the equality $P_x(\mathfrak{A}_0)=1$ a.e. Since $\mathfrak{A}_0$ is $\mathfrak{B}$-measurable, the second assertion is obtained by (6.1).

Now we define a system of probability measures $\tilde{\boldsymbol{M}}^{(\alpha)}=\{\tilde{P}_\xi^{(\alpha)}\}_{\xi\in V^*}$ on $\tilde{\mathfrak{B}}$ (and therefore on $\tilde{\mathfrak{F}}$) as follows:

$$\tilde{P}_\xi^{(\alpha)}(\tilde{\mathfrak{A}})=P_\xi(\pi_\alpha^{-1}(\tilde{\mathfrak{A}}))$$

for any $\xi \in V^*$. We shall often drop the suffix $\alpha$, that is, $\tilde{\boldsymbol{M}}=\tilde{\boldsymbol{M}}^{(\alpha)}$ $\tilde{P}_\xi=\tilde{P}_\xi^{(\alpha)}$ $\pi=\pi_\alpha$ $\tau=\tau_\alpha$ and $\Phi=\Phi_\alpha$.

THEOREM 6.5. $\tilde{\boldsymbol{M}}$ *satisfies the conditions:*

($\tilde{P}$. 1) $\tilde{P}_\xi(\xi_0=\xi)=1$.

19) $V^*=V\cup\{\partial\}$.

20) By the definition, we have $\tilde{\zeta}(\tilde{w}_\alpha)=\Phi_\alpha(\infty)$ if $w\in\mathfrak{A}$ and $\tilde{\zeta}(\tilde{w}_\alpha)=0$ if $w\in\mathfrak{A}$.

($\tilde{P}$. 2) *For any* $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{B}}$. $P_\xi(\mathfrak{A})$ *is a* $\tilde{\boldsymbol{B}}$*-measurable function in* $\xi$.

PROOF. Since $\xi \in V = V_r$ by the assumption $R^*$, $P_\xi(\tau(0)=0)=1$, which proves ($\tilde{P}$. 1). Since $\pi^{-1}(\tilde{\mathfrak{A}}) \in \mathfrak{B}$ by (6.4) and $P_x(\pi^{-1}(\tilde{\mathfrak{A}}))$ is $\boldsymbol{B}$-measurable in $x$, restricting $x$ on $V$, we obtain ($\tilde{P}$. 2).

PROPOSITION 6.6. $\pi^{-1}(\tilde{\mathfrak{B}}_t) \subset \mathfrak{F}_{\tau(t)} \wedge \mathfrak{B}$.

PROOF. By (6.4) $\pi^{-1}(\tilde{\mathfrak{B}}_t) \subset \mathfrak{B}$. For any $A \in \tilde{\boldsymbol{B}} \subset \boldsymbol{B}$ and $s \leqq t$, $\{w : \xi_s(\tilde{w}) \in A\} = \{w : x_{\tau(s)} \in A\}$ a.e. The right side of this equality is $\mathfrak{F}_{\tau(t)}$-measurable. Noting that $\tilde{\mathfrak{F}}_t$ is generated by $\{\xi_s \in A\}$ $(s \leqq t,\ A \in \tilde{\boldsymbol{B}})$, we obtain the proposition.

LEMMA 6.7. *Let* $\tilde{\mathfrak{A}}$ *be in* $\tilde{\mathfrak{B}}$ *and* $\mu$ *be in* $M(S)$. *Then* $\tilde{P}_{\tilde{\nu}}(\tilde{\mathfrak{A}})=0$ *implies* $P_\mu(\pi^{-1}(\tilde{\mathfrak{A}}))=0$, *where* $\tilde{\nu}(A)=\int_{S^*} \mu(dx) P_x(x_\sigma \in A)$ *for* $A \in \tilde{\boldsymbol{B}} \subset \boldsymbol{B}$.

PROOF. For any $\varepsilon > 0$, there exists

$$\tilde{\mathfrak{A}}_\varepsilon = \bigcup_n \{\xi_{t_1^n} \in A_1^n,\ \cdots,\ \xi_{t_{in}^n} \in A_{in}^n\}$$

such that $\tilde{\mathfrak{A}}_\varepsilon \supset \tilde{\mathfrak{A}}$ and $\tilde{P}_\nu(\tilde{\mathfrak{A}}_\varepsilon) < \varepsilon$, where $A_j^n$'s are in $\tilde{\boldsymbol{B}}$. Since, by (6.6)

$$\begin{aligned}\tau(t, w) &= \tau(0) + \tau(t, w_{\tau(0)}^+) \\ &= \sigma + \tau(t, w_\sigma^+) \text{ a.e.,}\end{aligned}$$

we have

$$\begin{aligned}P_\mu(\pi^{-1}(\tilde{\mathfrak{A}}_\varepsilon)) &= P_\mu(\bigcup_n x_{\tau(t_j^n)} \in A_j^n,\ j=1, 2, \cdots, i_n\}) \\ &= P_\mu(\bigcup_n \{x_{\tau(t_j^n, w_\sigma^+)}(w_\sigma^+) \in A_j^n,\ j=1, 2, \cdots, i_n\}) \\ &= E_\mu(P_{x_\sigma}(\bigcup_n \{x_{\tau(t_j^n)} \in A_j^n,\ j=1, 2, \cdots, i_n\})) \\ &= P_{\tilde{\nu}}(\pi^{-1}(\tilde{\mathfrak{A}}_\varepsilon)) = \tilde{P}_{\tilde{\nu}}(\tilde{\mathfrak{A}}_\varepsilon) < \varepsilon .\end{aligned}$$

Since $\pi^{-1}(\tilde{\mathfrak{A}}) \subset \pi^{-1}(\tilde{\mathfrak{A}}_\varepsilon)$, the proposition follows.

LEMMA 6.8. $\pi^{-1}(\tilde{\mathfrak{F}}_t^*) \subset \mathfrak{F}_{\tau(t)}$[21].

PROOF. For any $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{F}}_t^*$ and $\mu \in M(S)$, if we define $\tilde{\nu}(A) = \int \mu(dx) P_x(x_\sigma \in A)$ $(A \in \tilde{\boldsymbol{B}})$, there exists $\tilde{\mathfrak{A}}'$ and $\tilde{\mathfrak{A}}''$ in $\tilde{\mathfrak{B}}_t$ such that

$$\tilde{\mathfrak{A}} \ominus \tilde{\mathfrak{A}} = (\tilde{\mathfrak{A}} \wedge \tilde{\mathfrak{A}}'^c) \vee (\tilde{\mathfrak{A}}^c \wedge \tilde{\mathfrak{A}}') \subset \tilde{\mathfrak{A}}'' \text{ and } \tilde{P}_{\tilde{\nu}}(\tilde{\mathfrak{A}}'')=0 .$$

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21) Before proving ($\tilde{P}$. 3), we can not yet know $\tilde{\mathfrak{F}}_t^* = \tilde{\mathfrak{F}}_t$ where $\tilde{\mathfrak{F}}_t^* = \bigwedge_{\tilde{\nu} \in \tilde{M}(S)} \tilde{\mathfrak{B}}_{t\tilde{\nu}}$ and $\tilde{\mathfrak{F}}_t = \{A :\ A \in \tilde{\mathfrak{F}}_\infty^*,\ A \wedge \{t < s\} \in \tilde{\mathfrak{F}}_s^* \text{ for any } s\}$.

But $\pi^{-1}(\tilde{\mathfrak{A}}') \in \mathfrak{F}_{\tau(t)}$ by (6.6) and $P_\mu(\pi^{-1}(\tilde{\mathfrak{A}}''))=0$ by (6.7), and $\pi^{-1}(\tilde{\mathfrak{A}} \ominus \pi^{-1}(\tilde{\mathfrak{A}}') \subset \pi^{-1}(\tilde{\mathfrak{A}}'')$. Therefore $\pi^{-1}(\tilde{\mathfrak{A}})$ is in the $\mu$-completion $\mathfrak{F}^\mu_{\tau(t)}$ of $\mathfrak{F}_{\tau(t)}$. Since $\mu$ is arbitrary and $\mathfrak{F}_{\tau(t)} = \bigcap_{\mu \in M(S)} \mathfrak{F}^\mu_{\tau(t)}$, we obtain the lemma.

PROPOSITION 6.9. *Let $\tilde{\rho}$ be any Markov time (with respect to $\tilde{M}$). Then $\tau(\rho(\tilde{w}), w)$ is a Markov time (with respect to $M$) and $\pi^{-1}(\tilde{\mathfrak{F}}_{\tilde{\rho}}) \subset \mathfrak{F}_{\tau(\rho(\tilde{w}), w)}$.*

PROOF.

(i) Let $T_1 = \{r_n\}$ be a countable set in $[0, \infty]$ and the values of $\tilde{\rho}$ be in $T_1$. Then $\{\tilde{\rho} = r_n\} \in \bigcap_{\varepsilon>0} \tilde{\mathfrak{F}}^*_{r_n+\varepsilon}$, so $\{\tilde{\rho}(\tilde{w}) = r_n\} \in \bigcap_{\varepsilon>0} \pi^{-1}(\tilde{\mathfrak{F}}^*_{r_n+\varepsilon}) = \bigcap_{\varepsilon>0} \mathfrak{F}_{\tau(r_n+\varepsilon)} = \mathfrak{F}_{\tau(r_n)}$ by (6.8) and the right continuity of $\tau(s)$. Therefore

$$\{\tau(\tilde{\rho}(\tilde{w}), w) < t\} = \bigcup_n \{\tilde{\rho}(\tilde{w}) = r_n\} \cap \{\tau(r_n, w) < t\} \in \mathfrak{F}_t ,$$

which shows that $\tau(\tilde{\rho}) = \tau(\tilde{\rho}(\tilde{w}), w)$ is a Markov time ($M$). Moreover, if $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{F}}_{\tilde{\rho}}$, then for any $\varepsilon > 0$,

$$\tilde{\mathfrak{A}} \cap \{\tilde{\rho} = r_n\} = \tilde{\mathfrak{A}} \cap \{\tilde{\rho} = r_n\} \cap \{\tilde{\rho} < r_n + \varepsilon\} \in \tilde{\mathfrak{F}}^*_{r_n+\varepsilon} ,$$

and by using (6.8)

$$\pi^{-1}(\tilde{\mathfrak{A}}) \cap \{\tilde{\rho}(\tilde{w}) = r_n\} \in \bigcap_{\varepsilon>0} \mathfrak{F}_{\tau(r_n+\varepsilon)} = \mathfrak{F}_{\tau(r_n)} .$$

Therefore,

$$\pi^{-1}(\tilde{\mathfrak{A}}) \cap \{\tau(\tilde{\rho}) < t\} = \bigcup_n \{\pi^{-1}(\tilde{\mathfrak{A}}) \cap \{\tilde{\rho}(\tilde{w}) = r_n\} \cap \{\tau(r_n) < t\} \in \mathfrak{F}_t$$

which shows that $\pi^{-1}(\tilde{\mathfrak{A}}) \in \mathfrak{F}_{\tau(\tilde{\rho})}$.

(ii) For a general $\tilde{\rho}$ we approximate $\tilde{\rho}$ by such $\tilde{\rho}_n$ as

$$\tilde{\rho}_n = \begin{cases} 2^{-n}k & \text{if } \ 2^{-n}(k-1) \leqq \rho < 2^{-n}k \\ \infty & \text{if } \ \tilde{\rho} = \infty . \end{cases}$$

Then, $\tilde{\rho}_n \downarrow \tilde{\rho}$ and $\rho_n$'s are Markov times ($\tilde{M}$), so $\tau(\tilde{\rho}_n)$'s are Markov times ($M$) by (i). Therefore, $\tau(\tilde{\rho}) = \lim \tau(\tilde{\rho}_n) \downarrow$ is also a Markov time ($M$). Finally, if $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{F}}_{\tilde{\rho}}$, then $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{F}}_{\tilde{\rho}_n}$ for any $n$, so $\pi^{-1}(\tilde{\mathfrak{A}}) \in \bigcap_n \mathfrak{F}_{\tau(\tilde{\rho}_n)} = \mathfrak{F}_{\tau(\tilde{\rho})}$.

THEOREM 6.10. *$\tilde{M}$ satisfies the condition:*

($\tilde{P}$. 3) *Let $\tilde{\rho}$ be any Markov time ($\tilde{M}$) and $\tilde{\mathfrak{A}} \in \tilde{\mathfrak{B}}$. Then*

$$\tilde{P}_\xi(\tilde{w}^+_{\tilde{\rho}} \in \tilde{\mathfrak{A}} \mid \tilde{\mathfrak{F}}_{\tilde{\rho}}) = \tilde{P}_{\xi_{\tilde{\rho}}}(\tilde{w} \in \tilde{\mathfrak{A}}) \quad \text{a.e. } P_\xi , \textit{ for any } \xi \textit{ in } V.^{22)}$$

22) That is, $M$ is a strong Markov process.

PROOF. Setting $\rho(w)=\tau(\tilde{\rho}(\tilde{w}, w))$, we can easily show that $(\pi(w))^{+}_{\tilde{\rho}}=\pi(w^{+}_{\rho})$. Therefore, for any $\tilde{\mathfrak{A}}_1 \in \tilde{\mathfrak{F}}_{\tilde{\rho}}$ and $\xi \in V$

$$\tilde{P}_\xi(\tilde{w}^{+}_{\tilde{\rho}} \in \tilde{\mathfrak{A}},\ \tilde{w} \in \mathfrak{A}_1)=P_\xi(w^{+}_{\rho} \in \pi^{-1}(\tilde{\mathfrak{A}}),\ w \in \pi^{-1}(\tilde{\mathfrak{A}}_1))$$
$$=E_\xi(P_{x_\rho}(\pi^{-1}(\tilde{\mathfrak{A}})) :\ \pi^{-1}(\tilde{\mathfrak{A}}_1))$$

since $\pi^{-1}(\mathfrak{A}_1) \in \mathfrak{F}_\rho$. Noting $x_\rho \in V^*$ a.e. (by theorem 5.7) and $\xi_{\tilde{\rho}(\tilde{w})}(\tilde{w})=x_\rho(w)$ a.e., we have

$$E_\xi(P_{x_\rho}(\pi^{-1}(\tilde{\mathfrak{A}})) :\ \pi^{-1}(\tilde{\mathfrak{A}}_1))=E_\xi(\tilde{P}_{x_\rho}(\tilde{\mathfrak{A}}) :\ \pi^{-1}(\tilde{\mathfrak{A}}_1))$$
$$=\tilde{E}_\xi(\tilde{P}_{\xi_{\tilde{\rho}}}(\tilde{\mathfrak{A}}) :\ \mathfrak{A}_1)\,.$$

The theorem is proved.

LEMMA 6.11. *Let $\rho$ be a Markov time ($\boldsymbol{M}$). Then for such $w$ as $x_{\rho(w)}(w) \in V$*

$$\tau(\Phi(\rho))=\rho \qquad a.e.$$

PROOF. Since $\tau(\Phi(\rho))=\rho$ if and only if $\tau(0, w^{+}_{\rho})=0$, noting $P_\xi(\tau(0)=0)=1$ if $\xi \in V$ (by (5.5)), we have

$$P_x(x_\rho \in V,\ \tau(\Phi(\rho))=\rho)$$
$$=E_x(P_{x_\rho}(\tau(0)=0),\ x_\rho \in V)$$
$$=P_x(x_\rho \in V)$$

which proves the lemma.

THEOREM 6.12. *$\tilde{M}$ satisfies the conditions:*

($P$. 4) *Let $\tilde{\rho}_n$ be an increasing sequence of Markov times ($\tilde{\boldsymbol{M}}$) and $\tilde{\rho}=\lim \tilde{\rho}_n$. Then for any $\xi \in V^*$*

$$\xi_{\tilde{\rho}}=\lim \xi_{\tilde{\rho}_n} \ \textit{for such}\ \tilde{w}\ \textit{as}\ \tilde{\rho}(\tilde{w})<\infty\ \textit{a.e.}\ (\tilde{\boldsymbol{M}})^{23)}.$$

PROOF. If we set $\rho_n=\tau(\tilde{\rho}_n(\tilde{w}), w)$ and $\rho=\tau(\tilde{\rho}(\tilde{w}), w)$, $\{\rho_n\}$ is an increasing sequence of Markov times ($\boldsymbol{M}$) and $\rho_n \leqq \rho$. Setting $\rho^*=\lim \rho_n$, we have $\rho^* \leqq \rho$ and $\Phi(\rho^*)=\lim \Phi(\rho_n)=\tilde{\rho}$.

The following events occur almost everywhere ($\boldsymbol{M}$):

a) If $s<\Phi(\infty)$, then $x_{\tau(s)} \in V$.

b) If $\rho^*<\infty$, then $x_{\rho^*}=\lim x_{\rho_n}$ (by ($P$. 4) for $\boldsymbol{M}$).

c) If $x_{\rho^*} \in V$, then $\rho^*=\rho$.

23) That is, $\boldsymbol{M}$ is a quasi-left continuous process.

For, $\rho^*=\tau(\Phi(\rho^*))$ a.e. by (6.11) if $x_{\rho^*}\in V$, and $\tau(\Phi(\rho^*))=\tau(\tilde{\rho})=\rho$.

d) If $\Phi(\infty)<\infty$, there exists such $\bar{\zeta}(w)$ as $\bar{\zeta}<\infty$ and $\Phi(\bar{\zeta})=\Phi(\infty)$ (by theorem 5.10).

Henceforth, we only consider the paths which satisfy the conditions (a), (b), (c) and (d).

(Case 1) If $\tilde{\rho}(\tilde{w})<\Phi(\infty, w)$, then $\tilde{\rho}_n\leqq\tilde{\rho}<\Phi(\infty)$ and $x_{\rho_n}$, $x_\rho\in V$ (by (a)). Moreover, since $\rho_n\leqq\rho^*\leqq\rho<\infty$, $x_{\rho^*}=\lim x_{\rho_n}$ is in $V$ (by (b)) and so $\rho^*=\rho$ (by (c)). Thus we have proved $x_\rho=\lim x_{\rho_n}$.

(Case 2) If $\Phi(\infty)\leqq\tilde{\rho}<\infty$ and $\tilde{\rho}_n<\Phi(\infty)$ for all $n$, then $\rho=\infty$, $\rho_n<\bar{\zeta}<\infty$ (by (d)) and $x_{\rho_n}\in V$ (by (a)). Thus $\rho^*=\lim\rho_n\leqq\bar{\zeta}<\infty$ and $x_{\rho^*}=\lim x_{\rho_n}\in V$ (by (b)). Therefore $\rho=\rho^*<\infty$ (by (c)), which contradicts the assertion $\rho=\infty$. Namely, the case (ii) can not occur.

(Case 3) If $\Phi(\infty)\leqq\tilde{\rho}<\infty$ and $\tilde{\rho}_{n_0}\geqq\Phi(\infty)$ for some $n_0$, then $\rho_n=\rho=\infty$ for all $n$ such as $n\geqq n_0$. Therefore $x_\rho=\partial=\lim x_{\rho_n}$. The above consideration shows that

$$P_x(x_\rho=\lim x_{\rho_n},\ \tilde{\rho}(\tilde{w})<\infty)=P_x(\tilde{\rho}(\tilde{w})<\infty)$$

for any $x$ in $S^*$. Since $x_{\rho_n(w)}(w)=\xi_{\tilde{\rho}_n(\tilde{w})}(\tilde{w})$ a.e. and $x_{\rho(w)}(w)=\xi_{\tilde{\rho}(\tilde{w})}(\tilde{w})$ a.e., we have

$$\tilde{P}_\xi(\xi_{\tilde{\rho}}=\lim\xi_{\tilde{\rho}_n},\ \tilde{\rho}<\infty)=\tilde{P}_\xi(\tilde{\rho}<\infty)$$

for any $\xi$ in $V^*$. The theorem is proved.

Finally, we shall prove $\tilde{\boldsymbol{M}}$ has a reference measure.

LEMMA 6.13. *Let $K$ be a closed set of $V$. Then*

$$P_\xi(\sigma_K<\infty)=\tilde{P}_\xi(\tilde{\sigma}_K<\infty)$$

*for any $\xi$ in $V$.*

PROOF. Throughout the proof $\xi$ in $V$ is fixed. Then, neglecting the event of $P_\xi$-measure 0, the following statements holds.

(i) If $\tilde{\sigma}_K(\tilde{w})<\infty$, then there exists a $s(w)$ such that $s>\tilde{\sigma}_K(\tilde{w})$ and $x_{\tau(s)}=\xi_s(\tilde{w})\in K$. Therefore $\sigma_K<\tau(s)<\infty$, since $s>0$ means $\tau(s)>0$.

(ii) If $\sigma_K<\infty$ and $P_\xi(\sigma_K=0)=0$, then $\sigma_K>0$, $x_{\sigma_K}\in K$, $\tau(\Phi(\sigma_K))=\sigma_K$ by (6.11), and $\tau(0)=0$ by (5.5). Therefore, $\Phi(\sigma_K)>0$ and $x_{\tau(\Phi(\sigma_K))}\in K$ which proves that $\tilde{\sigma}_K(\tilde{w})<\infty$ on $\sigma_K<\infty$.

(iii) If $P_\xi(\sigma_K=0)=1$, let $\sigma_\varepsilon=\varepsilon+\sigma_K(w_\varepsilon^+)$ for any $\varepsilon>0$. Then $\varepsilon\leqq\sigma_\varepsilon<\infty$ implies $\tilde{\sigma}_K(\tilde{w})<\infty$ $P_\xi$ by the similar argument as (ii) in which we replace $\sigma_K$ by $\sigma_\varepsilon$. Since $\sigma_\varepsilon\downarrow\sigma_K=0$, we also have $\tilde{\sigma}_K(\tilde{w})<\infty$ on $\sigma_K<\infty$. From (i), (ii) and (iii), we have

$$P_\xi(\sigma_K<\infty)=P_\xi(\tilde{\sigma}_K(\tilde{w})<\infty)=\tilde{P}_\xi(\tilde{\sigma}_K<\infty)\,.$$

Let $\nu_0$ be in $\boldsymbol{M}(S)$ and $\nu_0(S-V)=0$, and let $\tilde{\nu}_0$ be a restriction of $\nu_0$ on $V^*$, (therefore $\nu_0$ is in $\tilde{M}(V)$). Then, integrating the both sides of the equality in (6.13), we have

$$(*) \qquad P_{\nu_0}(\sigma_K<\infty)=\tilde{P}_{\tilde{\nu}_0}(\tilde{\sigma}_K<\infty)\,.$$

LEMMA 6.14. *For $\nu_0$ and $\tilde{\nu}_0$ defined above,*

$$P_{\nu_0}(\sigma_B<\infty)=\tilde{P}_{\tilde{\nu}_0}(\tilde{\sigma}_B<\infty)$$

*for any $B$ in $\boldsymbol{B}$[24] ($B\subset V$).*

PROOF. Let $\{K_n\}$ and $\{L_n\}$ be increasing sequences of compact sets which are contained in $B$ and $\sigma_{K_n}\downarrow\sigma_B$ a.e. $P_{\nu_0}$ and $\tilde{\sigma}_{K_n}\downarrow\sigma_B$ a.e. $\tilde{P}_{\tilde{\nu}_0}$ ([3]). Then, setting $M_n=L_n\cup K_n$, $M_n$'s are compact sets $\sigma_{M_n}\downarrow\sigma_B$ a.e. $P_{\nu_0}$ and $\tilde{\sigma}_{M_n}\downarrow\sigma_B$ a.e. $\tilde{P}_{\tilde{\nu}_0}$. Since $\lim P_{\nu_0}(\sigma_{M_n}<\infty)=P_{\nu_0}(\sigma_B<\infty)$ and $\lim\tilde{P}_{\tilde{\nu}_0}(\sigma_{M_n}<\infty)=\tilde{P}_{\tilde{\nu}_0}(\rho_B<\infty)$, using (*), we have the lemma.

THEOREM 6.15. *$\tilde{\boldsymbol{M}}$ satisfies the condition:*

($\tilde{P}$. 5) *There exists $\tilde{\nu}_0$ in $\tilde{M}(V)$ which has the following properties: For any $B\in\tilde{\boldsymbol{B}}$, $\tilde{P}_{\nu_0}(\tilde{\sigma}_B<\infty)=0$ implies that $\tilde{P}_\xi(\tilde{\sigma}_B<\infty)=0$ for all $\xi\in V$.*[25]

PROOF.

(i) We can choose a measure $\nu_0$ in $M(S)$ such that $\nu_0(A)=0$ implies $E_x\left(\int_0^\infty e^{-\alpha t}\chi_A(x_t)d\Phi\right)=0$ for any $x$ in $S$, where $A\in\boldsymbol{B}$ (c.f. theorem 3.2 in [7]). Since $\chi_V\cdot\Phi\sim\Phi$ by theorem 3.10, we can assume $\nu_0(S-V)=0$ without loss of generality. Let $\tilde{\nu}_0$ be a restriction of $\nu_0$ on $V$.

(ii) Let $B$ be a subset of $V$ which is in $\tilde{\boldsymbol{B}}$, and $\tilde{P}_{\tilde{\nu}_0}(\tilde{\sigma}_B<\infty)=0$. Then, $P_{\nu_0}(\sigma_B<\infty)=0$ by (6.14). If we set

$$f(x)=E_x(e^{-\alpha\sigma_B})=H^\alpha_{\sigma_B}1(x)\,,$$

$f(x)$ is a bounded $\alpha$-excessive function ($\boldsymbol{M}$), and $\int_S f(x)\nu_0(dx)=E_{\nu_0}(e^{-\alpha\sigma_B})=0$ or $f(x)=0$ a.e. $\nu_0$. Therefore,

$$H^\alpha_{\tau(0)}f(x)=\lim_{\varepsilon\to0}E_x\left(\int_0^\infty e^{-\alpha t}g_\varepsilon(x_t)d\Phi\right)\,,$$

where $g_\varepsilon(x)=\dfrac{1}{\varepsilon}(f(x)-H^\alpha_{\tau(\varepsilon)}f(x))$ (lemma 3.7 in [7]). Since $0\leqq g_\varepsilon(x)\leqq\dfrac{1}{\varepsilon}f(x)=0$ a.e. $\nu_0$, we have $H^\alpha_{\tau(0)}f(x)=0$ for any $x$ in $S$. Noting $P_\xi(\tau(0)=0)=1$, we see $f(\xi)=E_\xi(e^{-\alpha\sigma_B})=0$ or $P_\xi(\sigma_B<\infty)=0$ for all $\xi$ in $V$.

24) We can regard $\tilde{B}$ as a subset of $\boldsymbol{B}$.

25) That is, $M$ has a reference measure.

By (6.14), it is followed by $\tilde{P}_\xi(\tilde{\sigma}_B<\infty)=0$ for all $\xi$ in $V$, which proves the theorem.

Thus, under the assumption $R^*$, we have proved that $\tilde{\boldsymbol{M}}$ is a Markov process satisfying $(\tilde{P}.\ 1)\sim(\tilde{P}.\ 5)$. We shall add the following theorem:

THEOREM 6.16. *If* $\boldsymbol{M}$ *is conservative*[26] *and* $P_x(\sigma<\infty)=1$ *for any* $x$ *in* $S$, *then* $\tilde{\boldsymbol{M}}$ *is conservative.*

PROOF. For any $N$ $(N=1, 2, \cdots)$ and any $x\in S$

$$P_x(\sigma(w_N^+)<\infty)=E_x(P_{x_N}(\sigma<\infty):\ N<\zeta)=1\,,$$

or

$$P_x(\sigma(w_N^+)<\infty,\ N=1, 2, \cdots)=1\,.$$

Therefore, $P_x(W_1)=1$, where $W_1$ is the set given in theorem 5.2. Using theorem 5.2, we see, for any $x$ in $S$, $P_x(\Phi(\infty)=\infty)=1$, that is, $P_x(\tau(s)<\infty$ for all $s<\infty)=1$. Thus $P_\xi(x_{\tau(s)}\in V$ for all finite $s)=1$ for any $\xi$ in $V$, which proves the theorem.

In the remainder of this section, we shall discuss the relation between the processes $\tilde{\boldsymbol{M}}^{(\alpha)}$ and $\tilde{\boldsymbol{M}}^{(\beta)}$. Without losing the generality, we assume $0<\alpha<\beta$. By (3.4)

$$\Phi_\alpha\sim\Phi_\beta+(\beta-\alpha)\widetilde{(g^0\cdot T)}_{(\beta)}\,,$$

where $g_\alpha^0(x)=E_x\left(\int_0^\sigma e^{-\alpha t}dT\right)\leqq\frac{1}{\alpha}$. Therefore $\widetilde{(g_\alpha^0\cdot T)}_{(\beta)}\ll\frac{1}{\alpha}\tilde{T}_{(\beta)}\equiv\frac{1}{\alpha}\Phi_\beta$, and we have

PROPOSITION 6.17. $\Phi_\beta\ll\Phi_\alpha\ll\frac{\beta}{\alpha}\Phi_\beta$.

Therefore by [7], there exists a function $k(x)$ in $F(S)$ such as $1\leqq k(x)\leqq\frac{\beta}{\alpha}$ and

$$\Phi_\alpha\sim k\cdot\Phi_\beta \quad\text{and}\quad \Phi_\beta\sim\frac{1}{k}\cdot\Phi_\alpha\,.$$

Let $\tilde{G}_\lambda^{(\alpha)}$ and $\tilde{G}_\lambda^{(\beta)}$ be $\lambda$-order Green operators of $\tilde{\boldsymbol{M}}^{(\alpha)}$ and $\tilde{\boldsymbol{M}}^{(\beta)}$ respectively. Then, for any $f$ in $\tilde{F}(V)$,

$$\tilde{G}_\lambda^{(\alpha)}f(\xi)=\tilde{E}_\xi^{(\alpha)}\left(\int_0^\infty e^{-\lambda s}f(\xi_s)ds\right)$$

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26) We say, $\boldsymbol{M}$ is conservative if and only if $P_x(\zeta=\infty)=1$ for any $x$ in $S$. Similarly, $\tilde{\boldsymbol{M}}$ is conservative if and only if $\tilde{P}_\xi(\tilde{\zeta}=\infty)=1$ for any $\xi$ in $V$.

$$=E_\xi\left(\int_0^\infty e^{-\lambda s}f(x_{\tau_\alpha(s)})ds\right)$$

$$=E_\xi\left(\int_0^\infty e^{-\lambda\Phi_\alpha(t)}f(x_t)d\Phi_\alpha\right)^{27)}$$

$$=E_\xi\left(\int_0^\infty e^{-\lambda k\cdot\Phi_\beta(t)}f(x_t)k(x_t)d\Phi_\beta\right)$$

$$=E_\xi\left(\int_0^\infty e^{-\int_0^s k(x_{\tau_\beta(u)})du}f(x_{\tau_\beta(s)})k(x_{\tau_\beta(s)})ds\right)$$

$$=\tilde{E}_\xi^{(\beta)}\left(\int_0^\infty e^{-\int_0^s k(\xi_u)du}f(\xi_s)k(\xi_s)ds\right),$$

and similarly,

$$G_\lambda^{(\beta)}f(\xi)=\tilde{E}_\xi^{(\alpha)}\left(\int_0^\infty e^{-\int_0^s\frac{du}{k(\xi_u)}}f(\xi_s)\frac{ds}{k(\xi_s)}\right),$$

for any $\xi$ in $V$. These equalities show that $\tilde{M}^{(\alpha)}$ *and* $\tilde{M}^{(\beta)}$ *can be transformed into each other by the classical time changes.*

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27) $f$ is considered to be extended to the function in $F(S)$. $f\cdot\Phi_\alpha$ is independent of the extension, since $\Phi_\alpha\sim\chi_V\cdot\Phi_\alpha$.