ON A SELECTION AND RANKING PROCEDURE FOR GAMMA POPULATIONS*

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Summary

The problem of selecting a subset of k gamma populations which includes the "best" population, i.e. the one with the largest value of the scale parameter, is studied as a multiple decision problem. The shape parameters of the gamma distributions are assumed to be known and equal for all the k populations. Based on a common number of observations from each population, a procedure R is defined which selects a subset which is never empty, small in size and yet large enough to guarantee with preassigned probability that it includes the best population regardless of the true unknown values of the scale parameters θ_i. Expression for the probability of a correct selection using R are derived and it is shown that for the case of a common number of observations the infimum of this probability is identical with the probability integral of the ratio of the maximum of $k-1$ independent gamma chance variables to another independent gamma chance variable, all with the same value of the other parameter. Formulas are obtained for the expected number of populations retained in the selected subset and it is shown that this function attains its maximum when the parameters θ_i are equal. Some other properties of the procedure are proved. Tables of constants b which are necessary to carry out the procedure are appended. These constants are reciprocals of the upper percentage points of $F_{\max}$, the largest of several correlated F statistics. The distribution of this statistic is obtained.

1. Introduction

In many problems of practical interest, the classical tests of homogeneity do not provide satisfactory answers. In recent years, some work has been done toward developing techniques which try to incorporate in the original statistical formulation of the problem the plans of the experimenter for further analysis after the hypothesis of homogeneity i.e., the hypothesis of equality of parameters is tested. In cases where

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the experimenter tests for homogeneity and regardless of the outcome ranks the populations on the basis of further analysis of the same data, it would, clearly, be more realistic to assume at the outset that parameters are unequal and formulate the main problem as a ranking problem.

The formulation considered in this and earlier papers [4], [5], [6], [7] is that of selecting a subset of k populations which contains the "best" population or all those populations which are better than a standard. The best population is usually defined as the one with the largest (or smallest) parameter value. Some further remarks on the motivation of this formulation are given in these papers.

In the present paper we are interested in the scale parameters $\theta_i (i=1, 2, \cdots, k)$ of the k gamma populations with a common known shape parameter r. The object is to select a subset which includes the population with the largest θ with a preassigned probability of at least P^*, regardless of the true values of the k parameters θ_i. The procedure R depends only on the sample means each of which has a gamma distribution.

It should be pointed out that this "selecting a subset" formulation is different from the *indifference* zone formulation for the problem of ranking means and variances of normal populations treated in [2] and [3], respectively. In the latter formulation, an indifference zone in the parameter space is preassigned, the common number of observations needed is tabulated and the final decision is the selection of *single* population which is asserted *to be the best population*. In the formulation of this paper the number of observations is given, constants needed for carrying out the procedure are tabulated, and the final decision is the selection of a *subset* of populations which is asserted *to contain the best population*.

There are several aspects of this formulation which have already been described in [4], [6], [7], and which also apply to the problem treated in this paper. One is that the procedure R can be regarded as an elimination or screening procedure. Another is that a confidence statement can be made after experimentation. A third is that the expected size of the selected subset can be regarded as a measure of the efficiency of the procedure.

The main problem is formally described in section 2 and the procedure R is defined in section 3. In section 4 we derive exact and asymptotic expressions for the probability of a correct selection using procedure R and the infimum of this probability over all points in the parameter space. Section 5 gives the expression for the expected number of populations retained in the selected subset. In section 6 we discuss and prove some other properties of the procedure R. The distribution of $F_{\max}$, the largest of several correlated F statistics is relevant to the procedure R and is discussed in section 7.

2. Formal statement of the problem

Let $\pi_1, \pi_2, \cdots, \pi_k$ denote k given gamma populations with density functions

$$\frac{1}{\Gamma(r)\theta_i}\exp(-x/\theta_i)(x/\theta_i)^{r-1}, \ x>0, \ \theta_i>0, \ i=1,\cdots,k \tag{2.1}$$

with a common parameter $r(>0)$ which is assumed to be known. The ordered parameters θ_i are denoted by

$$\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]} \tag{2.2}$$

(equalities being allowed for mathematical convenience only). It is assumed that there is no prior information available about the correct pairing of the k given populations and the ordered scale parameters $\theta_{[i]}$.

The population with the parameter θ_i equal to $\theta_{[k]}$ is called the best population. The goal is to select a subset of the k populations containing the best population. Any such selection will be called a correct selection (CS). Then the problem is to find a rule R such that for a preassigned probability P^*

$$P\{CS|R\} \geqq P^* \tag{2.3}$$

regardless of the true unknown values of the population parameters. It is assumed that the same number n of observations will be taken from each population.

From each population $\pi_i(i=1, 2, \cdots, k)$ we take n observations $x_{ij}(j=1, 2,\cdots, n)$ and compute $\bar{x}_i=\sum_j x_{ij}/n$. The sample means $\bar{x}_i$ form a set of sufficient statistics for the problem and the rule R (defined in Section 3) depends only on these statistics.

It is clear that we prefer rules which make the size S of the selected subset never empty and as small as possible, subject to satisfying (2.3). [One can always attain any specified P^*, even unity, by putting all the populations in the selected subset.]

3. Procedure *R*

Let the ordered values of the k observed sample means $\bar{x}_i(i=1, 2,\cdots, k)$ all based on a common number n of observations be denoted by

$$\bar{x}_{[1]} \leqq \bar{x}_{[2]} \leqq \cdots \leqq \bar{x}_{[k]}. \tag{3.1}$$

The procedure R is then defined as follows.

Procedure R: "Retain π_i in the selected subset if and only if

$$\bar{x}_i \geqq b\bar{x}_{[k]} \tag{3.2}$$

where $b=b(2nr, k, P^*)$ is a constant with $0<b\leqq 1$ which is determined in advance of experimentation."

The constant b is chosen to be the smallest number which satisfies the basic probability requirement in (2.3) for all true configurations $\boldsymbol{\theta}=(\theta_1, \theta_2, \cdots, \theta_k)$. Tables of the values of b for $\nu/2=nr=1$ (1) 25, $k=2$ (1) 11 and $P^*=.75$, .90, .95 and .99 are given at the end of the paper in Table IA, B, C, D; the reciprocals of the b values can also be regarded as percentage points of a largest Studentized χ^2-statistic or the percentage points of $F_{\max}$ as explained in section 4.

Illustration

From each of the $k=4$ gamma populations with parameter $r=2$, five observations were taken. The observed sample means based on $n=5$ are 2.01, 3.12, 4.13, 5.92. Then the common value of $\nu=2nr$ is 20. If $P^*=.75$, the value of b from Table IA is $b=b(20, 4, .75)=.565$. For this case $\bar{x}_{[k]}=5.92$ so that $b\bar{x}_{[k]}=3.34$. Applying the procedure R we find that the two populations with sample means 4.13 and 5.92 which fall in the interval [3.34, 5.92] are retained in the selected subset. At this point the experimenter can assert with confidence level .75 that one of these two populations has the largest value of θ among the 4 populations.

4. The probability of a correct selection

We will now derive exact and asymptotic expressions for the probability of a correct selection for general k, $\nu(=2nr)$ and any point in the parameter space and also for the infimum of this probability over all points in the parameter space. It will be convenient here to discuss the case of general sample sizes n_i from π_i so that the parameters $\nu_i/2=(n_i r)$ are associated with the distribution of the sample means which follow gamma distributions with scale parameters θ_i/n_i $(i=1, 2, \cdots, k)$. Under this general framework, we can just as easily write down the expression for the probability of a correct selection $P\{CS|R\}$ if we were to apply the procedure R of (3.1) with some fixed b in the general case when ν_i are not necessarily equal. This is done in section 4.1. In section 4.2, it is pointed out that if we keep ν_i fixed, the $P\{CS|R\}$ approaches its infimum as $\theta_{[i]}$ approach equality (with $\theta_{[k]}=\theta_{[k-1]}$ only in the limit). It should be noted that the infimum is not the same as the value of the $P\{CS|R\}$ when the parameters θ_i are equal (the latter has not been defined but by any reasonable extension of our definition should be unity); mathematically, we can use the configuration with all θ_i equal provided we "tag" a particular one of the populations and regard it as being the best. In section 4.3 the $P\{CS|R\}$ is shown to be equivalent to the cumulative distribution function (c.d.f.) of $F_{\max}$, or the c.d.f. of the largest of several correlated F or that of a Studentized largest chi-square statistic and this leads to an approximation of the $P\{CS|R\}$ based on asymptotic normality.

4.1 *Probability of a correct selection*

Let $\bar{x}_{(i)}$ denote the (unknown) sample mean that is associated with the ith smallest population parameter, $\theta_{[i]}$; let $\nu_{[i]}$ denote twice the value of the other parameter associated with $\bar{x}_{(i)}$. Then for this case the procedure R with some fixed $b(0<b\leqq 1)$ gives the following result.

THEOREM 1.

$$P\{CS|R\}=\int_0^\infty g_{\nu_{(k)}}(x)\prod_{\alpha=1}^{k-1}\left[G_{\nu_{(\alpha)}}\left(\frac{x\nu_{(\alpha)}\theta_{[k]}}{b\nu_{(k)}\theta_{[\alpha]}}\right)\right]dx \tag{4.1}$$

where $G_\nu(x)$ and $g_\nu(x)$ are the cumulative distribution function and the density, respectively, of a standardized gamma chance variable (i.e. with $\theta=1$) and with parameter $\nu/2$.

PROOF: The procedure R with some fixed $b(0<b\leqq 1)$ yields a correct selection if and only if the event

$$\bar{x}_{(k)}\geqq b\,\underset{\alpha}{\mathrm{Max}}\,\bar{x}_{(\alpha)} \qquad (\alpha=1,2,\cdots,k) \tag{4.2}$$

occurs. Since $0<b\leqq 1$, the occurrence of the event (4.2) is equivalent to the occurrence of the event

$$\bar{x}_{(k)}\geqq b\,\underset{\alpha}{\mathrm{Max}}\,\bar{x}_{(\alpha)} \qquad (\alpha=1,2,\cdots,k-1). \tag{4.3}$$

Hence, the probability of a correct selection is given by

$$\begin{aligned} P\{CS|R\} &= P\{\bar{x}_{(k)}\geqq b\,\underset{\alpha}{\mathrm{Max}}\,\bar{x}_{(\alpha)}\ (\alpha=1,2,\cdots,k-1)\} \\ &= P\left\{\frac{\nu_{(k)}\bar{x}_{(k)}}{\theta_{[k]}}\geqq b\frac{\nu_{(\alpha)}\bar{x}_{(\alpha)}}{\theta_{[\alpha]}}\cdot\frac{\nu_{(k)}\theta_{[\alpha]}}{\nu_{[\alpha]}\theta_{[k]}}\ (\alpha=1,2,\cdots,k-1)\right\} \\ &= \int_0^\infty g_{\nu_{(k)}}(x)\prod_{\alpha=1}^{k-1}\left[G_{\nu_{(\alpha)}}\left(\frac{x\nu_{(\alpha)}\theta_{[k]}}{b\nu_{(k)}\theta_{[\alpha]}}\right)\right]dx. \end{aligned} \tag{4.4}$$

This proves the theorem.

4.2 *Exact expressions for the infimum of the $P\{CS|R\}$*

It follows from (4.4) that for fixed b and fixed $\nu_{(\alpha)}$ $(\alpha=1,2,\cdots,k)$, the $P\{CS|R\}$ depends only on the ratios of the $\theta_{[\alpha]}$ and that it approaches its infimum by setting $\theta_{[1]}=\theta_{[2]}=\cdots=\theta_{[k-1]}$ letting $\theta_{[k-1]}\to\theta_{[k]}$, with equality only in the limit. Hence, for the case of a common ν [letting $\underset{\Omega}{\mathrm{Inf}}$ denote the infimum over all points in the parameter space $\theta_{[\alpha]}>0$ $(\alpha=1,2,\cdots,k)$], the theorem

THEOREM 2.

$$\text{(4.5)} \qquad \operatorname*{Inf}_{\Omega} P\{CS|R\} = \int_0^\infty G_\nu^{k-1}\left(\frac{x}{b}\right) g_\nu(x)dx.$$

It follows that if for given ν and k, we solve for b by equating the right hand side of (4.5) to P^* and use this value of b in the definition of R, then R will satisfy the basic probability requirement.

Suppose in the next to the last expression in (4.4) we make the transformation

$$\text{(4.6)} \qquad y_k = \frac{\nu_{(k)}\bar{x}_{(k)}}{\theta_{[k]}}, \quad y_\alpha = \frac{\nu_{(\alpha)}\bar{x}_{(\alpha)}\theta_{[k]}}{\nu_{(k)}\bar{x}_{(k)}\theta_{[\alpha]}} \quad (\alpha=1, 2, \cdots, k-1),$$

then the limits of y_k are from 0 to ∞ and upon integrating out y_k we obtain an alternative expression for the $P\{CS|R\}$ as a $(k-1)$ fold integral

$$\text{(4.7)} \qquad P\{CS|R\} = \int_0^{\Psi_1} \cdots \int_0^{\Psi_{k-1}} \frac{\Gamma(m_1+m_2+\cdots+m_k)}{\prod_{\alpha=1}^{k}\Gamma(m_\alpha)} \cdot \left[\frac{\prod_{\alpha=1}^{k-1}[y_\alpha^{m_{(\alpha)}-1}dy_\alpha]}{(1+y_1+\cdots+y_{k-1})^{m_1+\cdots+m_k}}\right]$$

where $m_\alpha = \nu_\alpha/2$ and $m_{(\alpha)} = \nu_{(\alpha)}/2$ and $\Psi_\alpha = \theta_{[k]}\nu_{(\alpha)}/[\theta_{[\alpha]}\nu_{(k)}b]$.

An alternative expression for (4.5) follows from (4.7), so that for the case of a common $\nu = 2m$, we have

$$\text{(4.8)} \qquad \operatorname*{Inf}_{\Omega} P\{CS|R\} = \int_0^{1/b} \cdots \int_0^{1/b} \frac{\Gamma(km)}{[\Gamma(m)]^k} \frac{\prod_{\alpha=1}^{k-1}[y_\alpha^{m-1}dy_\alpha]}{(1+y_1+\cdots+y_{k-1})^{km}}$$

4.3 *Approximation to inf $P\{CS|R\}$ based on asymptotic normality*

Let x_j $(j=0, 1, \cdots, p; p=k-1)$ denote k independent gamma chance variables with a common value of the parameter $=\nu/2$. It follows from (4.2) and the fact that the $P\{CS|R\}$ approaches its infimum as the θ_i approach equality that we can write the basic probability requirement in the form

$$\text{(4.9)} \qquad P\left\{\frac{\operatorname*{Max}_{j=1,2,\cdots,p} x_j}{x_0} \leqq \frac{1}{b}\right\} = P^*.$$

Hence, the determination of b to satisfy (2.3) for all points in the parameter space is equivalent to the determination of the reciprocal of the upper $\alpha=1-P^*$ percentage point of $F_{\max}=\max(F_1, F_2, \cdots, F_p)$ or, equivalently, the percentage point of the studentized largest chi-square statistic with ν degrees of freedom for all chi-squares.

Using the fact [1] that $\log x$ is approximately normally distributed for large values of $\nu=2m$, we can obtain a normal approximation to the infimum of the probability of a corret selection (the details of the derivation are same as in [3]) as

(4.10) $$\inf_{\Omega} P\{CS|R\} \cong \int_{-\infty}^{\infty} [F(x+d)]^{k-1} f(x)dx$$

where $F(x)$ and $f(x)$ are the standard normal c.d.f. and the density function, respectively, and

(4.11) $$d = -\sqrt{(\nu-1)/2}\ \log_e b$$

For large values of ν an approximate solution to the smallest value of b satisfying the probability requirement (2.3) can be obtained by equating the right hand side of (4.10) to P^* and using (4.11) to solve for b. Values of d are tabulated in Table I of [2] for $k=2$ (1) 10 and many values of P^* [our d corresponds to $\lambda\sqrt{N}$ of [2]) and also in Table AI of [4] for $k=2$ (1) 51 for $P^*=.75$, .90, .95, .975, and 99 [our $k-1$ and P^* correspond to n and $1-\alpha$ of [4], respectively].

Using (4.10) and (4.11) and the tables in [4] and [2] we have computed the approximate b values for $\nu=50$ and $P^*=.75$, .90, .95, and .99. These are listed at the bottom of Tables IA, B, C and D. In all cases the approximate values agree with the exact values to one unit in the second decimal place.

5. Expected size of the selected subset

For the procedure R the size S of the selected subset is a chance variable which can take on only integer values 1 to k, inclusive. For any fixed values of ν, k, and P^*, the expected size of the selected subset is a function of the true configuration $\boldsymbol{\theta}=\{\theta_1, \theta_2, \cdots, \theta_k\}$ and this function can be regarded as a criterion of the efficiency of any procedure which satisfies the basic probability requirement (2.3). In analogy with power function consideration, one secondary problem is to find the smallest common sample size n necessary to control $E(S)$ at some preassigned level for a particular alternative in the parameter space; alternatively, we may wish to control the maximum $E(S)$ over all parameter points in the subset $\Omega(\delta)$ of Ω given by $\delta\theta_{[i]} \leqq \theta_{[k]}$ $(i=1, 2, \cdots, k-1)$ with $\delta>1$.

5.1 *Exact expression for the expected size*

Let Y_i denote a chance variable which equals 1 if π_i is included in the selected subset and equals 0 otherwise. Then $S=\sum_{i=1}^{k} Y_i$ and hence for any values of ν, k, P^* and $\boldsymbol{\theta}$,

(5.1) $$\begin{aligned} E(S) &= E\left(\sum_{i=1}^{k} Y_i\right) = \sum_{i=1}^{k} E(Y_i) \\ &= \sum_{i=1}^{k} P(\pi_i \text{ is included in the selected subset}). \end{aligned}$$

Using an argument similar to the one in section 4.1 for obtaining the exact $P\{CS\}$, we obtain from (5.1), the following result:

THEOREM 3.

$$E(S)=\sum_{i=1}^{k}\int_0^\infty g_\nu(x)\prod_{\substack{j=1\\ j\neq i}}^{k}[G_\nu(\theta'_{ij}x/b)]dx \tag{5.2}$$

where $\theta'_{ij}=\theta_i/\theta_j$. [It should be noted that if $\theta_{ij}=\theta_{[i]}/\theta_{[j]}$ is used in (5.2), then the ith term on the right hand side of (5.2) is the probability that the population associated with $\theta_{[i]}$ is included in the selected subset and hence the sum is again $E(S)$.]

COROLLARY.

For the particular configuration

$$\theta_{[k]}=\delta\theta_{[\alpha]}, \qquad (\alpha=1,2,\cdots,k-1) \tag{5.3}$$

we obtain from (5.2) *the result*

$$E(S)=\int_0^\infty G_\nu^{k-1}(\delta x/b)g_\nu(x)dx + (k-1)\int_0^\infty G_\nu(x/b\delta)G_\nu^{k-2}(x/b)g_\nu(x)dx \tag{5.4}$$

where the first term on the right side of (5.4) *is the* $P\{CS\}$ *for the configuration* (5.3). *In particular, if all the* θ_i *are equal, then* $\delta=1$ *and* $E(S)$ *equals* k P^*.

5.2 *Maximum value of* $E(S)$

It will now be shown that the maximum value of $E(S)$ takes place when all the parameters θ_i are equal. If we set the m smallest parameters $\theta_i(1\leqq m<k)$ equal to a common value θ (say) and define $\theta_{i\cdot}=\theta_{[i]}/\theta=1/\theta_{\cdot i}$, then writing $Q=E(S|\theta_{[1]}=\theta_{[2]}=\cdots=\theta_{[m]}=\theta)$, we obtain from the remark following (5.2),

$$Q=\sum_{i=m+1}^{k}\int_0^\infty G_\nu^m(\theta_{i\cdot}x/b)g_\nu(x)\prod_{\substack{j=m+1\\ j\neq i}}^{k}G_\nu(\theta_{ij}x/b)dx + m\int_0^\infty G_\nu^{m-1}(x/b)g_\nu(x)\prod_{j=m+1}^{k}G(\theta_{\cdot j}x/b)dx \tag{5.5}$$

We now show that the right hand member of (5.5) is a non-decreasing function of θ for $1/k\leqq P^*\leqq 1$ (actually, the proof shows that it is a strictly increasing function of θ for $1/k<P^*<1$). This proves that it is maximum when $\theta=\theta_{[m+1]}$ and since this holds for any integer $m<k$, the desired result will follow.

To show that Q is monotonic we differentiate Q with respect to θ and show that the result is positive for $1/k<P^*<1$ and $\theta<\theta_{[m+1]}$. Differentiation gives

(5.6)
$$\frac{dQ}{d\theta}=\frac{m}{b}\sum_{i=m+1}^{k}\left\{-\frac{\theta_{[i]}}{\theta^2}\int_0^\infty xG_\nu^{m-1}(\theta_i.x/b)g_\nu(x)g_\nu(\theta_i.x/b)\cdot\left[\prod_{\substack{i=m+1\\ j\neq i}}^{k}G_\nu(\theta_{ij}x/b)\right]dx\right.$$
$$\left.+\frac{1}{\theta_{[i]}}\int_0^\infty xG_\nu^{m-1}(x/b)g_\nu(x)g_\nu(\theta_{.i}x/b)\prod_{\substack{j=m+1\\ j\neq i}}^{k}G_\nu(\theta_{.j}x/b)dx\right\}.$$

For $P^*<1$, it is clear that $b>0$. If we let $x=x'\ \theta_{i}.$ in the second integral and drop primes then (5.6) becomes

(5.7)
$$\frac{dQ}{d\theta}=-\frac{m}{b}\sum_{i=m+1}^{k}\frac{\theta_{[i]}}{\theta^2}\int_0^\infty xG_\nu^{m-1}(\theta_i.x/b)\prod_{\substack{j=m+1\\ j\neq i}}^{k}[G_\nu(\theta_{ij}x/b)].$$
$$\{g_\nu(x)g_\nu(\theta_i.x/b)-g_\nu(\theta_i.x)g_\nu(x/b)\}dx.$$

It suffices to prove that for each $x>0$ the expression in (5.7) is *negative* or that the ratio of the two terms in braces in (5.7) is less than unity.

(5.8) $\log\{g_\nu(x)g_\nu(\theta_i.x/b)\div g_\nu(\theta_i.x)g_\nu(x/b)\}=x\left(\frac{1}{b}-1\right)(1-\theta_i.),\quad i=m+1,\cdots,k.$

Since $b<1$ for $P^*>1/k$, it follows from (5.8) that for $\theta<\theta_{[m+1]}$ the right side of (5.8) is negative for each $x>0$ and hence the expected size of the selected subset is an increasing function of θ. Thus, we have proved the following theorem.

THEOREM 4.

For given k, P^* $(1/k<P^*<1)$, and common ν, the expected size of the selected subset

$$E(S|\theta_{[1]}=\theta_{[2]}=\cdots=\theta_{[m]}=\theta,\ m<k)$$

in using the procedure R is strictly increasing in θ.

COROLLARY 1.

(5.9)
$$\operatorname*{Max}_{\Omega}\ E(S)=k\int_0^\infty G_\nu^{p}(x/b)g_\nu(x)dx=k\ P^*.$$

COROLLARY 2.

In the subset $\Omega(\delta)$ defined by $\delta\theta_{[i]}\leqq\theta_{[k]}$ $(i=1,2,\cdots,k-1)$, the function $E(S)$ takes on its maximum value when $\delta\theta_{[i]}=\theta_{[k]}$ $(i=1,2,\cdots,k-1)$ and hence

(5.10)
$$\operatorname*{Max}_{\Omega(\delta)}\ E(S)=\int_0^\infty G_\nu^{k-1}(\delta x/b)g_\nu(x)dx+(k-1)\int_0^\infty G_\nu[x/(b\delta)]G_\nu^{k-2}(x/b)g_\nu(x)dx.$$

COROLLARY 3.

The smallest value of the common sample size $n=\nu/(2r)$ such that

$E(S) \leqq 1+\varepsilon$ for all points in $\Omega(\delta)$ where $\varepsilon>0$ and $\delta>1$ are preassigned is given by

$$(5.11) \quad \int_0^\infty G_\nu^{k-1}(\delta x/b) g_\nu(x) dx + (k-1)\int_0^\infty G_\nu[x/(b\delta)] G_\nu^{k-2}(x/b) g_\nu(x) dx = 1+\varepsilon .$$

5.3 *An approximation to $E(S)$ based on asymptotic normality*

Using the fact that $\log_e[\bar{x}_{(i)}/\theta_{[i]}]$, $(i=1, 2, \cdots, k)$ are independently and asymptotically normally distributed with a common mean and common variance, we now derive the following approximation for $E(S)$,

$$(5.12) \qquad E(S) \cong \sum_{i=1}^{k} \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{k} \left[F\left(x + \frac{\log(\theta_{ij}/b)}{\sqrt{2/(\nu-1)}} \right) \right] f(x) dx .$$

For the particular configuration (5.3) this reduces to

$$(5.13) \qquad E(S) \cong \int_{-\infty}^{\infty} \left[F\left(x + \frac{\log_e(\delta/b)}{\sqrt{2/(\nu-1)}} \right) \right]^{k-1} f(x) dx$$
$$+ (k-1)\int_{-\infty}^{\infty} \left[F\left(x - \frac{\log_e b}{\sqrt{2/(\nu-1)}} \right) \right]^{k-2} F\left(x - \frac{\log_e(b\delta)}{\sqrt{2/(\nu-1)}} \right) f(x) dx .$$

6. Properties of the procedure *R*

1. In section 5 we proved that

$$(6.1) \qquad \operatorname*{Max}_{\Omega} E(S|R) = k\ P^* .$$

Thus subject to the basic probability requirement, viz., $\operatorname*{Inf}_{\Omega} P\{CS|R\} = P^*$, the procedure R satisfies the condition that the expected size of the selected subject is $\leqq k\ P^*$ for all parameter points in Ω.

2. *Property of monotonicity*

THEOREM 5.

For $\theta_{[i]} \geqq \theta_{[j]}$, we have

$P\{$including the population with $\theta_{[i]}$ in the subset$\} \geqq P\{$including the population with $\theta_{[j]}$ in the selected subset$\}$.

PROOF.

We can write

$$(6.2) \quad P\{\text{selecting the population with } \theta_{[i]}\} = \int_0^\infty g_\nu(x) \prod_{\substack{\alpha=1 \\ \alpha\neq i}}^{k} [G_\nu(\theta_{i\alpha} x/b)] dx$$

where
$$\theta_{i\alpha} = \frac{\theta_{[i]}}{\theta_{[\alpha]}} .$$

Again the probability of selecting the population with $\theta_{[j]}$ is given by (6.2) with i replaced by j in the expression on the right hand side of that equation.

Noting the fact that $\theta_{i_\alpha} \geqq \theta_{j\alpha}$ for all α, we see from (6.2) that at each $x>0$ the integrand in (6.2) is $\geqq$ the integrand in the case when we are concerned with the selection of $\theta_{[j]}$.

This completes the proof of the theorem.

3. The subset selected by the procedure R has the maximum probability $P\{CS|R\}$ among all different subsets of size S (S not known in advance) which are $\binom{k}{S}$ in number.

Note that the procedure R selects populations with sample means $\bar{x}_{[k]} \geqq \bar{x}_{[k-1]} \geqq \cdots \geqq \bar{x}_{[k-S+1]}$. It follows from similar arguments as in 2 above that $P\{CS|R\}$ is maximized among all different subsets of size S.

7. Distribution of $V=x_{\max}/x_0=\chi^2_{\max}/\chi^2_0=F_{\max}$

As pointed out in section 3 and 4, the procedure R depends on the constants b. If r_i and n_i are such that $2r_i n_i=\nu_i=\nu$ $(i=1, 2, \cdots, k)$, then the constants b depend on ν, k and P^* and are the solutions of (4.9), which is

$$P\left\{\frac{\max\limits_{j=1,2,\cdots,p} x_j}{x_0} \leqq \frac{1}{b}\right\}=P^* \qquad \text{where } p=k-1. \tag{7.1}$$

Hence, the reciprocals of b-values are the upper $\alpha=1-P^*$ percentage points of

$$V=x_{\max}/x_0=\chi^2_{\max}/\chi^2_0=\max(F_1, F_2, \cdots, F_p) \tag{7.2}$$

where $x_1, x_2, \cdots, x_p, x_0$ are independent standard gamma chance variables with a common value of the parameter $v/2$ on all $(p+1)$ gamma variables and where $x_{\max}$ is the largest of $x_1, x_2, \cdots, x_p$.

From the above, it is clear that we are interested in the distribution of V which is the maximum of several correlated F-statistics. The theory and discussion pertaining to this distribution are similar as for the case of $F_{\min}$ which is treated in [8]. For the sake of brevity we shall give only a few basic results and formulas.

As shown earlier in sections 3 and 4, we can write (7.1) as

$$\int_0^\infty \left[G_\nu\left(\frac{x}{b}\right)\right]^p g_\nu(x)dx=P^*=1-\alpha \tag{7.3}$$

where $G_\nu(x)$ and $g_\nu(x)$ refer to the c.d.f. and the density, respectively, of a standard gamma chance variable with parameter $\nu/2$.

If ν is an even integer $=2m$ (say), then (7.3) reduces to

$$\frac{b^m}{\Gamma(m)}\int_0^\infty \left[1-\sum_{j=0}^{m-1}\frac{e^{-y}y^j}{j!}\right]^p e^{-by}y^{m-1}dy=1-\alpha. \tag{7.4}$$

For $m=1$, the gamma distribution reduces to an exponential distribution, and in this case (7.4) simplifies to

$$b\int_0^\infty [1-e^{-y}]^p e^{-by}dy=1-\alpha. \tag{7.5}$$

From (7.5) we find that

$$b\sum_{j=1}^{p}(-1)^{j-1}\binom{p}{j}\frac{1}{j+b}=\alpha. \tag{7.6}$$

Special cases for $m=1$ where we can write down the b-values explicitly are

$$\begin{cases} \text{For } p=1,\ b=\dfrac{\alpha}{1-\alpha}\,. \\ \text{For } p=2,\ b=-\dfrac{3}{2}+\dfrac{1}{2}\sqrt{9+\dfrac{8\alpha}{1-\alpha}}\,. \end{cases} \tag{7.7}$$

Alternatively, $b(0<b\leqq 1)$ is the solution of

$$\frac{b\Gamma(p+1)\Gamma(b)}{\Gamma(p+b+1)}=1-\alpha. \tag{7.8}$$

For large values of p the equation that results after taking logarithms of both sides in (7.8) is better suited than (7.8) for solving for b by iterative methods on high speed computers. Also (7.8) reduces to

$$\frac{\Gamma(p+1)}{(b+1)(b+2)\cdots(b+p)}=1-\alpha. \tag{7.9}$$

We will now show that the integral in (7.3) or (7.4) can be reduced to a finite series when ν is an even integer $=2\,m$ (say). If we expand the first factor in the integrand in (7.4) and collect the coefficients of y^r we obtain

$$\frac{b^m}{\Gamma(m)}\sum_{t=0}^{p}(-1)^t\binom{p}{t}\sum_{r=0}^{t(m-1)}\int_0^\infty a_r(m,t)e^{-(t+b)y}y^{r+m-1}dy=1-\alpha. \tag{7.10}$$

where

$$\begin{aligned} a_r(m,t) &= \text{coefficient of } y^r \text{ in } \left(\sum_{j=0}^{m-1}\frac{y^j}{j\,!}\right)^t \\ &= \sum_{\substack{0\leq j_i\leq m-1 \\ j_1+j_2+\cdots+j_t=r}} \frac{1}{\prod_{i=1}^{t} j_i\,!} \\ &= \frac{t^r}{r\,!}\,A_r(m,t), \end{aligned} \tag{7.11}$$

Note that the coefficients $A_r(m, t)$ are all positive and at most equal to one; in particular for $r \leqq m-1$, their value is one for any t. With a slight simplification we see that b is the solution of

$$(7.12) \quad \sum_{t=1}^{p}(-1)^{t-1}\binom{p}{t}\sum_{r=0}^{t(m-1)} A_r(m, t)\frac{\Gamma(r+m)}{r\,!\,\Gamma(m)}\;\frac{(t/b)^r}{(1+(t/b))^{r+m}}=\alpha.$$

Using iterative procedures on an IBM 7090 the b-values were computed. These are tabulated in Tables IA, B, C and D. The main procedure used is to write (7.12) as

$$(7.13) \quad b_{i+1}=\frac{1}{\alpha}\sum(-1)^{t-1}\binom{p}{t}\sum_{r=0}^{t(m-1)} A_r(m, t)\frac{\Gamma(r+m)t^r}{r\,!\,\Gamma(m)b_i^{r-1}(1+(t/b_i))^{r+m}}\ .$$

Starting with a guessed value of $b_0(0<b_0<1)$, we compute b_1, by using (7.13) and find the value of $|b_1-b_0|$. If $|b_1-b_0|$ is not small enough, we compute b_2 by substituting b_1 on the right hand side and obtain b_2. The process is discontinued after the ith iteration if $|b_i-b_{i-1}|$ is less than a preassigned constant. For fixed value of α and $\nu=2\,m$, it is easy to see from (7.3) that b decreases monotonically as p increases. The tables show that there is also a monotonicity in ν for each fixed value of p and α considered. For $p=1$, the entries are the reciprocals of the upper percentage points of the usual $F_{\nu_1,\,\nu_2}$ statistic with equal degrees of freedom $(\nu_1=\nu_2=\nu)$. Also for $m=1$, and $p=1, 2$ we can find the values exactly by using formulas (7.7). Thus one can always obtain a "good guess" value by starting with an entry in an adjoining row or column that has already been computed; this helps to speed up the convergence of the iterative procedure.

In a more generalized form one can consider the statistic

$$(7.14) \quad V=F_{\max}=\max_{i=1,2,\cdots,p}\left\{\frac{\nu_0}{\nu_i}\,\frac{x_{\nu_i}}{x_{\nu_0}}\right\}$$

where x_{ν_i} $(i=0, 1,\cdots, p)$ denote $(p+1)$ independent gamma or chi-squares chance variables with parameters $\nu_i/2$. For $\nu_1=\nu_2=\cdots=\nu_p=\nu=2\,m$, and $\nu_0=2\,m_0$, the probability integral of this more general form is

$$(7.15) \quad P\{V\leqq v\}=1 + \sum_{t=1}^{p}(-1)^t\binom{p}{t}\sum_{r=0}^{t(m-1)} A_r(m, t)\frac{\Gamma(r+m_0)}{r\,!\,\Gamma(m_0)}\;\frac{(vmt/m_0)^r}{(1+(vmt/m_0))^{r+m_0}}\ .$$

A very brief table of the percentage points of V when the numerator χ^2 all have equal degrees of freedom which are not necessarily the same as the degrees of freedom of the denominator χ^2 is given by Ramachandran [9] only for the case of $p=2$ and $\alpha=.05$. The values in

[9] are given to two decimal places and the reciprocals of 10 entries agree with the corresponding values in Tables IC to three decimal places except in the case $\nu=20$ where the agreement is only to two places.

Earlier Nair [8b] computed the upper 5 and 1 percent points of $V=F_{\max}$ (as defined in (7.14) of this paper) when $\nu_i=1$, $i=1,\cdots,10$ and $\nu_0=10, 12, 15, 20, 30, 60, \infty$. Nair's tables are given in Pearson and Hartley [8c, p. 164]. Approximations for the above maximum F ratio for the case $\nu_i=2$, $i=1,2,\cdots,p$, were considered by Hartley [8a] and Finney [3b] David [3a] also briefly discussed the distribution of $F_{\max}$ and gave tables of the upper 5 and 1 percentage points for selected values of ν_i and ν_0 that correspond to the degrees of freedom for the various mean square and error square that enter into a randomized block, Latin square and Graeco Latin square.

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TABLE IA

Reciprocals of Upper $\alpha(=1-P^*)$ Percentage Points of the Statistic $V=x_{\max}/x_0=\chi^2_{\max}/\chi^2_0=\max(F_1, F_2,\cdots, F_p)$ where $x_1, x_2,\cdots, x_p, x_0$ are Independent Standard Gamma Chance Variables with a Common Value of the Parameter $\nu/2$ on All $p+1$ Independent Gamma Chance Variables and where $x_{\max}$ is the Largest of $x_1, x_2, \cdots, x_p$.

$\alpha=1-P^*=.25$

ν \ p	1	2	3	4	5	6	7	8	9	10
2	.333	.208	.166	.145	.131	.122	.115	.109	.104	.101
4	.484	.350	.300	.271	.253	.239	.229	.221	.214	.208
6	.561	.431	.379	.349	.329	.314	.303	.294	.286	.280
8	.610	.486	.434	.404	.383	.368	.356	.347	.339	.332
10	.645	.526	.475	.445	.425	.409	.397	.388	.380	.372
12	.671	.557	.507	.478	.458	.442	.430	.421	.413	.406
14	.692	.582	.534	.505	.485	.470	.458	.448	.440	.433
16	.709	.603	.556	.528	.508	.493	.482	.472	.464	.457
18	.724	.621	.575	.547	.528	.513	.502	.492	.484	.477
20	.736	.637	.592	.565	.545	.531	.520	.510	.502	.495
22	.747	.650	.606	.580	.561	.547	.535	.526	.518	.511
24	.757	.662	.620	.593	.575	.561	.550	.540	.533	.526
26	.765	.673	.631	.605	.587	.573	.562	.553	.546	.539
28	.773	.683	.642	.616	.599	.585	.574	.565	.558	.551
30	.780	.692	.652	.627	.609	.596	.585	.576	.568	.562
32	.786	.700	.660	.636	.618	.605	.595	.586	.579	.572
34	.792	.708	.669	.644	.627	.614	.604	.595	.588	.582
36	.797	.715	.676	.652	.636	.623	.612	.604	.596	.590
38	.802	.721	.683	.660	.643	.630	.620	.612	.605	.598
40	.807	.727	.690	.667	.650	.638	.628	.619	.612	.606
42	.811	.733	.696	.673	.657	.644	.635	.626	.619	.613
44	.815	.738	.702	.679	.663	.651	.641	.633	.626	.620
46	.819	.743	.707	.685	.669	.657	.647	.639	.632	.626
48	.822	.748	.712	.690	.675	.663	.653	.645	.638	.632
50	.825	.752	.717	.696	.680	.668	.659	.651	.644	.638
50†	.825	.748	.712	.689	.672	.659	.649	.640	.633	.627

This table gives the values of b for which

$$\int_0^\infty \left[G_\nu\left(\frac{x}{b}\right)\right]^p g_\nu(x)dx=1-\alpha$$

where $G_\nu(x)$ and $g_\nu(x)$ refer to the c.d.f. and the density, respectively, of a standard gamma chance variable with parameter $\nu/2$.

† First term normal approximation based on (4.10) and (4.11)

TABLE IB

Reciprocals of Upper $\alpha(=1-P^*)$ Percentage Points of the Statistic $V=x_{max}/x_0=\chi^2_{max}/\chi^2_0=\max(F_1, F_2,\cdots, F_p)$ where $x_1, x_2,\cdots, x_p, x_0$ are Independent Standard Gamma Chance Variables with a Common Value of the Parameter $\nu/2$ on All $p+1$ Independent Gamma Chance Variables and where x_{max} is the Largest of $x_1, x_2, \cdots, x_p$.

$\alpha=1-P^*=.10$

ν \ p	1	2	3	4	5	6	7	8	9	10
2	.111	.072	.059	.052	.047	.044	.041	.039	.038	.036
4	.244	.183	.159	.145	.135	.128	.123	.119	.116	.113
6	.327	.260	.232	.215	.203	.195	.188	.183	.178	.174
8	.386	.317	.286	.268	.255	.246	.239	.232	.228	.223
10	.430	.360	.329	.310	.297	.287	.279	.273	.268	.263
12	.466	.396	.364	.345	.332	.321	.313	.307	.301	.296
14	.494	.426	.394	.374	.361	.350	.342	.336	.330	.325
16	.519	.451	.419	.400	.386	.376	.367	.360	.355	.350
18	.539	.472	.441	.422	.408	.398	.389	.382	.376	.371
20	.558	.492	.460	.441	.428	.417	.409	.402	.396	.391
22	.573	.508	.478	.459	.445	.434	.426	.419	.414	.408
24	.588	.524	.493	.474	.461	.450	.442	.435	.429	.424
26	.600	.537	.507	.488	.475	.465	.456	.450	.444	.439
28	.612	.550	.520	.502	.488	.478	.470	.463	.457	.452
30	.622	.561	.532	.514	.500	.490	.482	.475	.469	.464
32	.632	.572	.543	.525	.511	.501	.493	.486	.481	.476
34	.641	.582	.553	.535	.522	.512	.504	.497	.491	.486
36	.649	.591	.562	.544	.532	.522	.514	.507	.501	.496
38	.657	.599	.571	.553	.540	.531	.523	.516	.510	.506
40	.664	.607	.579	.562	.549	.539	.531	.525	.519	.514
42	.671	.614	.587	.570	.557	.547	.540	.533	.528	.523
44	.677	.622	.594	.577	.565	.555	.547	.541	.535	.530
46	.683	.628	.601	.584	.572	.562	.554	.548	.543	.538
48	.689	.634	.608	.591	.578	.569	.561	.555	.550	.545
50	.694	.640	.614	.597	.585	.576	.568	.562	.556	.552
50†	.693	.637	.609	.591	.578	.568	.560	.553	.547	.542

This table gives the values of b for which

$$\int_0^\infty \left[G_\nu\left(\frac{x}{b}\right)\right]^p g_\nu(x)dx=1-\alpha$$

where $G_\nu(x)$ and $g_\nu(x)$ refer to the c.d.f. and the density, respectively, of a standard gamma chance variable with parameter $\nu/2$.

† First term normal approximation based on (4.10) and (4.11)

TABLE IC

Reciprocals of Upper $\alpha(=1-P^*)$ Percentage Points of the Statistic $V=x_{\max}/x_0=\chi^2_{\max}/\chi^2_0=\max(F_1, F_2, \cdots, F_p)$ where $x_1, x_2, \cdots, x_p, x_0$ are Independent Standard Gamma Chance Variables with a Common Value of the Parameter $\nu/2$ on All $p+1$ Independent Gamma Chance Variables and where $x_{\max}$ is the Largest of $x_1, x_2, \cdots, x_p$.

$\alpha=1-P^*=.05$

ν \ p	1	2	3	4	5	6	7	8	9	10
2	.053	.035	.028	.025	.023	.021	.020	.019	.018	.018
4	.156	.119	.104	.095	.089	.085	.082	.079	.076	.074
6	.233	.188	.168	.156	.148	.142	.138	.134	.131	.128
8	.291	.242	.220	.206	.197	.190	.184	.180	.176	.173
10	.336	.285	.261	.247	.237	.229	.223	.218	.214	.210
12	.372	.320	.296	.281	.271	.263	.256	.251	.247	.243
14	.403	.350	.326	.310	.300	.291	.285	.279	.275	.271
16	.428	.376	.351	.336	.325	.316	.310	.304	.300	.296
18	.451	.399	.374	.358	.347	.339	.332	.326	.322	.317
20	.471	.419	.394	.378	.367	.359	.352	.346	.341	.337
22	.488	.437	.412	.396	.385	.377	.370	.364	.359	.355
24	.504	.453	.428	.413	.402	.393	.386	.380	.376	.371
26	.518	.468	.443	.428	.417	.408	.401	.395	.390	.386
28	.531	.481	.457	.442	.430	.422	.415	.409	.404	.400
30	.543	.494	.470	.454	.443	.434	.428	.422	.417	.413
32	.554	.505	.481	.466	.455	.446	.439	.434	.429	.424
34	.564	.516	.492	.477	.466	.457	.450	.445	.440	.436
36	.574	.526	.502	.487	.476	.468	.461	.455	.450	.446
38	.582	.535	.512	.497	.486	.477	.470	.465	.460	.456
40	.591	.544	.520	.506	.495	.486	.480	.474	.469	.465
42	.598	.552	.529	.514	.503	.495	.488	.483	.478	.474
44	.606	.560	.537	.522	.511	.503	.496	.491	.486	.482
46	.613	.567	.544	.530	.519	.511	.504	.498	.494	.490
48	.619	.574	.551	.537	.526	.518	.512	.506	.501	.497
50	.625	.580	.558	.544	.533	.525	.518	.513	.508	.504
50†	.625	.578	.554	.539	.528	.520	.512	.506	.501	.497

This table gives the values of b for which

$$\int_0^\infty \left[G_\nu\left(\frac{x}{b}\right)\right]^p g_\nu(x)dx=1-\alpha$$

where $G_\nu(x)$ and $g_\nu(x)$ refer to the c.d.f. and the density, respectively, of a standard gamma chance variable with parameter $\nu/2$.

† First term normal approximation based on (4.10) and (4.11)

TABLE ID

Reciprocals of Upper $\alpha(=1-P^*)$ Percentage Points of the Statistic $V=x_{\max}/x_0=\chi^2_{\max}/\chi^2_0=\max(F_1, F_2, \cdots, F_p)$ where $x_1, x_2, \cdots, x_p, x_0$ are Independent Standard Gamma Chance Variables with a Common Value of the Parameter $\nu/2$ on All $p+1$ Independent Gamma Chance Variables and where $x_{\max}$ is the Largest of $x_1, x_2, \cdots, x_p$.

$\alpha=1-P^*=.01$

ν \ p	1	2	3	4	5	6	7	8	9	10
2	.010	.007	.006	.005	.004	.004	.004	.004	.004	.003
4	.063	.048	.043	.039	.037	.035	.034	.032	.032	.031
6	.118	.097	.087	.082	.078	.074	.072	.070	.069	.067
8	.166	.140	.128	.121	.116	.112	.109	.106	.104	.102
10	.206	.178	.164	.156	.150	.146	.142	.139	.136	.134
12	.241	.210	.196	.187	.180	.175	.171	.168	.165	.163
14	.270	.239	.224	.214	.207	.202	.198	.194	.191	.188
16	.297	.264	.249	.238	.231	.226	.221	.218	.214	.212
18	.320	.287	.271	.260	.253	.247	.243	.239	.236	.233
20	.340	.307	.291	.280	.273	.267	.262	.258	.255	.252
22	.359	.326	.309	.298	.291	.285	.280	.276	.272	.269
24	.376	.343	.326	.315	.307	.301	.296	.292	.289	.286
26	.392	.358	.341	.330	.322	.316	.311	.307	.304	.301
28	.406	.372	.356	.345	.337	.330	.326	.321	.318	.315
30	.419	.386	.369	.358	.350	.344	.339	.334	.331	.328
32	.431	.398	.381	.370	.362	.356	.351	.347	.343	.340
34	.443	.410	.393	.382	.374	.368	.362	.358	.354	.351
36	.454	.420	.404	.393	.385	.378	.373	.369	.365	.362
38	.464	.431	.414	.403	.395	.388	.383	.379	.376	.372
40	.473	.440	.424	.412	.404	.398	.393	.389	.385	.382
42	.482	.449	.433	.422	.414	.407	.402	.398	.394	.391
44	.490	.458	.441	.430	.422	.416	.411	.407	.403	.400
46	.498	.466	.450	.439	.431	.424	.419	.415	.411	.408
48	.506	.474	.457	.446	.438	.432	.427	.423	.419	.416
50	.513	.481	.465	.454	.446	.440	.435	.430	.427	.424
50†	.514	.482	.464	.453	.445	.438	.433	.428	.424	.421

This table gives the values of b for which

$$\int_0^\infty \left[G_\nu\left(\frac{x}{b}\right)\right]^p g_\nu(x)dx=1-\alpha$$

where $G_\nu(x)$ and $g_\nu(x)$ refer to the c.d.f. and the density, respectively, of a standard gamma chance variable with parameter $\nu/2$.

† First term normal approximation based on (4.10) and (4.11)