A METHOD FOR GENERATING UNIFORMLY DISTRIBUTED POINTS ON N-DIMENSIONAL SPHERES

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Summary

In this note a computer procedure to transform uniform random variables into random points uniformly distributed on an N-dimensional sphere is presented. The procedure is much simpler than the ones thus far published.

1. Introduction

As a Monte Carlo Method for solving the Dirichlet Problem and other problems the application of spherical processes has been considered [1, 2]. In order to simulate N-dimensional spherical processes on computers, it is required to generate points which are distributed uniformly on N-dimensional spheres. Some procedures have been suggested [3, 4, 5], and the one described in this note is a modification of Muller's procedure: Standard normal devites normalized by root sum of squares are the Cartesian coordinates of a random point on a super sphere.

2. The procedure

The case where the dimension is even, say $N=2M$, is easier to treat than the odd-dimensional case. Devide at random a unit interval into M parts, that is, observe $M-1$ random variables distributed uniformly on (0, 1) and arrange them according to their magnitudes;

$$0=U_0<U_1<\cdots<U_{M-1}<U_M=1. \tag{1}$$

Put

$$Y_i=U_i-U_{i-1}, \qquad i=1,\cdots, M, \tag{2}$$

and let $R_1,\cdots,R_M$ be other independent uniform random variables. Then $(X_1, X_2,\cdots, X_{2M})$ defined by

$$\begin{cases} X_{2i-1}=\sqrt{Y_i}\cos 2\pi R_i\,, \\ X_{2i}\;\;\;=\sqrt{Y_i}\sin 2\pi R_i\,, \qquad i=1,\cdots, M, \end{cases} \tag{3}$$

is a random point on a $2M$-dimensional unit sphere with its center at the origin. As is well known, the pair of random variables ($\cos 2\pi R$, $\sin 2\pi R$) may be generated from uniform random variables R_1, R_2 by a rejection technique: Accept the pair

$$\left(\pm\frac{R_1}{\sqrt{R_1^2+R_2^2}},\ \pm\frac{R_2}{\sqrt{R_1^2+R_2^2}}\right) \text{ or } \left(\frac{R_1^2-R_2^2}{R_1^2+R_2^2},\ \pm\frac{2R_1R_2}{R_1^2+R_2^2}\right) \tag{4}$$

if

$$R_1^2+R_2^2\leqq 1,$$

where $\pm$'s are independent random signs.

In case $N=2M+1$ ($M=1, 2, \cdots$), one may take the first $2M+1$ components of random points on a $(2M+2)$-dimensional sphere and normalize them. Or, one may transform a random point $(X_1, X_2, \cdots, X_{2M})$ on a $2M$-dimensional unit sphere into a point $(X_1^*, X_2^*, \cdots, X_{2M+1}^*)$ on a $(2M+1)$-dimensional unit sphere according to the procedure in [4]: Generate S, a random variable which has the probability density function

$$f(s)=C_{2M}\frac{s^{2M-1}}{\sqrt{1-s^2}}, \qquad 0<s<1, \tag{5}$$

where

$$C_{2M}=(2M-1)\,!!/(2M-2)\,!!$$

Then, the necessary random transformation is

$$\begin{aligned} X_i^* &= SX_i, \qquad i=1,\cdots,2M \\ X_{2M+1}^* &= \pm\sqrt{1-S^2}. \end{aligned} \tag{6}$$

In order to get $(S, \pm\sqrt{1-S^2})$, the following rejection technique may be used for $M\geqq 2$. Accept

$$\begin{aligned} &(S=1-R_1^2,\ \pm R_1\sqrt{1+S}\,), \\ \text{if}\quad &2S^{4M-2}>R_2^2(1+S). \end{aligned} \tag{7}$$

The technique is based on the factorization

$$f(s)=\sqrt{2}\,C_{2M}\cdot\sqrt{2}\,\frac{s^{2M-1}}{\sqrt{1+s}}\cdot\frac{1}{2\sqrt{1+s}} \tag{8}$$

(see, for example [6]), and the sampling efficiency, the inverse of the probability of the acceptance in (7), is equal to $\sqrt{2}\,C_{2M}$. See Table 1.

Table 1

M	N	$\sqrt{2}C_{2M}$
2	5	2.1213
3	7	2.6517
4	9	3.0936
5	11	3.4803
6	13	3.8283
7	15	4.1473
8	17	4.4436
9	19	4.7213
10	21	4.9836

When $M=1$ (or $N=3$),

$$(\sqrt{1-R^2},\ \pm R) \tag{9}$$

is the required pair, and no rejection is necessary. The pair combined with (4) and (6) is a simpler procedure than that of Cook [3]. According to his method, if $R_1^2+R_2^2+R_3^2+R_4^2<1$ one accepts

$$\begin{cases} X_1=2(R_2R_4+R_1R_3)/\sum_{i=1}^{4} R_i^2, \\ X_2=2(R_3R_4-R_1R_2)/\sum_{i=1}^{4} R_i^2, \\ X_3=(R_1^2+R_4^2-R_2^2-R_3^2)/\sum_{i=1}^{4} R_i^2, \end{cases}$$

as a random point. The machine time will be, however, not so different since the latter needs no square root computation.

3. Analysis

It is heuristically explained by the following two facts that the method described above is a modification of Muller's procedure.

(i) If T and R are respectively exponential and uniform random variables, then

$$\begin{cases} \sqrt{2T}\cos 2\pi R \\ \sqrt{2T}\sin 2\pi R \end{cases} \tag{10}$$

are independent normal random variables.

(ii) If $T_1,\cdots,T_M$ are independent exponential random variables, then $T_1/\sum_{i=1}^{M} T_i,\cdots,T_M/\sum_{i=1}^{M} T_i$ are equivalent to intervals randomly partitioned from a unit interval.

We give a direct proof. Consider a polar coordinate system in N $(=2M)$-dimensional space:

$$\begin{cases} x_1 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\theta_{N-1}, \\ x_2 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos\theta_{N-1}, \\ &\vdots \\ x_{N-1} &= r\sin\theta_1\cos\theta_2, \\ x_N &= r\cos\theta_1 \end{cases} \tag{11}$$

In this system the area element on the unit sphere is expressed as

$$dA = \sin^{N-2}\theta_1 \sin^{N-3}\theta_2\cdots\sin\theta_{N-2}\prod_{i=1}^{N-2} d\theta_i \tag{12}$$

$|dA|$ is, expect for a multiplier, the probability element of the uniform distribution on the sphere. Denote a random point which has such a distribution by

$$\begin{cases} V_1 &= \sin\Theta_1\sin\Theta_2\cdots\sin\Theta_{N-2}\sin\Theta_{N-1}, \\ V_2 &= \sin\Theta_1\sin\Theta_2\cdots\sin\Theta_{N-2}\cos\Theta_{N-1}, \\ &\vdots \\ V_{N-1} &= \sin\Theta_1\cos\Theta_2, \\ V_N &= \cos\Theta_1 \end{cases} \tag{13}$$

It is easy to see that $(W_1, \ldots, W_{N-1})$ defined by

$$W_i = \sum_{j=1}^{i} V_j^2, \qquad (i=1,\cdots,N-1) \tag{14}$$

have the joint probability density

$$\text{const}\times\{W_1(W_2-W_1)(W_3-W_2)\cdots(W_{N-1}-W_{N-2})(1-W_{N-1})\}^{-1}\cdot\prod_{i=1}^{N-1} dW_i, \quad 0\leqq W_1\leqq\cdots\leqq W_{N-1}\leqq 1. \tag{15}$$

From this it follows that

$$\begin{aligned} &Y_1 = W_2 = V_1^2+V_2^2,\ Y_2 = W_4-W_2 = V_3^2+V_4^2,\cdots, \\ &\qquad Y_{M-1} = W_{N-2}-W_{N-4} = V_{N-3}^2+V_{N-2}^2\,; \\ &Z_1 = W_1/W_2,\ Z_2 = (W_3-W_2)/(W_4-W_2),\cdots, \\ &\qquad Z_{M-1} = (W_{N-3}-W_{N-4})/(W_{N-2}-W_{N-4}), \\ &\qquad Z_M = (W_{N-1}-W_{N-2})/(1-W_{N-2})\,; \end{aligned} \tag{16}$$

have the joint probability density

$$\text{const}\times[\prod_{i=1}^{M-1} dy_i][\prod_{i=1}^{M}\{z_i(1-z_i)\}^{-1/2}dz_i], \tag{17}$$

$$0 \leqq y_i, \qquad \sum_{i=1}^{M-1} y_i \leqq 1, \qquad 0 \leqq z_i \leqq 1,$$

which shows that Y's $(\sum_{i=1}^{M+1} Y_i \leqq 1)$ are equivalent to random partition, and that Z's are independent to Y's and have the same distribution as $(\cos 2\pi R)^2$. These facts justify the procedure.

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