MULTIDIMENSIONAL QUANTIFICATION OF THE DATA OBTAINED BY THE METHOD OF PAIRED COMPARISON

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1. Introduction

Paired comparison is an excellent method for judgement of N objects, especially when N is fairly large. However, it is not easy to quantify the obtained data suitably for a purpose set up, because the quantification is not always deduced directly from the underlying conditions of the experiments of paired comparison. Thus, various models of quantification are devised which correspond to the hypotheses to be imposed on the method of paired comparison. There are two types of models. One is the model given mainly by statistical psychologists or statistical sociologists, and the other is the model given by mathematical statisticians. Among models of the former type, Thurston's model (we call T-model), including Mosteller's one, and Guttman's model (we call G-model) are, as is well known, useful for data analysis. Many models by mathematical statisticians are mostly concerned only with statistical testing.* Here, another model of the former type will be shown.

Probabilistic responses are considered in one-dimensional continuum and the probability density function of responses is assumed to be Gaussian in the T-model. The model is too much restricted in assumption and has no validity in some practical problems. In the G-model the responses obtained by the method of paired comparison are quantified in one dimension to represent the discriminative judgements without any assumption on the underlying continuum. This model is very useful for the determination of rank order of objects, based on the data by paired comparison which usually show complicated figures. The aim of these methods is to give a one-dimensional numerical representation of the data obtained by paired comparison. The numerical values of the data are meaningful only in rank order scale or in interval scale. In paired comparison, we meet the perplexed situations in treatment as

* See, for example, the references in the book, H. A. David, *The Method of Paired Comparison*, No. 12 of Griffin's Statistical Monographs and Courses, 1963.

shown below. Let A, B, C be objects, and $\succ$ the preference-sign. $A \succ B$ means " A is preferable to B." Suppose $A \succ B$, $B \succ C$ but $A \prec C$. This occurs both in judgements of objects even by only one person and in the counting of preference-signs of objects in the universe of persons—that is, $A \succ B$ occurs more than $B \succ A$, $B \succ C$ occurs more than $C \succ B$ but $C \succ A$ occurs more than $A \succ C$. This inconsistent relation may be interpreted in probabilistic responses in the T-model, that is, as a sample from the simultaneous probability distribution of responses.

Both in the T-model and the G-model, the quantification of specification is performed by synthesizing the various relations in the data obtained. In the T-model, the quantification of mean and variance of the Gaussian distribution of each object and some correlation coefficients between objects is considered, and in the G-model, the quantification of objects themselves or their factors (categories in items) is considered. These are the characteristics of uni-dimensional quantification and the validity is to be evaluated by the efficiency of applicability to practical problems. We consider here the problem to orientate, in a space of some dimensions, the objects which show apparently inconsistent relations, and we intend to give a model of multidimensional quantification.

2. Model

When we are given relations $A \succ B$, $B \succ C$ and $A \prec C$ ($\succ$, $\prec$ are symbolically to bo interpreted, for example, as preferable, favourable, more beautiful, better, greater, etc.; $A \succ B$ or $B \prec A$ means that A is preferable to B etc.) and when A, B, C are represented as in Fig. 1 in a plane, the above

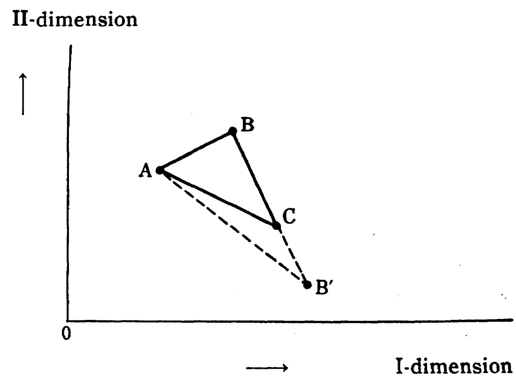


Fig. 1. One example of constellation

preference relations are reasonably interpreted in the II-dimensional space. That is the comparison between A and B is done in I-dimension in Fig. 1. We consider that the object with smaller magnitude in I-dimension is judged to be preferable to the object with larger magnitude in that dimension. The difference between the magnitudes in the metrical sense is not meaningful but the difference between the rank orders is meaningful; for example, the only essential thing concerning A and B is that A has higher rank order than B with respect to I-dimension, supposing that the object of higher rank order is judged to be preferable ($\succ$) to that of lower rank order. Then the expression is taken as A being in the left-hand side of B in I-dimension in the two-dimensional space. The comparison between B and C is done in I-dimension or in II-dimension. In Fig. 1, the comparison between B and C is done in I-dimension, while the comparison is done in II-dimension if B is located at the point B'. The comparison of A and C is done in II-dimension. Thus, the relations $A \succ B$, $B \succ C$ and $A \prec C$ are reasonably interpreted under the scheme mentioned above. In the following we shall give a model of multidimensional quantification of paired comparison based on this idea.

Let N be the number of objects and n that of judges. Then each judge compares $\binom{N}{2}$ pairs among N objects. For simplicity, suppose that only judgement $\succ$ or $\prec$ is made (equality is omitted). We shall reveal the constellation of the objects in rank order sense in an S-dimensional space, each dimension of which has rank order scale. The relation between judgement and rank order is as is shown above. Let O_i be the ith object ($i=1, 2 \cdots, N$). O_i corresponds to the point P_i in the S-dimensional space, which has rank orders $x_{i1}, x_{i2}, \cdots, x_{iS}$, where x_{is} is the rank order of O_i in the sth dimension, $s=1, 2, \cdots, S$, i.e. $x_{is}=1$ or 2 or $\cdots$ or N, the smaller number being of higher rank order. We assume that $x_{is} \neq x_{js}$ for $i \neq j$, and that $x_{is} < x_{js}$ means that O_i is preferable to O_j in the sth dimension. Here, we treat the problem to determine the number of dimension S, the constellation of objects in S-dimensional space (x_{is}, $i=1, 2 \cdots N$; $s=1, 2 \cdots S$) and the size n_s ($s=1, 2 \cdots S$) which are defined as below. We show two models to formulate the problem to be solved.

[Model A] Let n_s be the size of the sth class, ($n_s \geqq 1$). The judges who belong to the sth class compare paired objects only in the sth dimension, that is, paired objects O_i and O_j ($i, j=1, \cdots, N$) only by the rank order x_{is} and x_{js} in the sth dimension. We take $s=1, 2, \cdots S$, assuming that each judge belongs to only one class, $s=1$ or 2 or $\cdots$ or S. Then, clearly $\sum_{s=1}^{S} n_s = n$. If $x_{is} < x_{js}$, the judges who belong to the sth class judge that $O_i \succ O_j$, i.e., O_i being preferable to O_j. If $x_{is'} > x_{js'} (s' \neq s)$, the judges who belong to the s'th class judge that $O_i \prec O_j$, i.e., O_j being preferable to O_i. Let m_{ij} be the number of judges who judge that $O_i \succ O_j$, which is obtained in experiments and m_{ji} the number of judges who judge that $O_i \prec O_j$. Of course, $m_{ij} + m_{ji} = n$.

DEFINITION. We define $\delta_{ij}(s)$, for $i \neq j$, as

$$\delta_{ij}(s)=\begin{cases} 1, & \text{if } x_{is}<x_{js}, \\ 0, & \text{if } x_{is}>x_{js}, \end{cases}$$

and for $i=j$

$$\delta_{ii}(s)\equiv 0, \qquad i=1, \cdots, N.$$

Then we have the following:

$$(1) \qquad \begin{aligned} &\delta_{ij}(s)+\delta_{ji}(s)=1 \\ &m_{ij}=\sum_{s=1}^{S}\delta_{ij}(s)n_s, \end{aligned} \qquad i, j=1, 2, \cdots, N, \quad i\neq j,$$

where m_{ij}'s are given ($i, j=1, \cdots, N$; $i\neq j$) from experiments, and $\delta_{ij}(s)$, n_s, S ($s=1, \cdots, S$; $i, j=1, \cdots, N$; $i\neq j$) are unknown. We call (1) the fundamental equation. We require $\delta_{ij}(s)$, i.e., x_{is} ($i, j=1, \cdots, N$; $i\neq j$; $s=1, \cdots, S$) and n_s ($s=1, \cdots, S$) with the minimum dimension S. There are $N(N-1)/2$ $\delta_{ij}(s)$'s for fixed s. Since it is sufficient as is seen from the property of rank order, that only $(N-1)$ of x_{is}'s ($i=1, \cdots, N$) are determined, and since $\delta_{ij}(s)$'s are uniquely determined by x_{is}'s ($i=1, \cdots, N$), the number of unknown parameters is generally $S(N-1)+(S-1)=SN-1$, where $S(N-1)$ is the number of x_{is}'s to be required and $(S-1)$ is the number of classes, the size of which is to be determined so that $\sum_{s=1}^{S} n_s=1$. The smaller the number of S, the more favorable the situation becomes for analysis. In case $S=1$, all objects are in uni-dimension and the results by comparisons show no inconsistency.

The solution of (1) in model A is not always obtainable and will generally be determined approximately. As $(SN-1)\leqq N(N-1)/2$, i.e., $S\leqq(N-1)/2+1/N$, $[(N-1)/2+1/N]$ is given as a rough estimate of the upper bound of S, where [ ] is the Gaussian bracket.

This model is generalized as follows.

[Model B] Each judge who belongs to the tth group compares the paired objects in the sth dimension with probability w_{ts} ($s=1, 2, \cdots, S$), $\sum_{s=1}^{S} w_{ts}=1$, $t=1, 2, \cdots, T$, T being the number of groups. If the size of the tth group is large and each judge compares the paired objects independently in the sense of probability, w_{ts} is approximately equal to the proportion of judges in the tth group who compare the paired objects in the sth dimension. If $w_{ts}=\delta_{ts}$, where δ_{ts} is Kronecker symbol, and $S=T$, this model B is reduced to the model A mentioned above.

Let l_t be the size of the tth group. In this case, we have, as a fundamental equation, in expectation,

$$m_{ij}=\sum_{t=1}^{T}\sum_{s=1}^{S}\delta_{ij}(s)w_{ts}l_t\,, \qquad i, j=1, \cdots, N;\quad i\neq j\,.$$

If $T=1$, $t=1$, $l_t=n$, $l_tw_{ts}=lw_s=n_s$, we have

$$m_{ij}=\sum_{s=1}^{S}\delta_{ij}(s)n_s$$

which is formally equal to (1). However, $n_s=nw_{ts}$ has a different meaning from the n_s in Model A. In the latter case, the same judge compares objects in the sth dimension $s=1, \cdots, S$ with probability $n_s/n=w_s$, and in the former case, each judge belongs to only one class which is characterized by paired comparison in only one dimension corresponding to that class. Taking this probabilistic model and assuming that all the comparisons are done independently in the sense of probability, the simultaneous distribution of m_{ij} $(i, j=1, \cdots, N)$ is required when $\delta_{ij}(s)$'s are assumed to be known. The distribution of m_{ij}, $Q(m_{ij})$, is $\sum_{R_{ij}} P(y_1, y_2, \cdots, y_S)$, where $P(y_1, \cdots, y_S)=n\,!\prod_s \frac{w_s^{y_s}}{y_s\,!}$, y_s's being realized values of random variables as the number of judges who compares in the sth dimension with $\sum_{s=1}^{S} y_s=n$ and R_{ij} is the domain of $(y_1, y_2, \cdots, y_S)$ which satisfies the equation $m_{ij}=\sum_{s=1}^{S}\delta_{ij}(s)y_s$ when $\delta_{ij}(s)$'s $(s=1, \cdots, S)$ are known. The simultaneous distribution of $Q(m_{ij})$ $(i, j=1, \cdots, N;\ i\neq j)$ is $\prod_{i,j} Q(m_{ij})$. The patterns of m_{ij} are made by the Monte-Carlo method by giving $\delta_{ij}(s)$'s and w_s's $(s=1, \cdots, S)$ assuming that S is known.

For $T\neq 1$, it is useful to simplify the model B. One example of simplified model B:

$$T=S$$

$$\begin{array}{llll}
\text{for } t=1\,, & w_{11}=p\,, & w_{1s}=q\,, & s=2, 3, \cdots, S\,,\\
\text{for } t=2\,, & w_{22}=p\,, & w_{2s}=q\,, & s=1, 3, \cdots, S\,,\\
\vdots & \vdots & \vdots & \vdots\\
\text{for } t=T=S\,, & w_{SS}=p\,, & w_{Ss}=q\,, & s=1, \cdots, S-1\,.
\end{array}$$

As $\sum_{s=1}^{S} w_{ts}=1$, $q=(1-p)/(S-1)$. In this model, q may be regarded as probability of response error, as it were. We shall be able to estimate p by the same ideas as in the methodology for social research. For example, $p=0.95$ or 0.90. So, if we specify such probabilistic response patterns as mentioned above, the fundamental equation has a comparatively clear form to solve.

3. Solution to the model, (1)

Based on the consideration of section 2, we can devise various methods to give the solution to the model using the data, m_{ij}'s $(i, j=1, 2, \cdots, N)$, that is, to require the minimum dimension S and x_{is}'s $(i=1, 2, \cdots, N;\ s=1, 2, \cdots, S)$. We shall show one method here. Let (M) be the matrix $N\times N$, elements of which are m_{ij} $(i, j=1, \cdots, N;\ i\neq j)$ with $m_{ii}=0$ $(i=1, \cdots, N)$ and let $(D(s))$ be the matrix $N\times N$, elements of which are $\delta_{ij}(s)$ $(i, j=1, \cdots, N;\ i\neq j)$ with $\delta_{ii}(s)=0$ $(i=1, \cdots, N)$.

$$(M)=\begin{pmatrix} 0_1 & m_{12} & m_{13} & \cdot\ \cdot & m_{1N} \\ m_{21} & 0 & m_{23} & \cdot\ \cdot & m_{2N} \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ m_{N1} & m_{N2} & m_{N3} & \cdot\ \cdot & 0 \end{pmatrix}.$$

Let ${}^{U}\!A$ be the set of elements ${}^{U}\!A_{ij}$ $(i, j=1, \cdots, N)$ in the upper half of matrix (A) and ${}^{L}\!A$ be the set of elements ${}^{L}\!A_{ij}(i, j=1, \cdots, N)$ in the lower half of (A), i.e.,

$$(A)=\begin{pmatrix} 0 & & & {}^{U}\!A \\ & 0 & & \\ & & \ddots & \\ {}^{L}\!A & & & 0 \end{pmatrix}.$$

Matrix $(D(s))$ can be transformed into the rank order matrix $({}^{0}D(s))$, in which ${}^{U}D(s)_{ij}$'s are 1 $(i, j=1, \cdots, N)$, and ${}^{L}D(s)_{ij}$'s are 0 $(i, j=1, \cdots, N)$, changing the order of objects O_i $(i=1, \cdots, N)$, i.e.,

$$({}^{0}D(s))=\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ & & \ddots & & 1 \\ & 0 & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

This is easily proved by uni-dimensionality of x_{is}'s $(i=1, \cdots, N)$ with the definition of $\delta_{ij}(s)$.

DEFINITION. Matrix (A) is called positive, which we express as $(A)\geqq 0$, when all the elements $a_{ij}\geqq 0$ $(i, j=1, \cdots, N)$ and at least one strict inequality holds. Matrix $(A)=0$ if and only if $a_{ij}=0$ for all $i, j=1, \cdots, N$.

Thus, if matrix (A) is positive and matrix (B) is positive, $(A)+(B)=(C)$ is positive and $\alpha(A)$ is positive if real number $\alpha>0$.

Suppose that (M) is given by experiments. Proceed by the following steps to give the solution to model A.

1) Change the order of objects so as to maximize the sum of ${}^{U}M_{ij}$'s $(i, j=1, \cdots, N)$, that is, determine x_{i1} $(i=1, \cdots, N)$ so as to maximize the value $\Phi=-\sum_{\substack{i, j \\ i<j}} m_{ij} \operatorname{sgn}(x_{i1}-x_{j1})$, where x_{i1} is the rank order of the ith object, under the condition that any element of ${}^{U}M_{ij}$ $(i, j=1, \cdots, N)$ is not zero. As $\Phi=-\sum_{i, j} m_{ij} \frac{(x_{i1}-x_{j1})}{|x_{i1}-x_{j1}|}$, we can make a comparatively good first approximation to x_{i1} $(i=1, \cdots, N)$ by applying the method of quantification of e_{ij}-type* and then, obtain successively the required values of them by changing the order of objects step by step, i.e., by moving the position of one object and changing the order of objects so as to give the larger sum of ${}^{U}M_{ij}$'s in the new order than that in the previous order and repeating this step. We can always attain the maximum value of the sum of ${}^{U}M_{ij}$'s by finite steps.

Let $({}^{1}M)$ be the matrix obtained by this process and let ${}^{1}R$ be the rank order of objects O_i $(i=1, \cdots, N)$.

2) Let $({}^{0}D(1))$ be the rank order matrix by the rank order ${}^{1}R$. Find the positive integer n_1 which is the maximum integer satisfying the condition that $({}^{1}M)-n_1({}^{0}D(1))$ is positive. ${}^{1}R$ is the required order of the objects $\{x_{o_i 1}(i=1, \cdots, N)\}$ in I-dimension and $\delta_{ij}(1)$'s are given in $({}^{0}D(1))$. n_1 is the size of the judges who belong to the 1st-class.

3) Let $(M')=({}^{1}M)-n_1({}^{0}D(1))$. Change the order of objects to maximize the sum of ${}^{U}M'_{ij}$'s $(i, j=1, \cdots, N)$ by the idea mentioned above. Let $({}^{2}M)$ be the matrix obtained by this process of changing the order of objects and ${}^{2}R$ be the rank order of objects $O_i (i=1, \cdots, N)$.

4) Let $({}^{0}D(2))$ be the rank order matrix by the rank order ${}^{2}R$. Find the positive integer n_2 which is the maximum integer satisfying the condition that $({}^{2}M)-n_2({}^{0}D(2))$ is positive. ${}^{2}R$ is the required order of objects $\{x_{o_i 2}(i=1, \cdots, N)\}$ in II-dimension and $\delta_{ij}(2)$'s are given in $({}^{0}D(2))$. n_2 is the size of the judges who belong to the 2nd-class.

5) Let $(M'')=({}^{2}M)-n_2({}^{0}D(2))$. Repeat the same process as mentioned above. We stop the process when $(M^{\overbrace{\prime\prime\cdots\prime}^{p}})=(M^{(p)})=0$, and determine the dimensionality as $S=p$.

* C. Hayashi, "On the prediction of phenomena from qualitative data and the quantification of qualitative data from the mathematico-statistical point of view", *Ann. Inst. Stat. Math.*, 3 (1952), 69-98, and "Theory of quantification and its examples (VI)," *Proc. Inst. Stat. Math.*, 9 (1961), 29-35.

We can not always obtain the solution successfully by this process according to what property the matrix (M) obtained by experiments has. For example, there are some cases where we can not find $n_q(\neq 0)$, because some elements of ${}^U M_{ij}^{(q)}$ are equal to zero before $(M^{(p)})=0$ $(p<q)$, however we may change the order of objects. We can not find the solution as far as we do not recognize the equality of the rank order of objects, i.e., $x_{is}=x_{js}$ for some i, j. In such cases, our model is not valid for data analysis and another model must be adopted. However, in some situations, we can reasonably determine the unknown by using the idea of the least square method from the probabilistic point of view.

It should be noted that the solution of our model A above may be not always unique and, at least, the solutions obtained by the process mentioned above, and by changing the name of dimension of the solution are also solutions of our model A. As $n_r \geqq n_s$ $(r<s)$ generally holds, the order of dimension is to be determined according to magnitude of the size of the class. When the solution is not unique, what is the desired solution must be decided by the aim of analysis of data, for example, in relation to outside variables.

Example 1.

Preference table by paired comparison

Object \ Object	O_1	O_2	O_3	O_4	Total $=\sum_j m_{ij}$
O_1		8	4	4	16
O_2	2		6	6	14
O_3	6	4		4	14
O_4	6	4	6		16

By this preference table, it is very difficult to determine the rank order of objects (in uni-dimension) by the usual method, i.e., by the magnitude of $\sum_j m_{ij}$ $(i=1, \cdots, 4)$.

$$(M)=\begin{pmatrix} 0 & 8 & 4 & 4 \\ 2 & 0 & 6 & 6 \\ 6 & 4 & 0 & 4 \\ 6 & 4 & 6 & 0 \end{pmatrix}, \quad N=4 \text{ and } n=10 .$$

The number of judgement $O_1 \succ O_2$ is 8, that of judgement $O_2 \succ O_3$ is 6 and that of judgement $O_1 \prec O_3$ $(O_1 \succ O_3)$ is 6 (4). Thus we meet an inconsistent relation in judgements and can not determine the rank order of objects in uni-dimension.

$$({}^1M)=(M)\,,$$

$$\begin{array}{l} {}^1R=\{x_{O_{11}}<x_{O_{21}}<x_{O_{41}}<x_{O_{31}}\} \\ \quad\ \ \|\qquad\ \|\qquad\ \|\qquad\ \| \\ \quad\ \ 1\qquad\ 2\qquad\ 3\qquad\ 4\ \cdots \text{ rank order} \end{array}$$

$$({}^oD(1))=\begin{array}{c} \\ O_1 \\ O_2 \\ O_4 \\ O_3 \end{array}\begin{array}{c} \begin{array}{cccc} O_1 & O_2 & O_4 & O_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},\quad n_1=4\,,$$

$$(M')=\begin{array}{c} \\ O_1 \\ O_2 \\ O_3 \\ O_4 \end{array}\begin{array}{c} \begin{array}{cccc} O_1 & O_2 & O_3 & O_4 \end{array} \\ \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 2 \\ 6 & 4 & 0 & 2 \\ 6 & 4 & 4 & 0 \end{pmatrix} \end{array}.$$

Change the order, $\begin{array}{l} {}^2R=\{x_{O_{32}}<x_{O_{42}}<x_{O_{12}}<x_{O_{22}}\} \\ \quad\ \ \|\qquad\ \|\qquad\ \|\qquad\ \| \\ \quad\ \ 1\qquad\ 2\qquad\ 3\qquad\ 4\ \cdots \text{ rank order} \end{array}$

$$({}^2M)=\begin{array}{c} \\ O_3 \\ O_4 \\ O_1 \\ O_2 \end{array}\begin{array}{c} \begin{array}{cccc} O_3 & O_4 & O_1 & O_2 \end{array} \\ \begin{pmatrix} 0 & 4 & 6 & 4 \\ 2 & 0 & 6 & 4 \\ 0 & 0 & 0 & 4 \\ 2 & 2 & 2 & 0 \end{pmatrix} \end{array},$$

$$({}^oD(2))=\begin{array}{c} \\ O_3 \\ O_4 \\ O_1 \\ O_2 \end{array}\begin{array}{c} \begin{array}{cccc} O_3 & O_4 & O_1 & O_2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},\quad n_2=4\,,$$

$$(M'')=\begin{array}{c} \\ O_3 \\ O_4 \\ O_1 \\ O_2 \end{array}\begin{array}{c} \begin{array}{cccc} O_3 & O_4 & O_1 & O_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} \end{array}.$$

Change the order, ${}^3R=\{x_{O_23}<x_{O_43}<x_{O_33}<x_{O_13}\}$

$$\begin{array}{cccc} \Vert & \Vert & \Vert & \Vert \\ 1 & 2 & 3 & 4 \end{array} \cdots \text{ rank order}$$

$$({}^3M)=\begin{array}{c} \\ O_2 \\ O_4 \\ O_3 \\ O_1 \end{array}\begin{array}{c} \begin{array}{cccc} O_2 & O_4 & O_3 & O_1 \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$({}^3D(3))=\begin{array}{c} \\ O_2 \\ O_4 \\ O_3 \\ O_1 \end{array}\begin{array}{c} \begin{array}{cccc} O_2 & O_4 & O_3 & O_1 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad n_3=2\,,$$

$$(M''')=0\,, \qquad p=3\,, \qquad S=3\,.$$

Thus we obtain the constellation of objects in three-dimensional space (see Fig. 2).

	I	II	III	
O_1	(1	3	4)	$\cdots P_1$
O_2	(2	4	1)	$\cdots P_2$
O_3	(4	1	3)	$\cdots P_3$
O_4	(3	2	2)	$\cdots P_4$
n	4	4	2	

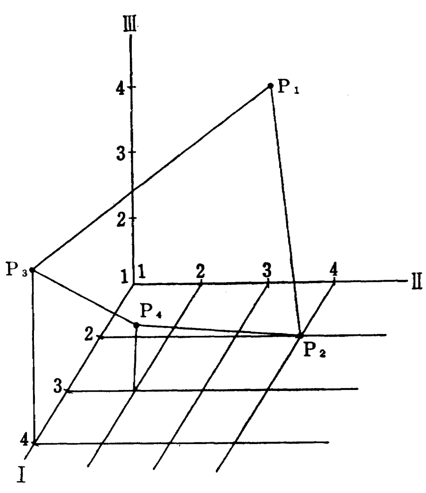


Fig. 2. Constellation of the objects

In this case, the solution is not unique. We have another solution

	I	II	III
O_1	(1	3	4)
O_2	(2	4	1)
O_3	(3	2	3)
O_4	(4	1	2)
n	4	4	3 .

Example 2. $N=7$, $n=10$.

Preference table

Object number	O_1	O_2	O_3	O_4	O_5	O_6	O^7	Total$=\sum_j m_{ij}$
O_1		8	4	4	8	8	4	36
O_2	2		6	6	8	10	6	38
O_3	6	4		4	8	10	6	38
O_4	6	4	6		4	10	6	36
O_5	2	2	2	6		10	6	28
O_6	2	0	0	0	0		6	8
O^7	6	4	4	4	4	4		26
								144

For example, the number of judgement $O_1 \succ O_2$ is 8, $O_2 \succ O_5$ is 8, $O_5 \succ O_6$ is 10, but the number of judgement $O_7 \succ O_1$ ($O_1 \succ O_7$) is 6 (4). Thus we meet an inconsistent relation in the judgements and can not determine the rank order of objects in uni-dimension. Remember the relation $S \leqq (N-1)/2-1/N$, and $S \leqq (10-1)/2-1/10=4.4$ when $N=10$.

$$(M)=\begin{pmatrix} 0 & 8 & 4 & 4 & 8 & 8 & 4 \\ 2 & 0 & 6 & 6 & 8 & 10 & 6 \\ 6 & 4 & 0 & 4 & 8 & 10 & 6 \\ 6 & 4 & 6 & 0 & 4 & 10 & 6 \\ 2 & 2 & 2 & 6 & 0 & 10 & 6 \\ 2 & 0 & 0 & 0 & 0 & 0 & 6 \\ 6 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix}.$$

$$\text{By } {}^1R=\{x_{O_1 1}<x_{O_2 1}<x_{O_3 1}<x_{O_5 1}<x_{O_4 1}<x_{O_6 1}<x_{O_7 1}\}$$

$$\begin{array}{ccccccc} \| & \| & \| & \| & \| & \| & \| \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \cdots \text{rank order} \end{array}$$

we can attain the maximum value of Φ in (M), Φ being 142 in (M) and 144 in $({}^1M)$. Thus $({}^0D(1))$ is the rank order matrix of the rank order ${}^1R$ of objects. The maximum value of n_1 is easily obtained and equal to 4.

$$(M')=({}^1M)-n_1({}^0D(1))= \begin{array}{c} \\ O_1 \\ O_2 \\ O_3 \\ O_5 \\ O_4 \\ O_6 \\ O_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} O_1 & O_2 & O_3 & O_5 & O_4 & O_6 & O_7 \end{array} \\ \begin{pmatrix} 0 & 4 & 0 & 4 & 0 & 4 & 0 \\ 2 & 0 & 2 & 4 & 2 & 6 & 2 \\ 6 & 4 & 0 & 4 & 0 & 6 & 2 \\ 2 & 2 & 2 & 0 & 2 & 6 & 2 \\ 6 & 4 & 6 & 4 & 0 & 6 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 6 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} \end{array} .$$

Change the order and we have

$${}^2R=\{x_{O_7 2}<x_{O_4 2}<x_{O_3 2}<x_{O_1 2}<x_{O_2 2}<x_{O_5 2}<x_{O_6 2}\} ,$$

$$\begin{array}{ccccccc} \| & \| & \| & \| & \| & \| & \| \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \cdots \text{rank order} \end{array}$$

in maximizing the value of Φ in (M'). Thus

$$({}^2M)= \begin{array}{c} \\ O_7 \\ O_4 \\ O_3 \\ O_1 \\ O_2 \\ O_5 \\ O_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} O_7 & O_4 & O_3 & O_1 & O_2 & O_5 & O_6 \end{array} \\ \begin{pmatrix} 0 & 4 & 4 & 6 & 4 & 4 & 4 \\ 2 & 0 & 6 & 6 & 4 & 4 & 6 \\ 2 & 0 & 0 & 6 & 4 & 4 & 6 \\ 0 & 0 & 0 & 0 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 0 & 4 & 6 \\ 2 & 2 & 2 & 2 & 2 & 0 & 6 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$${}^2R=\{x_{O_7 2}<x_{O_4 2}<x_{O_3 2}<x_{O_1 2}<x_{O_2 2}<x_{O_5 2}<x_{O_6 2}\} .$$

$$\begin{array}{ccccccc} \| & \| & \| & \| & \| & \| & \| \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \cdots \text{rank order} \end{array}$$

We have $({}^0D(2))$ by ${}^2R$.

$$(M'')=({}^{2}M)-n_2({}^{O}D(2)),\quad n_2=4.$$

Then

$$(M'')=\begin{array}{c} \\ O_7 \\ O_4 \\ O_3 \\ O_1 \\ O_2 \\ O_5 \\ O_6 \end{array}\begin{array}{c} \begin{array}{ccccccc} O_7 & O_4 & O_3 & O_1 & O_2 & O_5 & O_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

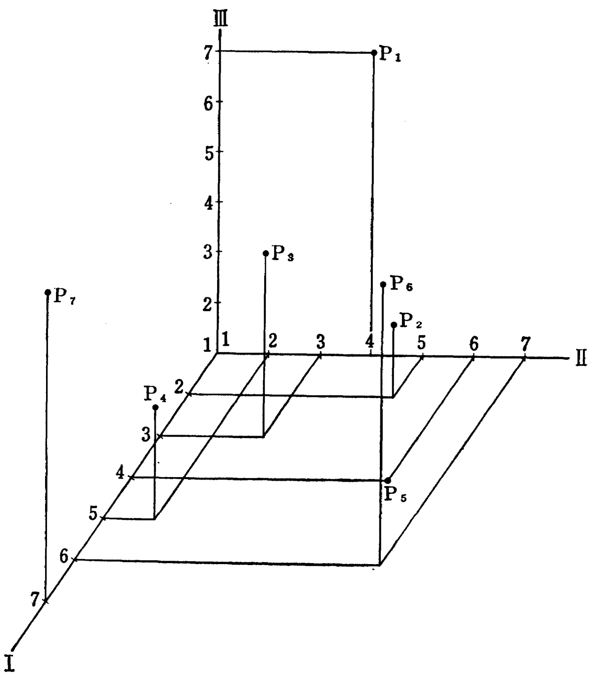
Change the order and we have

$$\begin{array}{l} {}^{3}R=\{x_{O_53}<x_{O_23}<x_{O_43}<x_{O_33}<x_{O_63}<x_{O_73}<x_{O_13}\}, \\ \qquad\ \ \Vert \qquad \Vert \qquad \Vert \qquad \Vert \qquad \Vert \qquad \Vert \qquad \Vert \\ \qquad\ \ 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6 \qquad 7 \cdots \text{rank order} \end{array}$$

$({}^{3}M)$ and $({}^{O}D(3))$ by ${}^{3}R$,

$$\begin{array}{c} \\ O_5 \\ O_2 \\ O_4 \\ O_3 \\ O_6 \\ O_7 \\ O_1 \end{array}\begin{array}{c} \begin{array}{ccccccc} O_5 & O_2 & O_4 & O_3 & O_6 & O_7 & O_1 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Then $M'''=M^{(3)}=({}^{3}M)-n_3({}^{O}D(3))=0$ for $n_3=2$. Thus the process finishes. We have $S=3$. In this case, the solution is unique except for the change of name of axes. Fig. 3 shows the constellation of objects in three-dimensional space, scale of which is given by rank order.



Rank order	4	4	2
	I	II	III
1	O_1	O_7	O_5
2	O_2	O_4	O_2
3	O_3	O_3	O_4
4	O_5	O_1	O_3
5	O_4	O_2	O_6
6	O_6	O_5	O_7
7	O_7	O_6	O_1

Fig. 3. Constellation of the objects

4. Solution to the model, (2)

There are some cases where we can not obtain any solution of our deterministic model A by using the data given by the method in section 3. It may be rather necessary to develop another new model, however, it is also useful to make the solution of our model in probabilistic treatment. In this case, we adopt the stochastic model (model B) for $T=1$ instead of model A as mentioned at the end of section 2, which includes the simplified model for $T\neq 1$, and in some practical problems, we require the solution by the idea of the least square method. When $N(N-1)/2>SN-1$, the least square method is of course available. The following procedure will be desirable:

1. First step:

Estimate S^1 and $\delta_{ij}(s)$'s $(i, j=1, \cdots, N;\ s=1, \cdots, S)$ as a first approximation by the method mentioned in section 3.

2. Second step:

Determine integer n_s $(n_s\geqq 1;\ s=1, \cdots, S)$ by the idea of the least square method, that is, determine integer n_s $(s=1, \cdots, S)$ so as to minimize $\sum\limits_{\substack{i,j\\i<j}}(m_{ij}-\sum\limits_{s}\delta_{ij}(s)n_s)^2=Q^2$ (or $\sum\limits_{\substack{i,j\\i<j}}(m_{ij}-\sum\limits_{t}\sum\limits_{i}\delta_{ij}(s)n_s w_{ts})^2$) under the condition $\sum\limits_{s} n_s=n,\ n_s\geqq 1$.

3. Third step:

Calculate the value Q^2 by obtained integers n_s ($s=1, \cdots, S$) and change the first approximation.

4. Fourth step:

Estimate S^1 and $\delta_{ij}(s)$ ($i=1, \cdots, N$; $s=1, \cdots, S$) (second approximation) by using Q^2 calculated.

5. And so on. Repeat the process until we obtain small value of Q^2, taking into account the decreasing behaviour of Q^2 (whether Q^2 gets to a constant of small magnitude or not).

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CORRECTIONS TO

"MULTIDIMENSIONAL QUANTIFICATION OF THE DATA OBTAINED BY THE METHOD OF PAIRED COMPARISON"

CHIKIO HAYASHI

In the above titled article (this Annals 16 (1964), 231–245) the following corrections should be made.

(i) On page 232, line 4 from bottom, replace

"the II-dimensional" by "the two-dimensional".

(ii) On page 241, line 5 from bottom, replace

"is 10, but" by "is 10 and $O_6 \succ O_7$ is 6, but".