ON THE LIMITING DISTRIBUTIONS IN MARKOV RENEWAL PROCESSES WITH FINITELY MANY STATES

BY YASUSHI TAGA

(Received April 19, 1963)

1. Introduction

Recently Markov renewal processes have been studied in connection with the reliability of systems of mechanisms (see [1], [2] and [3]). In this paper, several limiting distributions in Markov renewal processes with finitely many states will be treated by making use of the theorem in renewal processes given by Takács [4] and the central limit theorem with regard to Markov processes, extended to the multi-dimensional case from the one given by Doob [5].

2. Formulation of the problem

2.1 Definitions and notations

At first, necessary definitions and notations will be stated, which are just the same as those used by Pyke (see [1] and [2]).

$\boldsymbol{Q}(t)=(Q_{ij}(t))$: A matrix of transition distribution defined on $(-\infty, \infty)$ such that the mass functions $Q_{ij}(t)$ satisfy the conditions

(i) $Q_{ij}(t)=0$ for $t<0$ and

(ii) $\sum_{j=1}^{r} Q_{ij}(\infty)=1$ for $1 \leqq i \leqq r$.

$H_i(t)=\sum_{j=1}^{r} Q_{ij}(t)$

$\boldsymbol{a}=(a_1, \cdots, a_r)$: initial probability vector.

DEF. 2.1. (J, X)-processes: a two-dimensional stochastic process $\{(J_n, X_n);\ n \geqq 0\}$ such that $X_0=0$, $\Pr\{J_0=k\}=a_k$ and

$$\Pr\{J_n=k,\ X_n \leqq x/(J_0, X_0), \cdots, (J_{n-1}, X_{n-1})\}=Q_{J_{n-1},\, k}(x)$$

$$S_n=\sum_{l=0}^{n} X_l, \quad \text{for} \quad n \geqq 0$$

$$\boldsymbol{P}=(p_{ij}), \text{ where } p_{ij}=Q_{ij}(\infty)$$

DEF. 2.2. The integer-valued stochastic process $\{N(t);\, t \geqq 0\}$ and $\{N_j(t);\, t \geqq 0\}$ are defined by $N(t)=\sup\{n \geqq 0;\, S_n \leqq t\}$ and $N_j(t)=$number of times $J_k=j$ for $1 \leqq k \leqq N(t)$. The stochastic process $\boldsymbol{N}(t)=\{N_1(t), \cdots, N_r(t)\}$ will be called a Markov renewal process (M.R.P.) determined by $(r, \boldsymbol{a}, \boldsymbol{Q})$.

DEF. 2.3. The Z-process $\{Z_t;\, t \geqq 0\}$ defined by $Z_t=J_{N(t)}$ is called a Semi-Markov process (S.–M.P.) determined by $(r, \boldsymbol{a}, \boldsymbol{Q})$.

Remark. An M.R.P. is almost surely a Z-process if and only if $p_{ii}=0$ for any state i which can be reached with positive probability.

Notations.

$$G_{ij}(t)=\Pr\{N_j(t)>0/Z_0=i\}, \quad \text{for} \quad t \geqq 0$$

$$P_{ij}(t)=\Pr\{Z_t=j/Z_0=i\}, \quad \text{for} \quad t \geqq 0$$

$$g_{ij}(s)=\int_0^\infty e^{-st}dG_{ij}(t), \quad s>0$$

$$\pi_{ij}(s)=\int_0^\infty e^{-st}dP_{ij}(t), \quad s>0$$

$$q_{ij}(s)=\int_0^\infty e^{-st}dQ_{ij}(t), \quad s>0$$

$$\tau_i=\int_0^\infty t dH_i(t), \quad 1 \leqq i \leqq r$$

$$\sigma_i^2=\int_0^\infty (t-\tau_i)^2 dH_i(t), \quad 1 \leqq i \leqq r$$

$$\mu_{ij}=\int_0^\infty t dG_{ij}(t), \quad 1 \leqq i,\, j \leqq r$$

$$\rho_{ij}^2=\int_0^\infty (t-\mu_{ij})^2 dG_{ij}(t), \quad 1 \leqq i,\, j \leqq r$$

Remark. The mean recurrence time with respect to state j is denoted by μ_{jj}.

2.2. Formulation of the problem

In the present paper we restrict our attention to those M.R.P.'s determined by $(r, \boldsymbol{a}, \boldsymbol{Q})$ with $r<\infty$. Because of Lemma 4.1 in [1] all such M.R.P.'s are regular, i.e., almost all sample functions are finite valued step functions over $(-\infty, \infty)$. Moreover, we assume that every state j is recurrent, i.e., $G_{jj}(+\infty)=1$. Now we are interested in obtaining the asymptotic distributions of total sojourn times $S_j(t)(1 \leqq j \leqq r)$ spent in state j during $[0, t]$. Of course, the relation

$$\sum_{j=1}^r S_j(t)=t, \quad \text{for all } t>0\,, \tag{2.1}$$

holds. On sojourn time problems in renewal processes, Takács has proved the following theorem, in which the state space consists of two states A and B, and ξ_n or η_n denotes the nth duration times of states A, B and $\alpha(t)$, $\beta(t)$ denote the total sojourn times spent in state A, B respectively. (see [4]).

THEOREM (Takács) *If* $\{\xi_n\}$ *and* $\{\eta_n\}$ *are independent sequences of positive random variables with properties*

$$\lim_{n\to\infty} \Pr\left\{\frac{\sum_{i=1}^{n}\xi_i - A_1 n}{A_2 n^a} \leqq x\right\} = G(x) \tag{2.2}$$

and

$$\lim_{n\to\infty} \Pr\left\{\frac{\sum_{i=1}^{n}\eta_i - B_1 n}{B_2 n^b} \leqq x\right\} = H(x)\,, \tag{2.3}$$

where $A_1 \geqq 0$, $A_2 > 0$, $a > 0$, $B_1 \geqq 0$, $B_2 > 0$, $b > 0$ *and* $G(x)$, $H(x)$ *are non-degenerate distribution functions, then there exist the limiting distribution function of* $\beta(t)$ *such that*

$$\lim_{t\to\infty} \Pr\left\{\frac{\beta(t) - C_1 t}{C_2 t^c} \leqq x\right\} = P(x)\,, \tag{2.4}$$

where $P(x)$ *is a non-degenerate distribution function, and* C_1, C_2 *and* c *are constants which depend on* (2.2) *and* (2.3).

He has shown in Table I in his paper [4], various forms of $P(x)$ in accordance with combinations of values of A_1, B_1, a and b.

In the special case of $A_1 > 0$, $B_1 > 0$ and $0 < a = b < 1$, the limiting distribution of $\beta(t)$ is given by

$$\lim_{t\to\infty} \Pr\left\{\frac{\beta(t) - \dfrac{B_1}{A_1 + B_1} t}{\left(\dfrac{A_1}{A_1 + B_1}\right)^{1+a} t^a} \leqq x\right\} = \Pr\left\{\frac{A_1 B_2 \chi - A_2 B_1 \zeta}{A_1^{1+a}} \leqq x\right\}\,, \tag{2.5}$$

where ζ and χ are independent random variables with distribution functions $G(x)$ and $H(x)$, respectively. Further, if $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed independent positive random variables, then

$$\lim_{t\to\infty} \Pr\left\{\frac{\beta(t) - \dfrac{\beta}{\alpha + \beta} t}{\sqrt{\dfrac{t}{\alpha + \beta}}} \leqq x\right\} = \Phi\left(\frac{\dfrac{1}{\alpha} + \dfrac{1}{\beta}}{\sqrt{\dfrac{\sigma_\alpha^2}{\alpha^2} + \dfrac{\sigma_\beta^2}{\beta^2}}} x\right) \tag{2.6}$$

where $\alpha=E(\xi_n)$, $\beta=E(\eta_n)$, $\sigma_\alpha=D(\xi_n)$, $\sigma_\beta=D(\eta_n)$ and $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$.

Remark. In the last case, it is easily seen that the distribution functions $G(x)$ and $H(x)$, shown in (2.2) and (2.3), certainly exist and are equal to $\Phi(x)$, and $A_1=\alpha$, $A_2=\sigma_\alpha$, $B_1=\beta$, $B_2=\sigma_\beta$ and $a=b=1/2$.

Making use of the results obtained by Takács [4] and Pyke [2], it is easily seen that the limiting distributions of total sojourn times $S_j(t)$ $(j=1, \cdots, r)$ can be obtained only if the mean and the variance of duration time and of recurrence time with regard to each state are known. Namely, if, in the theorem of Takács, we take two states B and A for state j and others respectively, then we can obtain immediately the limiting distribution of $S_j(t)$. Further, when the matrices of transition distributions have a special form, we can obtain explicitly the limiting distributions concerning an M.R.P., which will be stated in section 3.

3. The limiting distributions of total sojourn time in each state

LEMMA 3.1. *Suppose the first and second moments of the distribution function $G_{ij}(t)(i, j=1, \cdots, r)$ exist and are finite. Then mean τ_{jj} and variance ρ_{jj}^2 of recurrence time with regard to state j are obtained to be*

$$(3.1)\qquad {}_d(\boldsymbol{\mu})=\lim_{S\to 0}\{{}_d[s(\boldsymbol{I}-q(s))^{-1}]\}^{-1}$$

$$(3.2)\qquad {}_d(\boldsymbol{\rho}^2)=\lim_{S\to 0} s^{-2}(2\{{}_d[\boldsymbol{I}-\boldsymbol{q}(s))^{-1}]^{-1}-\{{}_d[\boldsymbol{I}-\boldsymbol{q}(2s))^{-1}]\}^{-1}) -(\lim_{S\to 0}\{{}_d[s(\boldsymbol{I}-\boldsymbol{q}(s))^{-1}]\}^{-1})^2$$

where $\boldsymbol{\mu}=(\mu_{ij})$, $\boldsymbol{\rho}^2=(\rho_{ij}^2)$ and ${}_d\boldsymbol{A}$ denotes the diagonal of a matrix $\boldsymbol{A}$.

PROOF. The relation (3.1) was shown in the remark of Theorem 4.2 given by Pyke [2]. The relation (3.2) may be easily obtained by considering that the relation

$$(3.3)\qquad \lim_{S\to 0} s^{-2}(1-2g_{ij}(s)+g_{ij}(2s))=\int_0^\infty t^2 dG_{ij}(t)$$

holds, because $\lim_{S\to 0} s^{-2}(1-2e^{-st}+e^{-2st})=t^2$.

THEOREM 3.1. *Let the transition distribution functions $Q_{ij}(t)$ be expressed in the form*

$$(3.4)\qquad Q_{ij}(t)=p_{ij}H_i(t), \qquad (1\leqq i, j\leqq r)$$

where the matrix $\boldsymbol{P}=(p_{ij})$ is regular, and $H_j(t)$'s have the finite means and variances which will be denoted by

$$\tau_j=\int_0^\infty t dH_j(t), \qquad \sigma_i^2=\int_0^\infty (t-\tau_j)^2 dH_j(t).$$

Then the limiting distribution of the total sojourn time, $S_j(t)$ spent in state j in $[0, t]$, certainly exists for any initial state so that

$$\lim_{t\to\infty} \Pr\left\{\frac{S_j(t)-\frac{\tau_j}{\mu_{jj}}t}{\sqrt{t/\mu_{jj}}}\leq x\right\}=\Phi\left(\frac{\frac{1}{\tau_j}+\frac{1}{\mu_{jj}-\tau_j}}{\sqrt{\frac{\sigma_j^2}{\tau_j^2}+\frac{\rho_{jj}^2-\sigma_j^2}{(\mu_{jj}-\tau_j)^2}}}x\right), \tag{3.5}$$

where μ_{jj} and ρ_{jj}^2 are given by (3.1) *and* (3.2) *respectively, and $\Phi(u)$ denotes the unit normal distribution function.*

PROOF. By the assumption that the transition matrix $\boldsymbol{P}$ is regular, we may consider the initial state J_0 to be j without loss of generality. Now we shall denote the events being in state j or in states other than j by E_j or $\overline{E}_j$, and their nth duration times by ξ_n or η_n respectively. Then, it is easily seen that the events E_j and $\overline{E}_i$ occur alternatively, and that their duration times $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of asymptotically identically distributed independent positive random variables, with means τ_j, $\mu_{jj}-\tau_j$ and variances σ_j^2, $\rho_{jj}^2-\sigma_j^2$ respectively. Therefore, we can obtain the relation (3.5) by making use of the theorem of Takács slightly modified.

Remark. To obtain the same result of the general M.R.P. as for the special M.R.P. given in theorem 3.1, we have to use central limit theorems for additive random functions shown in [6], which will be discussed in detail in near future.

4. The limiting distribution of $N(t)$

In the following, we shall consider the problems under a special transition distribution functions such that

$$Q_{ij}(t)=p_{ij}H_i(t), \qquad 1\leq i,\ j\leq r \tag{4.1}$$

where $H_i(t)$ denotes the distribution function of duration time of state i, and p_{ij} denotes the conditional probability that the transition from state i to j occur at the end of duration of state i.

Such transition distribution functions may be useful in practical applications and easily treated because the M.R.P. generated by them has similar characters to the Markov chain. Moreover, transition probabilities p_{ij}'s may be estimated from observed frequency counts and distribution functions $H_i(t)$'s are also estimated from observed duration times in each states.

Now, it will be shown that we can take the (J, X)-process, stated in section 2, for a discrete Markov process with two-dimensional state space. Let us consider a Markov process $\{(\xi_n, X_n), n \geqq 0\}$, where $\xi_n = J_{n-1}$, with initial probability vector $\boldsymbol{a}$, a matrix of transition distribution $\boldsymbol{Q}$ given by (4.1) and two dimensional state space $(\boldsymbol{I}_r, \boldsymbol{R})$, where $\boldsymbol{I}_r = \{1, 2, \cdots, r\}$ and $R=[0, \infty)$. Then we can determine the probability measure on such processes based on $\boldsymbol{a}$ and $\boldsymbol{Q}$, and the condition D_0, given by Doob [5, p. 221], is clearly satisfied only if the stochastic matrix $\boldsymbol{P} = (p_{ij})$ is regular —— there is only a single ergodic set and this set contains no cyclically moving subsets. When we consider a real-valued function $f(\cdot)$ of $(j, x) \in (\boldsymbol{I}_r, R)$ such that

$$f(j, x) = x \,, \tag{4.2}$$

the central limit theorem on Markov processes, given by Doob [5], can be applicable.

LEMMA 4.1. *Suppose that the transition matrix* $\boldsymbol{P} = (p_{ij})$ *is regular and that each distribution function* $H_i(t)$ *satisfies the condition*

$$\int_0^\infty t^{2+\delta} dH_i(t) < \infty \,, \qquad 1 \leqq i \leqq r \,,$$

for some $\delta > 0$. *Then*

$$\lim_{n\to\infty} E_{\boldsymbol{a}} \left\{ \left[\frac{1}{\sqrt{n}} \sum_{m=1}^{n} (f(J_m, X_m) - E\{f(J_m, X_m)\}) \right]^2 \right\} = \sigma^2 \tag{4.3}$$

exists, where $E_{\boldsymbol{a}}$ *denotes the expectation for the initial probability vector* $\boldsymbol{a}$ *and if* $\sigma^2 > 0$, *for any initial probability vector* $\boldsymbol{a}$

$$\lim_{n\to\infty} \mathrm{P}_{\boldsymbol{a}} \left\{ \frac{1}{\sqrt{n}\,\sigma} \sum_{m=1}^{n} \left[f(\xi_m, X_m) - E\{f(\xi_m, X_m)\} \right] \leqq x \right\} = \Phi(x) \tag{4.4}$$

uniformly in x, *where* $f(\cdot)$ *is given by* (4.2) *and* $\Phi(x)$ *denotes unit normal distribution function. The formula for* σ^2 *in* (4.3) *is given by*

$$\sigma^2 = \sum_{j=1}^{r} \alpha_j \sigma_j^2 + \sum_{j,k=1}^{r} \gamma_{jk} \tau_j \tau_k \,, \tag{4.5}$$

where τ_j, σ_j^2 *and* γ_{jk} *are given by*

$$\tau_j = \int_0^\infty t\, dH_j(t) \,, \qquad 1 \leqq j \leqq r \,, \tag{4.6}$$

(4.7) $$\sigma_j^2=\int_0^\infty (t-\tau_j)^2 dH_j(t)\,, \qquad 1\leqq j\leqq r\,,$$

and

(4.8) $$\gamma_{jk}=\delta_{jk}\alpha_j-\alpha_j\alpha_k+\alpha_j\sum_{n=1}^{\infty}(p_{jk}^{(n)}-\alpha_k)$$
$$+\alpha_k\sum_{n=1}^{\infty}(p_{kj}^{(n)}-\alpha_j)\,, \qquad 1\leqq j,\ k\leqq r\,,$$

δ_{jk} *is Kronocker's delta and* $\boldsymbol{\alpha}=(\alpha_1, \cdots, \alpha_r)$ *denotes the stationary probability vector such that*

(4.9) $$\boldsymbol{\alpha}P=\boldsymbol{\alpha}\,.$$

PROOF. Since conditions in the central limit theorem, given by Doob, are clearly satisfied by assumption, we can obtain the results (4.3) and (4.4) directly from that theorem. Therefore it remains to show the formula (4.5) for σ^2. If we take $\boldsymbol{a}$ for $\boldsymbol{\alpha}$, which can be done without loss of generality, then

$$E_{\boldsymbol{\alpha}}\{\sum_{m=1}^{n}f(\xi_m,\ X_m)\}^2=E_{\boldsymbol{\alpha}}\{\sum_{m=1}^{n}X_m^2\}+E_{\boldsymbol{\alpha}}\{2\sum_{l<m}X_lX_m\}$$
$$=n\sum_{j=1}^{r}\alpha_j(\tau_j^2+\sigma_j^2)+n(n-1)(\sum_{j=1}^{r}\alpha_j\tau_j)^2$$
$$+2\sum_{j,k=1}^{r}\sum_{l=1}^{n}(n-l)(p_{jk}^{(l)}-\alpha_k)\alpha_j\tau_j\tau_k\,.$$

Therefore we obtain the relation,

$$\lim_{n\to\infty}E_{\boldsymbol{\alpha}}\left\{\frac{1}{\sqrt{n}}\left[\sum_{m=1}^{n}f(\xi_m,\ X_m)-E_{\boldsymbol{\alpha}}\{\sum_{m=1}^{n}f(\xi_m,\ X_m)\}\right]\right\}^2$$
$$=\sum_{j=1}^{r}\alpha_j\sigma_j^2+\sum_{j,k=1}^{r}\gamma_{jk}\tau_j\tau_k\,,$$

when we consider both $\sum_{n=1}^{\infty}(p_{jk}^{(n)}-\alpha_k)$ and $\sum_{n=1}^{\infty}n(p_{jk}^{(n)}-\alpha_k)$ are convergent as is wellknown in the theory of Markov chains.

Remark. Noting that

(4.10) $$S_n=\sum_{m=1}^{n}f(\xi_m,\ X_m)=\sum_{m=1}^{n}X_m\,,$$

(4.4) shows that

(4.11) $$\lim_{n\to\infty}\mathrm{P}_{\boldsymbol{\alpha}}\left\{\frac{S_n-n\tau}{\sqrt{n}\,\sigma}\leqq x\right\}=\Phi(x)\,,$$

where $\tau=\sum_{j=1}^{r}\alpha_j\tau_j$. Moreover $\sum_{n=1}^{\infty}(p_{jk}^{(n)}-\alpha_k)$ can be obtained as the (j, k)th element of the matrix

$$\{(I-P)^{-1}-I\}(I-A),$$

where $P=(p_{ij})$, $A=\begin{pmatrix}\alpha\\ \vdots\\ \alpha\end{pmatrix}$, $\alpha=(\alpha_1, \cdots, \alpha_r)$ and I denotes the unit matrix.

More generally, the following theorem can be shown.

THEOREM 4.1. *Under the same assumptions in lemma* 4.1, *the relation* (4.4) *holds for each function* $f_j(\cdot)$ *such that*

$$f_j(k, x)=\begin{cases}x & \text{if } k=j\\ 0 & \text{otherwise,}\end{cases} \qquad (1\leqq j\leqq r).$$

Here we have to take $\alpha_j\sigma_j^2+\gamma_{jj}\tau_j^2$ *for* σ^2 *in Lemma 4.1. More generally, the random vector*

$$\left\{\frac{1}{\sqrt{n}}(S_n^{(j)}-n\alpha_j\tau_j),\ 1\leqq j\leqq r\right\}, \tag{4.12}$$

is asymptotically normally distributed, i.e.

$$\lim_{n\to\infty}\mathrm{P}\alpha\left\{\frac{1}{\sqrt{n}}(S_n^{(j)}-n\alpha_i\tau_j)\leqq y_j,\ 1\leqq j\leqq r\right\}=\Phi_r(y_1, \cdots, y_r), \tag{4.13}$$

where $S_n^{(j)}$*'s are defined by*

$$S_n^{(j)}=\sum_{m=1}^{n}f_j(\xi_m, X_m), \tag{4.14}$$

and $\Phi_r(\cdot)$ *denotes the r-dimensional normal distribution function with mean* **0** *and covariance matrix* $(\delta_{jk}\alpha_j\sigma_j^2+\gamma_{jk}\tau_j\tau_k)$, $1\leqq j, k\leqq r$.

PROOF. The first past of the assertion can be proved in the same way as for lemma 4.1, when we notice that

$$\begin{aligned} &E\alpha\{f_j(\xi_n, X_n)\}=\alpha_j\tau_j\\ &\lim_{n\to\infty}E\alpha\left\{\frac{1}{\sqrt{n}}(S_n^{(j)}-n\alpha_j\tau_j)\right\}^2=\alpha_j\tau_j^2+\gamma_{jj}\tau_j^2.\end{aligned} \tag{4.15}$$

The second part may be obtained, by applying the same argument to the function $g(\cdot)$ defined by

$$g(k, x; \boldsymbol{u})=\boldsymbol{u}'\boldsymbol{f}(k, x)=\sum_{j=1}^{r} u_j f_j(k, x)$$

where $\boldsymbol{u}'=(u_1, \cdots, u_r)$ is any r-dimensional real vector on the unit sphere and $\boldsymbol{f}'=(f_1(\cdot), \cdots, f_r(\cdot))$. In fact, for any fixed $\boldsymbol{u}$, the random variable

$$(4.16) \quad V_n=\sum_{j=1}^{n} u_j \frac{1}{\sqrt{n}}\left(S_n^{(j)}-n\alpha_j\tau_j\right)=\frac{1}{\sqrt{n}}\left\{\sum_{m=1}^{n} g(\xi_m, X_m; \boldsymbol{u})-n\sum_{j=1}^{r} u_j\alpha_j\tau_j\right\}$$

is asymptotically normally distributed with mean 0 and variance $\sum_{j,k=1}^{r}(\delta_{jk}\alpha_j\sigma_j^2+\gamma_{jk}\tau_j\tau_k)u_ju_k$, because of (4.15) and the definition of $g(\cdot)$. Therefore, the random vector (4.12) has the r-dimensional normal distribution $\Phi_r(\cdot)$ given by (4.13) as its limiting distribution.

Remark. If we take $U_1(t)$ for $H_j(t)(1\leqq j\leqq r)$, where $U_1(t)=0$ for $t<1$ and $=1$ for $t\geqq 1$, then the Markov process $\{(\xi_n, X_n), n\geqq 1\}$ becomes a Markov chain and $S_n^{(j)}$ is identical to the number of times for the process being in state j. In this case, theorem 4.1 is reduced to the following results:

the random vector $\left\{\frac{1}{\sqrt{n}}(S_n^{(j)}-n\alpha_j), 1\leqq j\leqq r\right\}$ has the r-dimensional normal distribution with mean $\boldsymbol{0}$ and covariance matrix (γ_{ij}). (This result is obtained by P. Billingsley [7]).

THEOREM 4.2. *Under the same conditions of lemma 4.1, the limiting distribution of $N(t)$ certainly exists, and is given by*

$$(4.17) \qquad \lim_{t\to\infty} \mathrm{P}_{\boldsymbol{a}}\left\{\frac{N(t)-t/\tau}{(\sigma/\tau)\sqrt{t/\tau}}\leqq x\right\}=\Phi(x),$$

where $\tau=\sum_{1}^{r}\alpha_j\tau_j$ *and* σ^2 *is given by* (4.5).

PROOF. By the definitions of S_n and $N(t)$ (see section 2), it is easily seen that

$$(4.18) \qquad \{N(t)<n\}\equiv\{S_n\geqq t\},$$

for all $t\geqq 0$ and $n\geqq 1$. Therefore,

$$\Pr\{N(t)<n\}=\Pr\{S_n\geqq t\}, \text{ for all } t\geqq 0 \text{ and } n\geqq 1, \text{ and}$$

$$(4.19) \qquad \Pr\left\{\frac{N(t)-\frac{t}{\tau}}{\frac{\sigma}{\tau}\sqrt{\frac{t}{\tau}}}<\frac{n-\frac{t}{\tau}}{\frac{\sigma}{\tau}\sqrt{\frac{t}{\tau}}}\right\}=\Pr\left\{\frac{S_n-n\tau}{\sigma\sqrt{n}}\geqq\frac{t-n\tau}{\sigma\sqrt{n}}\right\}.$$

Putting

$$n=\left[\frac{t}{\tau}+\frac{\sigma}{\tau}\sqrt{\frac{t}{\tau}}x\right],$$

we obtain

$$\lim_{t\to\infty}\frac{n-\dfrac{t}{\tau}}{\dfrac{\sigma}{\tau}\sqrt{\dfrac{t}{\tau}}}=x \quad \text{and} \quad \lim_{t\to\infty}\frac{t-n\tau}{\sigma\sqrt{n}}=-x\,. \tag{4.20}$$

Substituting (4.20) into (4.19), we obtain

$$\lim_{t\to\infty}\Pr\left\{\frac{N(t)-\dfrac{t}{\tau}}{\dfrac{\sigma}{\tau}\sqrt{\dfrac{t}{\tau}}}<x\right\}=\lim_{n\to\infty}\Pr\left\{\frac{S_n-n\tau}{\sigma\sqrt{n}}\geqq -x\right\}=1-\Phi(-x)\,. \tag{4.21}$$

Since $1-\Phi(-x)=\Phi(x)$ our assertion (4.17) is proved.

5. Conclusion

We are interested in some limiting distributions of M.R.P. $N(t)=\{N_j(t),\ 1\leqq j\leqq r\}$ or the total sojourn times in each state for a special M.R.P., in which transition distribution functions have the form $Q_{ij}(t)=p_{ij}H_i(t)$. To obtain the same results for the general M.R.P. and the limiting joint distribution of $\{S_j(t),\ 1\leqq j\leqq r\}$, we have to use a different method, which will be stated in a paper, to appear in the near future, with examples in practical applications.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] R. Pyke, "Markov renewal process; definitions and preliminary properties," *Ann. Math. Stat.*, Vol. 32 (1961), pp. 1231-1242.

[2] R. Pyke, "Markov renewal processes with finitely many states," *Ann. Math. Stat.*, Vol. 32 (1961), pp. 1243-1259.

[3] G. H. Weiss, "The reliability of components exhibiting cumulative damage effects," *Technometrics*, Vol. 3 (1961), pp. 413-422.

[4] L. Takács, "On a sojourn time problem in the theory of stochastic processes," *Trans. Amer. Math. Soc.*, Vol. 93 (1959), pp. 531-540.

[5] J. L. Doob, "*Stochastic Processes*," 1952, John Wiley, pp. 221-232.

[6] Yu. A. Rozanov, "An application of the central limit theorem," *Proc. of the 4th Berkeley Simposium on Math. Stat. and Probability*, Vol. II (1961), pp. 445-454.

[7] P. Billingsley, "Statistical methods in Markov chains," *Ann. Math. Stat.*, Vol. 32 (1961) pp. 12-35.