A NOTE ON AN IDENTITY INVOLVING BINOMIAL COEFFICIENTS[1]

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1. Introduction and summary

Certain identities involving binomial coefficients have been found to be useful in probability theory and statistics. Feller [2] gives a list of such identities in his celebrated book on probability theory. While reading the paper by Birnbaum and Tingey [1] we found an identity which is a generalization of the identity used by Birnbaum and Tingey in order to prove an integral formula. We are not aware this identity has appeared in the literature and we also hope that this has some applications.

2. An identity involving binomial coefficients

RESULT 2.1. For any real δ we have

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\delta-j)^m=\begin{cases}0, & 0\leqq m\leqq k-1\\ k!, & m=k\\ \sum_{l=0}^{i}\binom{k+i}{k+l}\delta^{i-l}g_k^{(k+l)}(0), & m=k+i\\ & i=1,\ 2,\cdots,\end{cases}$$

where $g_k^{(k+l)}(0)=\sum\frac{k!}{\gamma_0!\gamma_1!\cdots\gamma_l!}\left(\frac{1}{2!}\right)^{\gamma_1}\left(\frac{1}{3!}\right)^{\gamma_2}\cdots\left(\frac{1}{(l+1)!}\right)^{\gamma_l}(-1)^l(k+l)!$,

taking the summation over all $\gamma_0, \gamma_1, \cdots, \gamma_l$ (non-negative integers) which satisfy $\sum_{i=1}^{l} i\gamma_i=l$ and $\sum_{i=0}^{l}\gamma_i=k$.

PROOF. Consider

$$(1\qquad f_0(x)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{(\delta-j)x}=e^{\delta x}(1-e^{-x})^k .$$

Define

$$(2)\qquad f_m(x)=\frac{d^m}{dx^m}f_0(x)\,,\qquad m=1,\ 2,\cdots .$$

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Then

$$f_m(x)=e^{\delta x}\sum_{j=0}^{m}\binom{m}{j}\delta^{m-j}\frac{d^j}{dx^j}(1-e^{-x})^k$$

or

$$f_m(x)=e^{\delta x}\sum_{j=0}^{m}\binom{m}{j}\delta^{m-j}g_k^{(j)}(x)\,, \tag{3}$$

where $g_k^{(j)}(x)$ is the jth derivative of $g_k(x)=(1-e^{-x})^k$. Now, we have

$$g_k(x)=\sum_{j=0}^{\infty}\frac{g_k^{(j)}(0)}{j!}x^j \tag{4}$$

and also

$$g_k(x)=x^k\left(1-\frac{x}{2!}+\frac{x^2}{3!}-\frac{x^3}{4!}+\cdots\right)^k. \tag{5}$$

Expanding the right side of (5) in power series, we obtain

$$g_k(x)=\sum_{j=0}^{\infty}a_jx^j\,, \tag{6}$$

where $a_0=a_1=\cdots=a_{k-1}=0\,,\qquad a_k=1$ and

for $l>1$ (integer)

$$a_{k+l}=\sum_{\substack{\gamma_1+2\gamma_2+\cdots+l\gamma_l=l\\ \gamma_0+\gamma_1+\gamma_2+\cdots+\gamma_l=k}}\frac{k!}{\gamma_0!\gamma_1!\cdots\gamma_l!}\left(\frac{1}{2!}\right)^{\gamma_1}\left(\frac{1}{3!}\right)^{\gamma_2}\cdots\left(\frac{1}{(l+1)!}\right)^{\gamma_l}(-1)^l\,.$$

Comparing (4) and (6), we have

$$\begin{aligned}
&g_k^{(j)}(0)=0\,, \qquad j=0,\ 1,\ 2,\cdots,\ k-1\,,\\
&g_k^{(k)}(0)=k! \qquad (7)\\
&g_k^{(k+l)}(0)=\sum_{\substack{\gamma_1+2\gamma_2+\cdots+l\gamma_l=l\\ \gamma_0+\gamma_1+\cdots+\gamma_l=k}}\frac{k!}{\gamma_0!\gamma_1!\cdots\gamma_l!}\left(\frac{1}{2!}\right)^{\gamma_1}\left(\frac{1}{3!}\right)^{\gamma_2}\cdots\left(\frac{1}{(l+1)!}\right)^{\gamma_l}\\
&\qquad\times(-1)^l(k+l)!\,.
\end{aligned}$$

From (3) and (7), we have

$$\begin{aligned}
&f_m(0)=0, \qquad m=0,\ 1,\ 2,\cdots,\ k-1\\
&f_k(0)=k! \qquad (8)
\end{aligned}$$

$$f_{k+i}(0)=\sum_{l=0}^{i}\binom{k+i}{k+l}\delta^{i-l}g_k^{(k+l)}(0), \qquad i=1,\ 2,\ \cdots .$$

On the other hand, starting from the series definition of $f_0(x)$, we obtain

$$f_m(x)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\delta-j)^m e^{(\delta-j)x}, \qquad m=0,\ 1,\ 2,\ \cdots ,$$

and so

$$(9) \qquad f_m(0)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\delta-j)^m, \qquad m=0,\ 1,\ 2,\ \cdots .$$

From (8) and (9) we have the desired result.

COROLLARY 2.1.1. *For* $0\leqq m\leqq k-1$ *and any real* δ *we have*

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\delta-j)^{(m)}=0$$

where $(\delta-j)^{(m)}$ *denotes the product* $(\delta-j)(\delta-j-1)\cdots(\delta-j-m+1)$.

PROOF. Since $(\delta-j)^{(m)}=\sum_{l=0}^{m-1}S_{l,m}(\delta-j)^{m-l}$, with $S_{0,m}=1$

where $S_{l,m}$ denote the sum of the products of l integers taken from the first $m-1$ positive integers, for $l=1,\ 2,\ \cdots,\ m-1$. This expansion together with result 2.1 will yield the desired result.

Remark 2.1.1. Birnbaum and Tingey [1] use the identity in result 2.1 with $m=k-1$ and $\delta=N\varepsilon+k+1$ in order to show by induction that

$$\int_0^{\varepsilon}\int_{y_1}^{(1/N)+\varepsilon}\cdots\int_{y_k}^{(k/N)+\varepsilon}dy_{k+1}\cdots dy_2dy_1=\frac{\varepsilon}{(k+1)!}\left(\varepsilon+\frac{k+1}{N}\right)^k .$$

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