ON WEIGHTED RANK-SUM TESTS FOR DISPERSION

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(Received Feb. 11, 1963)

Summary

Here is proposed a class of two sample non-parametric scale tests (for both the situations, viz., with and without assuming the identity of the locations) based on a family of congruent inter-quantile numbers, and their various properties have been studied. A class of appropriate weight-functions has also been proposed here, and the corresponding weighted rank-sum tests appear to be asymptotically more (power-) efficient, in many cases, than most of the other ones available in the literature.

1. Introduction

Let \( X_1, \cdots, X_m \) and \( Y_1, \cdots, Y_n \) be two independent samples of \( m \) and \( n \) independent observations from two populations with continuous cumulative distribution functions (cdf's) \( F_i(x) \) and \( F_j(x) \) respectively, where

\[
F_i(x) = F((x - \mu_i)/\delta_i) \quad \text{for } i = 1, 2;
\]

\( \mu_i \) and \( \delta_i \) being respectively the location and the scale parameters of the \( i \)-th population, for \( i = 1, 2 \). We are interested in testing the null hypothesis \( H_0: \delta_1 = \delta_2 \) against the set of alternatives that they are not equal (alternatives being one sided or both sided). And, in testing this hypothesis, we are faced with two different situations, according as we assume the homogeneity of the locations \( \mu_1 \) and \( \mu_2 \), or we want to test for the identity of \( \delta_1 \) and \( \delta_2 \) without any regard to the possible heterogeneity of \( \mu_1 \) and \( \mu_2 \). In the first case, various tests have been proposed by Lehmann [10], Wilks-Rosenbaum [13], Mood [12], Kamat [8], Barton and David [2], Sukhatme ([18], [20]), Tamura [21], Ansari and Bradley [1], Siegel and Tukey [17], Capon [4], Klotz [9], among others. Location-free scale-tests has mainly been studied by Sukhatme [19], while Tamura, and Ansari and Bradley have also considered such tests. A discussion of the relative properties of some of these tests, along with some other ones, has also been made by the author [15], elsewhere. Here the following type of tests has been proposed, their properties studied, and contrasted with those of the other ones, referred to above.

Let \( X_{(1)} < \cdots < X_{(m)} \) be the first sample ordered variables. We then define a system of (first sample) congruent inter-quantile cells as
\[ I_j : X_{(j)} \leq x \leq X_{(m-j+1)} \quad \text{for } j = 0, 1, \ldots, [m/2] ; \]

where conventionally \( X_{(0)} = -\infty \) and \( X_{(m+1)} = \infty \), and where \([s]\) stands for the largest integer contained in \(s\). Let us also define the corresponding ring-cells as

\[ L_j : X_{(j)} < x < X_{(j+1)} ; \quad X_{(m-j)} < x \leq X_{(m-j+1)} \]

i.e.,
\[ L_j = I_j - I_{j+1} , \quad \text{for } j = 0, 1, \ldots, [m/2] - 1 , \]

and
\[ L_{[m/2]} = I_{[m/2]} . \]

Let now in the second sample, \( r_j \) observations belong to the ring-cell \( L_j \), and \( R_j \) observations to the cell \( I_j \), so that \( R_j = \sum_{i=j}^{[m/2]} r_{ji} \) for \( j = 0, \ldots, [m/2] \) and \( R_0 = n \). Let us also select a sequence \( \{a(j, m)\} \) or real and finite elements, satisfying the following conditions:

\[ |a(j, m)| \leq K \{(j+1)(m-j)/m^3\}^{\delta - 1/2} , \]

for some \( \delta > 0 \), and for all \( j = 0, \ldots, [m/2] \). Let then

\[ \bar{a}(m) = \sum_{j=0}^{[m/2]} a(j, m)/\{[m/2]+1\} , \quad \text{and} \]

\[ \sigma^2(a(m)) = \frac{1}{[m/2]+1} \sum_{j=0}^{[m/2]} \left( a^2(j, m) - [\bar{a}(m)]^2 \right) . \]

It then follows that
\[ \frac{1}{[m/2]+1} \sum_{j=0}^{[m/2]} a^2(j, m) \leq K^2 \sum_{j=0}^{[m/2]} \left( \frac{j+1}{m} \right)^{2\delta - 1} /[\{[m/2]+1\} , \]

and as
\[ \lim_{n \to \infty} \sum_{j=0}^{n} (j/n)^{2\delta - 1}/n = \int_0^1 x^{2\delta - 1} dx = \frac{1}{2\delta} < \infty , \]

it follows that \( \bar{a}(m) \) and \( \sigma^2(a(m)) \) have finite limits as \( m \to \infty \). We thus write

\[ \bar{a} = \lim_{m \to \infty} \bar{a}(m) \quad \text{and} \quad \sigma^2 = \lim_{m \to \infty} \sigma^2(a(m)) , \]

and assume further that

\[ \sigma^2(a(m)) > 0 \quad \text{for all } m, \quad \text{and } \sigma^2 > 0 . \]
Naturally, (1.6) implies that \( a(j, m) \) is not a constant.

(c.2) If we write \( F=j/m \), then \( a(j, m)(=a(F, m)) \) is an explicit function of \( F \) and \( m \), and for any given \( m \), \( a(F, m) \) is a strictly monotonic function of \( F \). This condition, as would be shown later on, is required only for the scale tests of this type.

(c.3) \( a(F, m) \) has a continuous first order derivative (with respect to \( F \)) \( G(F, m) \), which is finite for all \( F : 0 < F < 1 \), and let

\[
G(F) = \lim_{m \to \infty} G(F, m).
\]

We then assume that \( G(F) \) is a continuous and integrable function of \( F \).

Then our proposed test is based on

\[
S_n = n^{-1} \sum_{j=0}^{[m/2]} a(j, m)r_j,
\]

where \( N = m + n \). Also, if we write \( b(j, m) = a(j, m) - a(j-1, m) \) for \( j = 1, \ldots, [m/2] \), and \( b(0, m) = a(0, m) \), we can then equivalently write

\[
S_n = n^{-1} \sum_{j=0}^{[m/2]} b(j, m)R_j.
\]

In this paper, we have considered first, various properties of the \( S_n \)-test for arbitrary sequences \( \{a(j, m)\} \) of weight-functions, satisfying the conditions (c.1), (c.2) and (c.3). Subsequently, we have proposed a class of such weight-functions, and have shown that the corresponding \( S_n \)-test, in many cases, is asymptotically more (power-) efficient than most of the other ones, referred to earlier. Now, the procedure sketched so far, is obviously based upon the assumed identity of the locations \( \mu_i \) and \( \mu_t \). In order to make the tests location-free, we, following Sukhatme [19], centre the sample observations at the respective sample medians, and investigate the regularity conditions under which the \( S_n \)-test based on these centred variables would be asymptotically distribution-free. These conditions are essentially the same as those of Sukhatme [19].

2. Null distribution and moments of \( S_n \)

Let us first consider the joint distribution of \( r, \ldots, r_{[m/2]-1} \) under the null hypothesis \( H_0: F_1(x) = F_2(x) \) (i.e., \( \mu_1 = \mu_2 \) and \( \delta_1 = \delta_2 \) in (1.1)). It then follows that if \( m \) be even, this joint distribution, after some simplification, reduces to

\[
\binom{m+n}{m}^{-1} \prod_{j=0}^{(m-2)/2} (r_j + 1),
\]
where \( r_j \geq 0 \) for all \( j \), and \( \sum_{j=0}^{\frac{m}{2}} r_j = n \). If on the other hand, \( m \) be odd, the joint distribution of \( r_0, \ldots, r_{(m-3)/2} \) comes out as

\[
(2.2) \quad \left( \frac{m+n}{m} \right)^{-\frac{1}{2}(\frac{m-1}{2})} \prod_{j=0}^{\frac{m-1}{2}} (r_j + 1),
\]

where \( r_j \geq 0 \) for all \( j \), and \( \sum_{j=0}^{\frac{m-1}{2}} r_j = n \).

Now, we can always enumerate the set of points \( (r) = (r_0, \ldots, r_{\frac{m}{2}}) \) for which \( S_r = n^{-\frac{m}{2}} \sum_{j=0}^{\frac{m}{2}} a(j, m) r_j \leq S_r \) and the probability of each element \( r \) in this set can be evaluated by using (2.1) or (2.2), according as \( m \) is even or odd. Thus we get the values of \( P\{S_r \leq S_r | H_r\} \) for any \( S_r \). This procedure yields the null distribution of \( S_r \). However, for large samples (even for moderately large ones) the labour in this computation increases so much, that one is naturally inclined to adopt some simple limiting form of distribution, for the purpose, and the same has been accomplished in the following section. But before that, we consider here the first two moments of \( S_r \) under \( H_r \).

It follows by a straightforward computation that for an odd \( m \),

\[
(2.3) \quad E\left\{ \prod_{i=1}^{k} r_j^{s_i} | H_r \right\} = \left[ n^{(s_j)} \right] / (m+s)^{\left(\frac{s_j}{2}\right)} \prod_{i=1}^{k} (s_i + 1)!
\]

where \( a^{(r)} = p(p-1) \cdots (p-r+1) \), \( s_1, \ldots, s_k \) are non-negative integers, \( k \geq 1 \), and \( s = \sum_{i=1}^{k} s_i \). Whence, we get

\[
E\{r_j | H_r\} = 2n/(m+1),
\]
\[
V \{r_j | H_r\} = 2n(m-1)(n+m+1)/(m+1)^{q}(m+2). \quad \text{and}
\]
\[
Cov \{r_j, r_j' | H_r\} = -4n(n+m+1)/(m+1)^{q}(m+2).
\]

Thus,

\[
E(S_r | H_r) = \frac{2}{m+1} \sum_{j=0}^{(m-1)/2} a(j, m) = \bar{a}(m), \quad \text{and}
\]
\[
V(S_r | H_r) = n(n+m+1)/(m+2).
\]

If on the other hand, \( m \) is even, \( r_0, \ldots, r_{(m-2)/2} \) will have the same moments as in (2.3) and (2.4), while \( r_{m/2} \) will have mean \( n/(m+1) \), variance \( nm(n+m+1)/(m+1)^{q}(m+2) \) and covariance with any other \( r_j \) equal to \( -2n(n+m+1)/(m+1)^{q}(m+2) \). Thus, the mean and variance of \( S_r \) would be changed to
\[ E(S_n|H_0) = \frac{2}{m+1} \sum_{j=0}^{(m-3)/2} a(j,m) + \frac{1}{m+1} a(m/2, m), \]

(2.6) \[ V(S_n|H_0) = \frac{n+m+1}{n(m+2)} \left\{ \frac{2}{m+1} \sum_{j=0}^{(m-3)/2} a^2(j, m) - \left( \frac{2}{m+1} \sum_{j=0}^{(m-3)/2} a(j, m) \right)^2 ight\} + \frac{m}{(m+1)^2} a^2(m/2, m) - \frac{4a(m/2, m)^{(m-2)/2}}{(m+1)^2} \sum_{j=0}^{(m-3)/2} a(j, m) \} . \]

Hence, from (2.5) and (2.6), we get that the difference of \( E(S_n|H_0) \) between the cases with \( m \) being odd and even is of the order \( m^{-1} \) (as \( a(m/2, m) < \infty \), by (c.1)) and that of \((m+1) V(S_n|H_0) \) between those cases is also of the order \( m^{-1} \). Hence, (2.6) and (2.5) are asymptotically equal.

3. Asymptotic normality of \( S_n \)

The asymptotic normality of the distribution of \( S_n \) (standardised by appropriate location and scale parameters) under the null as well as a family of alternative hypotheses, has been established here through the use of an elegant theorem by Blum, Chernoff, Rosenblatt and Teicher ([3], theorem 3; corollary) on interchangeable processes. For our purpose, let us first state the theorem.

For each positive integer \( n \), let \( \{X_{ni}, i=1, 2, \ldots\} \) be an interchangeable process with mean zero and variance unity. If then

(3.1) \[ \lim_{n \to \infty} E_n(X_{n1}^2 X_{n2}^2) = 1 , \]

(3.2) \[ E_n \{ |X_{ni}| \} = o(n^{1/2}) , \text{ and} \]

(3.3) \[ \lim_{n \to \infty} n^{k/2} E_n \{ X_{n1} \cdots X_{nk} \} = a_k , \text{ for } k=1, 2, \ldots , \]

where \( a_k \) is the \( k \)-th moment of a normal distribution with mean \( a_1 \) and variance \( b \), then \( n^{-1/2} \sum_{i=1}^{n} X_{ni} \) has asymptotically a normal distribution with mean \( a_1 \) and variance \((1+b)\).

Let now \( N=m+n, m/N=\lambda_n \), and we consider values of \( m \) and \( n \) such that for some \( \lambda_0 \leq 1/2, 0 < \lambda_0 \leq \lambda_n \leq 1-\lambda_0 < 1 \). Let then \( X_m \) stand for the order statistics \( X_{(1)} < \cdots < X_{(m)} \), and we adopt the definition of the ring-cells \( L_j : j=0, 1, \ldots, \lfloor m/2 \rfloor \) given in (1.3). We then define the counter functions \( \phi(Y_j|X_m) \) for \( j=1, \ldots, n \), by

(3.4) \[ \phi(Y_j|X_m) = a(i, m) , \text{ if } Y_j \in L_i \text{ for } i=0, \ldots, \lfloor m/2 \rfloor . \]

It then follows readily from (1.8) and (3.4) that

(3.5) \[ S_n = n^{-1} \sum_{j=1}^{n} \phi(Y_j|X_m) . \]
Since $Y_1, \ldots, Y_n$ are all independently and identically distributed, it follows that whatever be the two cdf's $F_1(x)$ and $F_2(x)$, $\phi(Y_j|X_m)$ has the same marginal distribution for all $j=1, \ldots, n$, and further the joint distribution of any subset of these variables is a symmetric function of them. Thus for any two arbitrary cdf's $F_1(x)$ and $F_2(x)$, the sequence of variables $\{\phi(Y_j|X_m) : j=1, 2, \ldots\}$ forms an interchangeable process (cf. Chernoff and Teicher [6], case b). Thus, we require to prove only that for this sequence of interchangeable random variables, the theorem stated earlier holds true. We propose to prove the theorem first in the null case (i.e., when $F_1(x) \equiv F_2(x)$) and later append a lemma by which the theorem can be readily extended to the more general case of a family of alternative hypotheses. Further, we would consider here only the case of $m$ being an odd integer. The case with an even $m$ would follow precisely on the same line.

We now adopt the notations and definitions in (1.4), (1.5) and (1.6). We then write $\Psi_m(Y_j) = \phi(Y_j|X_m) - a(m)$, for $j=1, \ldots, n$, and $a'(j, m) = a(j, m) - a(m)$ for $j=0, 1, \ldots, [m/2]$, and propose to prove that under $H_0: F_1(x) \equiv F_2(x)$,

$$E_{N-1_N} \{ |\Psi_m(Y_j)|^3 \} = O(N^{1/3}) = o(N^{1/2}),$$

$$\lim_{N \to \infty} E_{N-1_N} \{ \Psi_m^2(Y_j) | Y_j \neq Y_{j'} \} = \sigma_a^4,$$ and

$$\lim_{N \to \infty} E_{N-1_N} \{ n^{k/2} \Psi_m(Y_{i_1}) \cdots \Psi_m(Y_{i_k}) \} = \left( \frac{1-\lambda_N}{\lambda_N} \right)^k \sigma_a^k a_k$$

where

$$a_k = 0, \quad \text{if } k \text{ is odd}$$

$$= \frac{k!}{2^{k/2}(k/2)!}, \quad \text{if } k \text{ is even},$$

and $E_N \lambda_N$ denotes the expectation for a given value of $N$ and $\lambda_N = m/N$, satisfying $0 < \lambda_N \leq 1 - \lambda_N < 1$ for some $\lambda \leq 1/2$.

The proof of (3.6) follows from (c.1) in section 1. We have

$$E_{N-1_N} \{ |\Psi_m(Y_j)|^3 \} \leq 2^k E_{N-1_N} \{ |\phi(Y_j|X_m)|^3 \} \leq 4k m^{1/2} \lambda_N \lambda_N^{1/4} E_{N-1_N} \{ |\phi(Y_j|X_m)|^3 \}$$

$$= O(N^{1/3}) = o(N^{1/2}),$$

as by (c.1)

$$E_{N-1_N} \{ |\phi(Y_j|X_m)|^3 \} < \infty.$$  

While, to prove (3.7) and (3.8), let us first prove the following lemma.

**Lemma 3.1.** Let $P \{ 1 \leq \phi(Y_j|X_m) = a(i_j, m) | H_0 \} = P_{i_1, \ldots, i_k}$. Let now $s_n$
of the \( i \)'s be equal to \( i_u \) for \( u = 1, 2, \ldots, l \), so that \( \sum_{u=1}^{l} s_u = k \). Then
\[
p_{i_1 \ldots i_l}^{(k)} = \prod_{u=1}^{l} (s_u + 1)!/(m+k)^{s_u},
\]
where
\[
p^{(k)} = p(p-1) \cdots (p-q+1),
\]
and \( l \) may range from 1 to \( k \).

The lemma can readily be proved by straightforward computation, and hence the proof is left to the reader.

Now
\[
E_{N, j_x} \{ \prod_{i=1}^{l} (Y_{i_1} Y_{i_2}) \} = \sum_{i_1=0}^{(m-1)/2} \sum_{i_2=0}^{(m-1)/2} \alpha'(i_1, m) \alpha'(i_2, m) \prod_{i} p_{i_1 i_2}^{(k)},
\]
and hence by lemma 3.1, the above reduces to
\[
\frac{1}{m+2} \left\{ \frac{2}{m+1} \sum_{i=0}^{(m-1)/2} \alpha'(i, m) \right\}^l + m+1 \left\{ \frac{2}{m+1} \sum_{i=0}^{(m-1)/2} \alpha'(i, m) \right\}^l = \sigma^2_{a(m)} + O(N^{-3l}),
\]
as
\[
\frac{2}{m+1} \sum_{i=0}^{(m-1)/2} \alpha'(i, m) \leq \sigma^2_{a(m)} k^2 m^{1-3l},
\]
by condition (c.1). Hence (3.7) is proved.

Let us next prove (3.8) for an even \( k \). Thus, we have for \( 2k \) variables,
\[
E_{N, j_x} \{ n^k \prod_{i=1}^{l} (Y_{i_1} Y_{i_2}) \} = n^k \sum_{i_1=0}^{(m-2)/2} \cdots \sum_{i_{2k}=0}^{(m-2)/2} \left\{ \prod_{j=1}^{2k} \alpha'(i_j, m) \right\}^l \prod_{i} p_{i_1 i_2}^{(k)} \cdots p_{i_1 i_2}^{(k)},
\]
where \( \sum_{j=1}^{l} \alpha_j = 2k \), and where the summation \( S \) extends over all possible \( i_1 \neq \cdots \neq i_l = 0, 1, \ldots, [m/2] \), and \( (\alpha_1, \ldots, \alpha_l) \) with \( \alpha_i \geq 1 \) for \( i = 1, \ldots, l \).

Let us next prove the following lemma.

**Lemma 3.2.** Let \( A_i(\alpha_1, \ldots, \alpha_l) = n^k \sum \left( \prod_{j=1}^{l} (\alpha'(i_j, m)) \right)^{l} \prod_{i} p_{i_1 i_2}^{(k)} \cdots p_{i_1 i_2}^{(k)} \) where the summation \( S_i \) extends over all possible \( i_1 \neq \cdots \neq i_l = 0, \ldots, (m-1)/2 \). Then
\[
A_i(\alpha_1, \ldots, \alpha_l) = O(N^{-3l}), \text{ at most, if sup.}(\alpha_1, \ldots, \alpha_l) > 2.
\]
PROOF. Let us first consider the case \( l \geq k \). Let \( l = k + s \), and thus \( s \) may range from 0 to \( k - 2 \), as \( \operatorname{sup}(\alpha_1, \ldots, \alpha_l) > 2 \). It then follows by simple arguments that if \( l = k + s \), and if \( \operatorname{sup}(\alpha_1, \ldots, \alpha_l) > 2 \) then at least \( 2s + 1 \) of the \( \alpha_j \)'s are equal to one. It may be further noted that

\[
\sum_{i=0}^{(m-1)/2} a'(i, m) = 0.
\]

Hence, we have for \( l = k + s \)

\[
A_l(\alpha_1, \ldots, \alpha_l)
\]

\[
= \frac{n^k \prod_{i=1}^l (\alpha_j + 1)!}{(m + 2k)^{\frac{2k^2}{2k}}} C_l(\alpha_1, \ldots, \alpha_l) \sum_{i_1, \ldots, i_{l-1} = 0}^{(m-1)/2} \prod_{j=1}^{2s+1} \prod_{j=2+1}^{k+s} [a'(i, j, m)]^s \]

\[
= (-1)^{l-1} \sum_{j=2}^{l} \frac{C_l(\alpha_1, \ldots, \alpha_l) A_{l-1}(\alpha_2, \ldots, \alpha_l, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_l)}{C_{l-1}(\alpha_2, \ldots, \alpha_l, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_l)},
\]

where

\[
C_l(\alpha_1, \ldots, \alpha_l) = \frac{(2k)!}{\alpha_1! \cdots \alpha_l! r_1! \cdots r_l!},
\]

where \( r_i \) of the \( \alpha_j \)'s are equal to \( \alpha_i \) for \( i = 1, \ldots, l \), and thus it is a finite quantity, independent of \( (m, n) \). Using this relation repeatedly, we ultimately get that \( A_l(\alpha_1, \ldots, \alpha_l) \) is a linear function of a finite number of \( A_j(\beta_1, \ldots, \beta_p) \)'s where \( p \leq l - 1, \sum_{i=1}^p \beta_i = 2k \), and none of \( \beta_1, \ldots, \beta_p \) are equal to 1, and the compounding coefficients are all finite. In the case of \( l < k \), we also similarly arrive at that \( A_l(\alpha_1, \ldots, \alpha_l) \) is a linear function of \( A_p(\beta_1, \ldots, \beta_p) \)'s with \( p \leq l, \sum_{i=1}^p \beta_i = 2k \) and none of \( \beta_1, \ldots, \beta_p \) are equal to 1. Thus, we require only to show that

\[
A_l(\beta_1, \ldots, \beta_p) = O(N^{-2l}),
\]

at most, for all \( p \leq k - 1, \sum_{i=1}^p \beta_i = 2k \) and none of \( \beta_1, \ldots, \beta_p \) being equal to 1. Let now \( p = k - s \), where \( s \) may range from 1 to \( k - 1 \). Then we have

\[
A_p(\beta_1, \ldots, \beta_p) = n^k \sum_{s=0}^{p} \left( \prod_{j=1}^p [a'(i, j, m)]^p \right) p_1^{p_1} \cdots p_p^{p_p}
\]

\[
= \frac{\prod_{j=1}^p (\beta_j + 1)! n^k}{(m + 2k)^{\frac{2k^2}{2k}}} \sum_{s=0}^{p} \left( \prod_{j=1}^p [a'(i, j, m)]^p \right)
\]

\[
\leq \frac{C_p(\beta_1, \ldots, \beta_p) n^k \prod_{j=1}^p (\beta_j + 1)!}{(m + 2k)^{\frac{2k^2}{2k}}} \prod_{j=0}^{(m-1)/2} [a'(i, j, m)]^s
\]

\[
\leq \frac{C_p(\beta_1, \ldots, \beta_p) n^k \prod_{j=1}^p (\beta_j + 1)! (m + 1)^p}{(m + 2k)^{\frac{2k^2}{2k}} 2^p}
\]

\[
= O(N^{-2l}),
\]
for any given \( \lambda_N : 0 < \lambda \leq \lambda_N \leq 1 - \lambda < 1 \).

Hence, the lemma follows through (3.12).

In view of this, in (3.10), we need consider only terms \( A_i(\alpha_1, \cdots, \alpha_l) \)
with \( \alpha_i \leq 2 \) for all \( i = 1, \cdots, l \). Consequently, here \( l \geq k \). Let now \( l = k + s \).
Then among these \( \alpha_j \)'s \( (k - s) \) are equal to 2, while the remaining 2s \( \alpha_j \)'s are equal to 1. Since, the summation \( S_i \) extends over all possible \( \alpha_i = \cdots = \alpha_{k+s} = 0, \cdots, (m-1)/2 \), the possible number of such sets of \( \alpha_j \)'s will be equal to

\[
\binom{2k}{2s} \frac{(2k)!}{(k-s)!} \frac{(2s)!}{2^{k-s}(2s)!} = \frac{(2k)!}{2^{k-s}(2s)!}.
\]

Hence

\[
A_i(\alpha_1, \cdots, \alpha_l) = \frac{(2k)!}{2^{k-s}(2s)!} \sum_{i_1, \cdots, i_k = 0}^{(m-1)/2} \prod_{j=1}^{2s} [a'(i_j, m)]^j \prod_{j=k+1}^{k+s} [a'(i_j, m)]^j
\]

\[
= \frac{(-1)(2s-1)(2k)!}{2^{k-s}(2s)!} \sum_{i_1, \cdots, i_k = 0}^{(m-1)/2} \prod_{j=1}^{2s} [a'(i_j, m)]^j + O(N^{-s}) \quad \text{(by lemma (3.2))}
\]

(3.14)

\[
= \frac{(-1)(2s-1)(2s-3) \cdots 3 \cdot 1(2k)!}{2^{k-s}(2s)!} \sum_{i_1, \cdots, i_k = 0}^{(m-1)/2} \prod_{j=1}^{2s} [a'(i_j, m)]^j + O(N^{-s})
\]

\[
= \frac{(-1)(2s)! (2k)!}{2^{k-s}(2s)!} \sum_{i_1, \cdots, i_k = 0}^{(m-1)/2} \prod_{j=1}^{2s} [a'(i_j, m)]^j + O(N^{-s})
\]

\[
= \frac{(-1)n^k (2k)!}{(m + 2k)^{2k}} \frac{\sigma_{2k}^{\alpha_m} (m+1)^k}{2k} + O(N^{-s}).
\]

Thus summing over \( s \) from 0 to \( k \), we get from (3.10) through (3.14) that

\[
E_{\Sigma_{\alpha}} \left\{ n^k \varphi_m(Y_{i_1}) \cdots \varphi_m(Y_{i_k}) \right\}
\]

(3.15)

\[
= \left( \frac{n}{m} \right)^k \frac{(2k)!}{2^{k}k!} \sum_{i=0}^{k} \frac{(-1)^i}{s} \binom{k}{s} 2^u 6^{k-t} + O(N^{-s})
\]

\[
= \left( \frac{1 - \lambda_N}{\lambda_N} \right)^k \frac{(2k)!}{2^{k}k!} \sigma_{2k}^{\alpha_m} + O(N^{-s}),
\]

whence, (3.8) is proved for an even \( k \). The case with an odd \( k \) follows precisely on the same line. Thus, under the null hypothesis, all the regularity conditions (3.1), (3.2) and (3.3) for the applicability of the Blum-Chernoff-Rosenblatt-Teicher theorem hold true. Hence, under \( H_0 \),
\[ \left( \frac{nm}{n+m} \right)^{1/2} \{ S_n - \overline{a}(m) \} / \sigma_a(m) \]

has asymptotically a normal distribution with zero mean and unit variance.

Let us now consider the family of alternative hypotheses, as specified by
\begin{equation}
H : F_i(x) = F\left( (x - \mu) / \delta_{i,N} \right), \quad \text{for } i = 1, 2
\end{equation}

where \( \{ \delta_{i,N} \} \) is a sequence of real and positive elements, converging to unity as \( N \to \infty \). For our purpose, we would assume that the cdf \( F \) possesses a continuous density function \( f \), everywhere. This restriction on the cdf \( F \), will be required only for the asymptotic normality of \( S_n \) under \( H \), but will not be required for the consistency of the \( S_n \)-test (as would be shown later on). We then propose to prove the following lemma first.

**Lemma 3.3.** Let \( P \{ \bigwedge_{j=1}^k \{ \phi(Y_j | X_m) = a(j, m) \} | H \} = p_{i_1 \ldots i_k}^H \). Then for all \( H \in H \), and all possible \( (\alpha_1, \ldots, \alpha_s) \)
\[
\lim_{m \to \infty} m^k |p_{i_1 \ldots i_k}^H - p_{i_1 \ldots i_k}^0| = 0.
\]

**Proof:** Let us now write
\begin{equation}
\begin{align*}
\mu_{i(1)} &= F_i(X_{m-i+1}) - F_i(X_{m-i}) + F_i(X_{i+1}) - F_i(X_i), \quad \text{and} \\
\mu_{i(2)} &= F_i(X_{m-i+1}) - F_i(X_{m-i}) + F_i(X_{i+1}) - F_i(X_i).
\end{align*}
\end{equation}

It then follows that
\begin{equation}
\begin{align*}
p_{i_1 \ldots i_k}^H &= E \left\{ \prod_{j=1}^k \mu_{i(j)} | F_i \right\} \quad \text{and} \\
p_{i_1 \ldots i_k}^0 &= E \left\{ \prod_{j=1}^k \mu_{i(j)} | F_i \right\},
\end{align*}
\end{equation}
where \( E \{ |F_i| \} \) indicates that the expectation extends over \( X_{i(1)} < \cdots < X_{i(m)} \) and is operated by the parent cdf \( F_i \). Since by definition \( \mu_{i(j)} \)'s are all finite and lie between zero and one, it follows that
\[
m_k |p_{i_1 \ldots i_k}^H - p_{i_1 \ldots i_k}^0| \to 0, \quad \text{as } m \to \infty,
\]
if we can show that
\begin{equation}
m |u_{i(1)} - u_{i(1)}| \to 0,
\end{equation}
uniformly in \( i = 0, \ldots, \lfloor m/2 \rfloor \), and in \( H \).

Now, using (3.16) and some simple but lengthy computations, it can be readily shown that
(3.20) \[ mu_{\{0\}} = \alpha_m \theta_N \{ mu_{\{1\}} \} \]

uniformly in \( i = 0, \ldots, [m/2] \); where \( \theta_N \) converges to unity as \( N \to \infty \) for all alternative in \( H \), and \( \alpha_m \) converges in probability to unity. Also, by the well-known results on the elementary sample coverages, \( mu_{\{i\}} \) has a limiting non-degenerate distribution with mean 2 and variance 6, for all \( i = 0, 1, \ldots, [m/2] \). Hence, (3.19) follows from (3.20), and the lemma is proved.

Combining now lemma 3.3 with lemmas 3.1 and 3.2, the proof of (3.6), (3.7) and (3.8) follows precisely on the same line as in (3.9) through (3.15). Hence, the asymptotic normality of \( S_N \), under the family of alternative hypotheses in \( H \), sketched in (3.16). It may be noted in this connection that the mean and variance of \( S_N \) under such a hypothesis will be asymptotically equal to \( \bar{a}_{(m)} \) and \( \{(nm/(n+m))^{-1}\sigma_{a(m)}\} \) respectively, but the distribution of

\[
\left( \frac{nm}{n+m} \right)^{1/2} \frac{S_N - \bar{a}(m)}{\sigma_{a(m)}}
\]

will then be normal with unit variance, and mean equal to

(3.21) \[ A_H = \left( \frac{nm}{n+m} \right)^{1/2} \sum_{i=0}^{(m-1)/2} a(i, m) \{ p^H_i - p_i \} / \sigma_{a(m)} \]

and thus \( |A_H| \) may be finite, zero or indefinitely large, depending on the particular alternative specification.

Remark One. There is some analogy of our statistic \( S_N \) with the one by Chernoff and Savage [5]. For our purpose, let us first introduce the following notations.

Let

\[ N = m + n, \quad m/N = \lambda_N, \quad 0 < \lambda \leq \lambda_N \leq 1 - \lambda < 1 \]

for some \( \lambda \leq 1/2 \). Let us also define the empirical cdf’s as

(3.22) \[ F_{1m}(x) = \text{(Number of } X_i \leq x) / m, \]

\[ F_{2m}(x) = \text{(Number of } Y_i \leq x) / n, \] and

\[ H_N(x) = \lambda_N F_{1m}(x) + (1 - \lambda_N)F_{2m}(x); \quad H(x) = \lambda_N F_1(x) + (1 - \lambda_N)F_2(x). \]

A statistic \( T_N \) has then been defined by Chernoff and Savage [5] as

(3.23) \[ T_N = \int_{-\infty}^{\infty} J_N[H_N(x)] dF_{1m}(x), \]

where \( J_N = J_N(i/N) \) is an explicit function of \( (i, N) \) and need be defined
only at \(1/N, \cdots, N/N\), but may have its domain of definition extended to (0,1], by some convention. Then under the following regularity conditions:

(a) \(J(H) = \lim_{N \to \infty} J_N(H)\) exists for all \(0 < H < 1\), and is not constant,

(b) \(\int_{J_N} [J_N(H_N) - J(H_N)] dF_{i_m}(x) = O_p(N^{-1/2})\) where \(I_N : 0 < H_N(x) \leq 1\),

(c) \(J_N(1) = O(\sqrt{N})\), and

(d) \(|J^{(i)}(H)| = \left| \frac{d^i J(H)}{dH^i} \right| \leq k[H(1 - H)]^{i-\frac{1}{2}+\delta}\)

for almost all \(H : 0 < H < 1\) and, for \(i = 0, 1, 2\) and for some \(\delta > 0\); \((T_N - \mu_N)/\sigma_N\) (where \(\mu_N\) and \(\sigma_N\) has been defined by them in [5], theorem 1) has asymptotically a normal distribution with zero mean and unit variance, for any given \(F_i, F_1\) and \(\lambda_N\). It may be noted that the statistic \(T_N\) may also be written as

\[
T_N = m^{-1} \sum_{j=1}^{m} J_N[H_N(x_{(j)})] = m^{-1} \sum_{i=1}^{N} E_{Ni} Z_{Ni}
\]

where \(Z_{Ni} = 1\), or 0, according as the \(i\)-th smallest in the combined sample is a \(X\) or \(Y\), \(E_{Ni} = J_N(i/N)\) are pure numbers, and \(X_{(1)} < \cdots < X_{(m)}\) are the first sample ordered variables. Adopting now the definition of \(b(j, m)\) in section 1, we let

\[
b(j, m) = b(m - j + 1, m) = m^{-1} G_1(j/m) \quad \text{for} \quad j = 1, \cdots, [m/2],
\]

where the function \(G_1\) satisfies the conditions (c.1), (c.2) and (c.3), and thus

\[
\lim_{m \to \infty} G_1 = G \quad \text{for all} \quad 0 < F < 1, \quad G \text{ being defined in (1.7). We then let}
\]

\[
J_N[H_N(x_{(j)})] = G_1(j/m) [H_N(x_{(j)}) - 1/2] \quad \text{for} \quad j = 1, \cdots, m.
\]

It can then be shown with some lengthy but direct computations that

\[
N^{1/2} \left\{ T_N - \left[ (1 - \lambda_N)S_N + \lambda_N C_N \right] \right\} \overset{P}{\to} 0
\]

uniformly in \(H\), defined in (3.16), where

\[
C_N = \bar{a}(m) + 1/m \{(m+1)a(1, m) + 2a(0, m) - 2\bar{a}(m)\}
\]
\[
\to \bar{a} + a_1 \text{ as } m \to \infty; \quad \bar{a} = \lim_{m \to \infty} \bar{a}(m), \text{ and } a_1 = \lim_{m \to \infty} a(1, m).
\]
We thus arrive at a linear relationship (almost sure) of our statistic \( S_N \) and their \( T_N \). But unfortunately, the function \( J_N \), defined in (3.26) fails to satisfy the basic requirement of Chernoff-Savage theorem, since their \( J_N \) depends only on \( (i/N) \) but our \( J_N \) depends on \( (j/m) \) as well as \( (i/N) \). Thus we fail to arrive at the asymptotic normality of \( S_N \), through that of \( T_N \).

However, it is well-worth comparing the regularity conditions pertaining to the asymptotic normality of \( S_N \) and of \( T_N \). It follows from (c.1), (c.2) and (c.3) that we do not require the regularity conditions (a), (b), (c) of the Chernoff-Savage theorem, though (a) is implicit in (c.1). Our condition (c.1) is in a sense, less restrictive than (d) of theirs. In (c.1), we do not require the existence of the second order derivative of our weight-function, as well as the restriction on the first order derivative of it, while the two weight-functions satisfy a similar condition. However, their theorem applies to any arbitrary \( F_1 \) and \( F_1 \), while ours only to a class of alternative hypotheses \( H \), considered in (3.16); though for large samples this class \( H \) covers the entire domain of interest.

4. Consistency and asymptotic efficiency of the \( S_N \)-test

The consistency of the \( S_N \)-test against any difference of the scale parameters (assuming, of course, the homogeneity of the locations) can be proved by Blum-Chernoff-Rosenblatt-Teicher theorem [3], provided we assume that the cdf \( F \) admits a continuous density function \( f \) everywhere. However, as for the consistency, we do not require the existence and finiteness of the density function, and the same may be established as follows.

It follows from (2.5) and (2.6) that under \( H_0 \): \( F_1(x) \equiv F_0(x) \).

\[
S_N \overset{P}{\to} \bar{a} \quad \text{where} \quad \bar{a} = \lim_{m \to \infty} \bar{a}(m).
\]

(4.1)

It follows from the definition of \( S_N \) in (1.9) and the condition (c.3) that under condition which are even less stringent than (c.1), (c.2) and (c.3), and for arbitrary \( F_1 \) and \( F_1 \), \( |S_N - E(S_N|F_1,F_1)| \overset{P}{\to} 0 \). The proof is somewhat lengthy, and for the intended brevity of our discussion here, is omitted. Thus, we require to prove only that for any \( H \): \( F_1(x) = F_1(\theta x) \) with \( \theta \neq 1 \), (conventionally, under the assumption of the homogeneity of locations, we take \( \mu = 0 \)), \( |E(S_N|\theta) - \bar{a}(m)| \) tends to a non-zero limit, as \( N \to \infty \). Again, with the definition of \( S_N \) in (1.9), conditions (c.2) and (c.3), and by an application of the Glivenko-Cantelli's theorem (cf. Loève [11], pp. 20-21), it can be shown readily
that

\begin{equation}
\lim_{N \to \infty} E(S_N | \theta) = \int_0^1 G(F') F'(\theta x) \, dF(x), \quad |G(F)| > 0
\end{equation}

where \( G(F) \) has the same sign for all \( 0 < F < 1 \), (by (c.2) and (c.3)). Consequently,

\begin{equation}
\lim_{N \to \infty} |E(S_N | \theta) - \bar{a}(m)| = \int_0^1 |G(F')| |F'(\theta x) - F(x)| \, dF(x) > 0
\end{equation}

for any \( \theta \neq 1 \).

Hence, we have the consistency of the \( S_N \)-test.

As the \( S_N \)-test has been shown to be consistent for any \( \theta \neq 1 \), the power of the test will be asymptotically equal to unity. Hence, we relate \( \theta \) with \( N \), in such a way that the power asymptotically lies between 0 and 1. Thus, as in (3.16), we take

\begin{equation}
F_i(x) = F([x - \mu]/\delta_{i,N}) \quad \text{for} \quad i = 1, 2,
\end{equation}

where \( \theta = \delta_{1,N}/\delta_{2,N} = 1 + \gamma / \sqrt{N} \), with a real and finite \( \gamma \).

It then follows from the definition of Pitman-efficiency (cf. Fraser [7], pp. 271–273) that the necessary regularity conditions are all satisfied here, provided we assume further that the cdf \( F \) has a continuous density function \( f \), everywhere. Hence, the relative Pitman-efficiency of the \( S_N \)-test with respect to the variance ratio (\( F \)-) test is given as

\begin{equation}
E_{S_F} = \frac{\beta_1 - 1}{4\sigma_0^2} \left[ \int_{-\infty}^{\infty} |x G(F)| f^2(x) \, dx \right]^2,
\end{equation}

where

\[ \sigma_0^2 = \lim_{m \to \infty} \sigma_{2(m)}^2, \quad F = F(x) \quad \text{and} \quad \beta_1 = E[X - E(X)]^4/[E[X - E(X)]^2]^2 \]

is the conventional measure of Kurtosis of the cdf \( F \).

5. Location-free \( S_N \)-test

Our \( S_N \)-test is based on the postulated identity of the locations. In order to test for the difference in the scale parameters, without presuming the identity of the locations, we, following Sukhatme [19], express the sample observations as deviations from the respective sample medians, and thus base the test on these centred observations. We thus require to study the regularity conditions, under which, our \( S_N \)-test thus modified, will be distribution-free. Now, for small samples, the distribution of this modified test criterion will be highly involved and will depend appreciably on the parent cdf, through the sampling distribution of the
sample medians. Hence, we will confine ourselves only to large samples, and study the regularity conditions, under which the modified $S_{N}$-test would be distribution-free.

Let $\tilde{X}$ and $\tilde{Y}$ denote respectively the first and the second sample medians. We then define $X_{i}^{*} = X_{i} - \tilde{X}$, $X_{(i)}^{*} = X_{(i)} - \tilde{X}$ for $i = 1, \ldots, m$, $X_{(0)}^{*} = -\infty$, $X_{(m+1)}^{*} = \infty$, and $Y_{i}^{*} = Y_{i} - \tilde{Y}$ for $i = 1, \ldots, n$. The definitions of the modified inter-quantile cells ($I_{j}^{*}$: $j = 0, \ldots, [m/2]$) and the modified ring-cells ($L_{j}^{*}$: $j = 0, \ldots, [m/2]$) will then go precisely on the same line as in (1.2) and (1.3) respectively, where only $X_{(i)}$ and $x$ are to be replaced by $X_{(i)}^{*}$ and $x^{*}$ for $i = 0, \ldots, m+1$. Let then $r_{j}^{*}$ of the $Y_{i}^{*}$ values in the second sample belong to $L_{j}^{*}$ and $R_{j}^{*}$ to $I_{j}^{*}$, for $j = 0, \ldots, [m/2]$. Then the modified test criterion $S_{N}^{*}$ may be written as

\begin{equation}
S_{N}^{*} = n^{-1} \sum_{j=0}^{[m/2]} a(j, m)r_{j}^{*} = n^{-1} \sum_{j=0}^{[m/2]} b(j, m)R_{j}^{*}.
\end{equation}

The asymptotic distribution-freeness of $S_{N}^{*}$ would be established, provided we can show that $N^{1/2}\{S_{N} - S_{N}^{*}\} \xrightarrow{P} \frac{P}{0}$ for any given $\lambda_{N}$ and under $H_0$.

Now

\begin{equation}
\sqrt{n}\{S_{N} - S_{N}^{*}\} = n^{-1/2} \sum_{j=0}^{[m/2]} b(j, m)(R_{j} - R_{j}^{*})
\end{equation}

and hence, it can be shown following some lengthy computations, that (5.2) converges in probability to zero, provided that (i) the cdf $F$ has everywhere a continuous density function $f$, which is symmetrical about the population median, and (ii) the density function at the population median is non-zero. The details of the deduction are omitted here for the intended brevity of the discussion. The deduction, however, follows precisely on the same line as in the case with modified inter-quantile tests, considered by the author [16] elsewhere. It may be noted that Sukhatme [19] has put forward the same set of regularity conditions, while studying the same property of his test [18] based on a $U$-statistic, though the other tests by Mood [12] or by Lehmann [10] fail to be asymptotically distribution-free (when thus modified) even under such regularity conditions. The modified form of the test by Tamura [21] is also asymptotically distribution-free under the same set of regularity conditions, while Ansari and Bradley [1] guessed that under these conditions their test might also be distribution-free.

The consistency of the $S_{N}^{*}$ test for any difference of the scale parameters, without any regard to the possible difference of the locations, follows precisely on the same line as in the preceding section, and fur-
ther $S_k^*$ and $S_N$ have the same asymptotic normal distribution, and hence, the same Pitman-efficiency.

6. A proposed class of weight-functions

Here a class of weight-functions has been proposed, and the resulting Pitman-efficiency of the corresponding $S_N$-test has been studied and contrasted with those of the others. We now let

\begin{equation}
(6.1) \quad a(j, m) = \left( \frac{m + 1}{2} - j \right)/(m + 1)^k \quad \text{for} \quad j = 0, \ldots, [m/2];
\end{equation}

where the possible values of $k$ ($> 0$) generate the class. It then follows that

(i) \quad \bar{a} = \lim_{m \to \infty} a(m) = 1/2^k(k + 1),

(ii) \quad \sigma^2_{a} = \lim_{m \to \infty} \sigma^2_{a(m)} = k/(k+1)^{1/2}v(2k+1), \quad \text{and}

(iii) \quad G(F) = \lim_{m \to \infty} G(F, m) = k|F - 1/2|^k-1.

Consequently, from (4.5) we get that the Pitman-efficiency of the corresponding $S_N$-test with respect to the variance ratio ($F$-) test is

\begin{equation}
(6.2) \quad E^{(s, F)}_{S_N} = \frac{\hat{\beta}_1 - 1}{4} \left[ \int_{-\infty}^{\infty} |x| k|F(x) - 1/2|^{k-1}f^2(x) \, dx \right]^{1/2} \frac{2^{2k}(2k+1)(k+1)^4}{k^3}
\end{equation}

\[= 2^{k-1}(2k+1)(k+1)^4(\hat{\beta}_1 - 1) \left[ \int_{-\infty}^{\infty} |x| |F(x) - 1/2|^{k-1}f^2(x) \, dx \right]^{1/2}.\]

Now we have compared (6.2) with the corresponding values of the tests by Mood [12], Sukhatme ([18], [20]), Tamura [21], and Ansari and Bradley [1]. The test by Lehmann [10], unfortunately, fails to be distribution-free, even under $H_0$. The present author [14] has utilized a distribution-free and consistent estimate of the variance of $U$-statistic, to render the studentized form of Lehmann's statistic asymptotically distribution-free, and to have the same asymptotic power properties as that of the original test. Sukhatme's [18] first test and the test by Ansari and Bradley [1] are asymptotically power-equivalent to our $S_N$-test in the particular case $k=1$. Sukhatme's [20] second test and Lehmann's [10] test have more or less the same power efficiency, and they are more-power efficient than the former ones, while they are less efficient than the ones by Mood [12] and by Tamura [21], these latter two being again asymptotically power-equivalent, and also power-equivalent to our $S_N$-test in the particular case: $k=2$. The test by Siegel and Tukey [17] is again power-equivalent to Ansari and Bradley's test. The
asymptotic power-efficiency of the tests by Kamat [8] and by Wilks-Rosenbaum [13] has been considered in detail by the author [15], elsewhere; where it has been shown that the asymptotic efficiency of such tests (with respect to either of the above tests) would be equal to zero, if the range of the parent cdf extends to infinity, in at least one extremity, while for distributions having finite extremities these two tests would be suitable only if the order of terminal contact be either zero or unity. In fact, for all distributions with non-zero density functions at both the extremities, the asymptotic efficiency of any one of the earlier tests with respect to either of these two tests would be equal to zero. Capon [4] and Klotz [9] have considered locally most powerful rank-sum tests based on the principle of likelihood ratio, and their tests are also asymptotically power-equivalent, and are most efficient for certain specified type of parent cdf (under the alternative specifications). But the main drawback of these two tests is that the weight-functions depend explicitly on the assumed parent density function, for which we intend to maximise locally the power function. Consequently, if such a density function can not fairly be guessed (so as to be really appropriate for the given data), application of such a test seems to be dubious from the point of view of the power of the test, as the locally most powerful test for a particular parent cdf, may lose its power efficiency considerably for other cdf's. Further, for the computation of the weight-functions in these two tests, we require a tabulation of the expected values of the order statistics for various sample sizes, and this may be quite tedious in computation for many parent cdf's (other than the normal one). From these considerations, it is clear that we should use some rank-sum tests, which are reasonably efficient for many parent cdf's (and more particularly in the vicinity of the normal distribution), whose weight functions do not depend on the particular cdf, (which may be assumed to fit adequately the given data), and which are not computationally cumbersome. The class of $S_r$-test appears to be very suitable from this standpoint. The following table illustrates this.

Thus, in many cases, our $S_r$-test with $k \geq 3$ has the power-efficiency substantially higher than that of the others. For $k > 4$, the efficiency of the $S_r$-test decreases, for normal alternatives, with $k$ increasing. In view of this, the $S_r$-test with $k = 4$ appears to be very suitable for a broad class of parent cdf's, including the normal one.

7. A few concluding remarks

The test proposed here is based on a family of congruent inter-quartile cells of the first sample. We may, however, use similarly a family of such cells of the pooled sample. We will have then $[N/2]$
TABLE 1
The Pitman-efficiency of some two sample scale tests with respect to the variance ratio test for certain common type of cdf's.

<table>
<thead>
<tr>
<th>Tests</th>
<th>Relative Pitman-efficiency with respect to the variance ratio test for the parent cdf.</th>
<th>Normal</th>
<th>Laplace</th>
<th>Uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sukhatme's [18] first test, Ansari and Bradley's [1] test, $S_N$-test with $k=1$</td>
<td>$6/\pi^2=0.61$</td>
<td>0.94</td>
<td>0.60</td>
<td></td>
</tr>
<tr>
<td>Sukhatme's [20] second test</td>
<td>$=0.69$</td>
<td>1.03</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>Mood's [12] test, Tamura's [21] test, $S_N$-test with $k=2$</td>
<td>$15/2\pi^2=0.76$</td>
<td>1.08</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>$S_N$-test with $k=3$</td>
<td>$=0.84$</td>
<td>1.14</td>
<td>1.40</td>
<td></td>
</tr>
<tr>
<td>$S_N$-test with $k=4$</td>
<td>$=0.88$</td>
<td>1.16+</td>
<td>1.80</td>
<td></td>
</tr>
<tr>
<td>$S_N$-test with $k=5$</td>
<td>$=0.85$</td>
<td>1.16−</td>
<td>2.20</td>
<td></td>
</tr>
</tbody>
</table>

+ (or −) indicates that the actual value is slightly more (or less) than the tabulated one.

(where $N$ is the pooled sample size) such cells, and the test may then be based on the number of observations of either of the samples, belonging to these cells. The asymptotic properties of such tests may then be studied by the application of the well-known theorem by Chernoff and Savage [5], following the lines of Ansari and Bradley [1]. If we use the weight function $a(j, N)$ (i.e., the same as $a(j, m)$ with $m$ replaced by $N$), then it can be shown that our $S_N$-test and this one would be asymptotically power equivalent. As the ordering of the observations (in order of magnitude) in any single sample is always less tedious than that in the samples pooled, from computational aspect, our $S_N$-test appear to be comparatively more suitable. There is a further point in favour of our $S_N$-test. The null distribution of $S_N$, for small samples, as has been considered in section 2, also appears to be more simple than the one, for the later test, which may be found by the same technique as by Ansari and Bradley [1]. Thus the finding out of the critical values of $S_N$, for small samples, also appears to be comparatively simpler, in relation to the later class of tests.

The extension of the findings to the case of more than two samples is kept pending, as it requires first of all, a $c$-sample extension of the theorem by Blum et al [3] or by Chernoff and Savage [5].
Acknowledgements

The author remains indebted to Prof. H. K. Nandi, Reader in Statistics, Calcutta University, for his generous help in course of this investigation. Thanks are also due to Prof. H. Chernoff, for some useful references.

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