CUTTING OUT PROCEDURES FOR MATERIAL WITH POISSON DEFECTS

BY MASAAKI SIBUYA

(Received Dec. 5, 1960)

1. Introduction

With the development of automation in industry, more and more products are produced rather continuously than lot by lot. In many cases such semi-manufactured materials as wire and sheet extend in one-dimension, and they are sent to the market after being cut into parts of a definite length. If the cut out part contains more than c (say) defects on it, it is rejected as a bad part, and if it has defects less than or equal to c it is accepted.

We assume that the distribution of defects on the material may be regarded as the Poisson process with parameter λ, that is, the number of defects on an interval of unit length, which is the length of parts sent to the market, is distributed according to the Poisson distribution

$$e^{-\lambda}\lambda^m/m!\,, \qquad m=0,1,2,\cdots, \quad (1)$$

and the distances between the adjacent defects are independently distributed with the probability density

$$\lambda \exp(-\lambda x)\,, \qquad 0\leqq x<\infty. \quad (2)$$

To cut out the parts the following four procedures can be considered:

Procedure 1. Simple cutting out.

Starting from the end point we cut out the intervals of unit length, and inspecting defects we rejects the intervals with more than c defects.

Procedure 2. Sequential cutting out.

From the end point we measure the interval of unit length and count the number of defects, if it is less than or equal to c, we cut out the interval, and otherwise we move the origin of measurement to the position of the first defect and count again the number of defects on the interval of unit length. In this way we continue to measure the interval of unit length until it does not contain more than c defects.

Let us denote the distance between the end point and the first defect by X_0, and the distance between the first and the second defects by

X_1, and so on. Then the procedure is to cut out the interval of unit length from the end point, if

$$\sum_{i=0}^{c} X_i > 1 ,$$

and generally to cut out the interval of unit length measured from the k-th defect if

$$\left.\begin{array}{l} \sum_{i=j}^{j+c} X_i < 1, \quad j=0, 1, \cdots, k-1, \\ \text{and} \\ \sum_{i=k}^{k+c} X_i > 1. \end{array}\right\} \qquad (3)$$

Procedure 3. Cutting out the interval of length l $(1<l<2)$.

We cut off intervals in the same way as in Procedure 1, but those of length l. If the defects are situated near the end points, we cut the end parts to get a unit interval containing not more than c defects.

Procedure 4. Cutting out the interval of length l $(2<l<3)$.

In the same way as in Procedure 3, we cut out, if possible, two unit intervals to be accepted, and if it is not possible to get the two, we try to obtain an interval.

Our problem is to investigate how far the rather troublesome Procedures 2–4 improve the yields, that is, the expected proportions of accepted parts to original material, comparing with the simple Procedure 1, and how to determine the length l in Procedures 3 and 4.

The above model and procedures are simplified ones, and there are many difficulties in real and practical situations. a. Materials might extend in two or three-dimension. b. In some cases parts are classified into A class if the number of defects is between 0 and c_1, and into B class if it is between c_1+1 and c_2, and are rejected if it is more than c_2. c. The parts not only of unit length, but also of half length are sent to the market. Then, if it is impossible to cut out a unit interval, one may obtain a part of half length.

2. Statistical analysis of the cutting out procedures

Procedure 1.

The yield is

$$\sum_{m=0}^{c} e^{-\lambda}\lambda^m/m! , \qquad (4)$$

and there is no difficulty.

Procedure 2.

Put

$$\sum_{j=i}^{i+c} X_j = S_i \,, \qquad i=0,1,2,\cdots. \quad (5)$$

λS's are distributed according to the χ^2 distribution with d.f. $2(c+1)$, and are not independent of each other.

When the event

$$A_i:\ T_i > 1 > T_{i-1}\,,$$

occurs, where (6)

$$T_i = \max_{1\leq j\leq i} S_j\,,$$

we obtain a part from the material of length

$$\sum_{j=0}^{i-1} X_j + 1\,.$$

Then, in this case, the yield is

$$\left[E\left\{\sum_{j=0}^{i-1} X_j + 1 \,\middle|\, A_i\right\} \Pr\{A_i\}\right]^{-1}. \quad (7)$$

The event A_i is equivalent to the first passage of the moving average over some value, and the yield is connected with the mean pass before reaching the value.

Procedure 3.

We find for the distribution function of $X_0+\cdots+X_{k-1}$ under the condition that m ($\geq k$) defects fall in the interval of length l

$$\begin{aligned}\Pr\{X_0+\cdots+X_{k-1}<x \mid Y=m\} &= \frac{e^{\lambda l} m!}{(\lambda l)^m}\int_0^x e^{-\lambda(l-t)}\frac{\lambda^{m-k}(l-t)}{(m-k)!}\cdot\lambda^k t^{k-1} e^{-\lambda t}\,dt \\ &= m^{(k)}\int_0^{x/l} s^{k-1}(1-s)^{m-k}\,ds\,. \end{aligned} \quad (8)$$

That is, $(X_0+\cdots+X_{k-1})/l$ is distributed according to the beta distribution. The original simultaneous distribution of $X_0, \cdots, X_{k-1}$ is symmetric with respect to X_i, and k is arbitrary, therefore, $(X_0/l, X_1/l, \cdots, X_{m-1}/l)$ has the same probability law as the order statistics $U_1 \leqq \cdots \leqq U_m$ of a sample of size m from the uniform distribution on (0, 1) (see [1]).

The probability $p_{\mathrm{III}}^{(m)}$ that we can cut out an acceptable unit interval from the interval of length l with m defects, is

$$p_{\mathrm{III}}^{(m)}=\begin{cases}1\,, & m\leqq c,\\ \Pr\{\max\limits_{c+1\leqq j\leqq m+1}(U_j-U_{j-c-1})\geqq 1/l\}\,, & m>c,\end{cases} \tag{9}$$

where $U_0=0$, $U_{m+1}=1$; and the yield of Procedure 3 is

$$P_{\mathrm{III}}=\frac{1}{l}\sum_{m=0}^{\infty}p_{\mathrm{III}}^{(m)}\frac{e^{-\lambda l}(\lambda l)^m}{m!}\,. \tag{10}$$

Procedure 4.

The distribution of $X_0, X_1, \cdots$ is the same as in the above case. Denote by A_i the event that we can cut out an acceptable unit interval with the "left" end at the i-th defect:

$$\begin{cases}U_{k+c+1}-U_k\leqq 1/l\,, & k=0,1,\cdots,i-1\,,\\ U_{i+c+1}-U_i>1/l\,,\end{cases} \tag{11}$$

and by B_j the event for an unit interval with the "right" end at the j-th defect,

$$\begin{cases}U_k-U_{k-c-1}\leqq 1/l\,, & k=m+1,m,\cdots,j+1\,,\\ U_j-U_{j-c-1}>1/l\,.\end{cases} \tag{12}$$

Then we can write the probability that two acceptable intervals may be cut out, as

$$p_{\mathrm{IV}}^{(m)}=\Pr\{A_i\cap B_j\cap\{U_j-U_i>2/l\}\} \tag{13}$$

and the probability that only one interval may be cut out, as

$$p_{\mathrm{IV}}^{\prime(m)}=\Pr\{A_i\cap B_j\cap\{U_j-U_i\leqq 2/l\}\} \tag{14}$$

and finally the yield of Procedure 4, as

$$P_{\mathrm{IV}}=\frac{1}{l}\sum_{m=0}^{\infty}(2p_{\mathrm{IV}}^{(m)}+p_{\mathrm{IV}}^{\prime(m)})\frac{e^{-\lambda l}(\lambda l)^m}{m!}\,. \tag{15}$$

3. Case $c=0$

In the case $c=0$, that is, when only the intervals without defect are accepted, all the expressions become simple and are summarized in this section.

Procedure 1.

$$P_{\mathrm{I}}=e^{-\lambda}\,. \tag{16}$$

Procedure 2.

As $S_i = X_i$ $(i=0, 1, \cdots)$ are independent,

$$\Pr\{A_i\} = pq^i ,$$

where (17)

$$p = \Pr\{X_i > 1\} = e^{-\lambda} ,$$

and

$$E\Big(\sum_{j=0}^{i-1} X_j \mid A_i\Big) = \sum_{j=0}^{i-1} E(X_j \mid X_j < 1)$$
$$= \frac{i}{\lambda}(1 - e^{-\lambda} - \lambda e^{-\lambda}) . \tag{18}$$

From these we obtain

$$P_{\mathrm{III}} = \lambda/(e^{\lambda}-1) . \tag{19}$$

Procedure 3.

The length l of the interval is between 1 and 2, and only one of $m+1$ differences, "elementary coverages", $V_i = U_i - U_{i-1}$ may be larger than $1/l$.

As elementary coverages have the distribution function $F(v) = (1-v)^m$, we find

$$\Pr\Big\{\max V_i \geqq \frac{1}{l}\Big\} = (m+1)\Big(\frac{l-1}{l}\Big)^m , \qquad m = 0, 1, \cdots, \tag{20}$$

and

$$P_{\mathrm{III}} = \frac{1}{l}\sum_{m=0}^{\infty}(m+1)\Big(\frac{l-1}{l}\Big)^m \frac{e^{-\lambda l}(\lambda l)^m}{m!}$$
$$= e^{-\lambda}\Big\{\lambda + \frac{1-\lambda}{l}\Big\} . \tag{21}$$

We choose the length l according to the value of λ:

$$\sup_{1<l<2} P_{\mathrm{III}} = \begin{cases} e^{-\lambda}, & \lambda < 1, \quad l = 1 . \\ \dfrac{1+\lambda}{2} e^{-\lambda}, & \lambda > 1, \quad l = 2-0 . \end{cases} \tag{22}$$

We write here $l=2-0$, because in Procedure 2 we cut out only an interval even if $l=2$ and no defect occurs on the interval. Notice that, if $l=2-0$, $P_{\mathrm{III}}=1/2$ when $\lambda=0$.

Procedure 4.

Let us denote again elementary by V_i. As the length of the interval is between 2 and 3, the events $V_i>2/l$ and V_i, $V_j>2/l$ do not occur simultaneously, and we have

$$p_{\mathrm{IV}}^{(m)}=(m+1)\Pr\left\{V_i>\frac{2}{l}\right\}+\binom{m+1}{2}\Pr\left\{V_i>\frac{1}{l},\ V_j>\frac{1}{l}\right\} \tag{23}$$

$$p_{\mathrm{IV}}^{\prime(m)}=(m+1)\left[\Pr\left\{V_i>\frac{1}{l}\right\}-\Pr\left\{V_i>\frac{1}{l},\ V_j>\frac{1}{l}\right\}\right] \tag{24}$$

and putting into (15)

$$P_{\mathrm{IV}}=e^{-\lambda}\left\{\lambda+\frac{1-\lambda}{l}\right\}+e^{-2\lambda}\left\{\lambda+\frac{1-2\lambda}{l}\right\}. \tag{25}$$

In Procedure 4 also the two extreme lengths is optimal:

$$\sup_{2<l<3} P_{\mathrm{IV}}=\begin{cases}\dfrac{1+\lambda}{2}e^{-\lambda}+\dfrac{1}{2}e^{-2\lambda}, & \lambda<0.76,\ l=2\,,\\[2mm] \dfrac{1+2\lambda}{3}e^{-\lambda}+\dfrac{1+\lambda}{3}e^{-2\lambda}, & \lambda>0.76,\ l=3-0\,,\end{cases} \tag{26}$$

where 0.76 is the solution of equation $e^{-\lambda}=(1-\lambda)/(2\lambda-1)$, and the meaning of $l=3-0$ is the same as that of $l=2-0$.

Five curves for $l=1$, $2-0$, 2, $3-0$ and for Procedure 2 are shown in Fig. 1.

4. Case $c\geqq 1$

When c is not zero, the expressions for yields are complicated except for some quantities. We carried out the Monté-Carlo computation for the case $c=1$ and 2. The results are summarized in Figures 2 and 3, in which the broken lines show the results of the Monté-Carlo computation.

In the following we analyze only Procedure 3 for $c=1$. Let us denote the sum of two successive elementary coverage by

$$W_i=U_{i+2}-U_i \qquad i=0,1,2,\cdots,m-1.$$

As the length l of the interval satisfies $1<l<2$, the events $W_i\geqq 1/l$ and $W_j\geqq 1/l$ are disjoint if $j-i>1$, and

$$\Pr\left\{\max_{0\leq i\leq m-1} W_i\geqq 1/l\right\}=m\Pr\{W_i\geqq 1/l\} -(m-1)\Pr\{W_i\geqq 1/l,\ W_{i+1}\geqq 1/l\}\,, \qquad m\geqq 2. \tag{27}$$

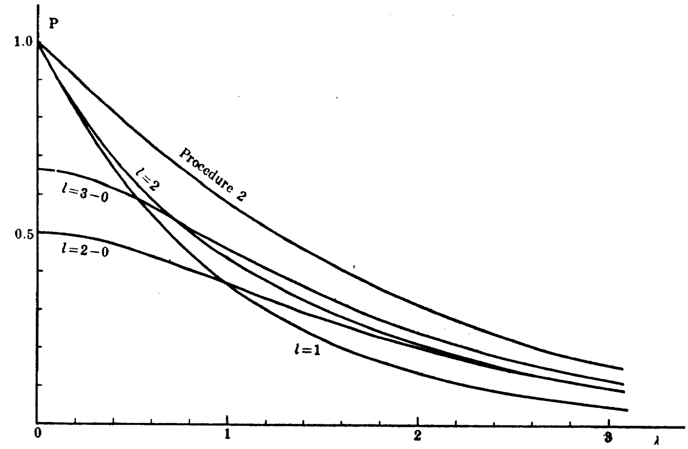

Fig. 1. The yield P for $c=0$

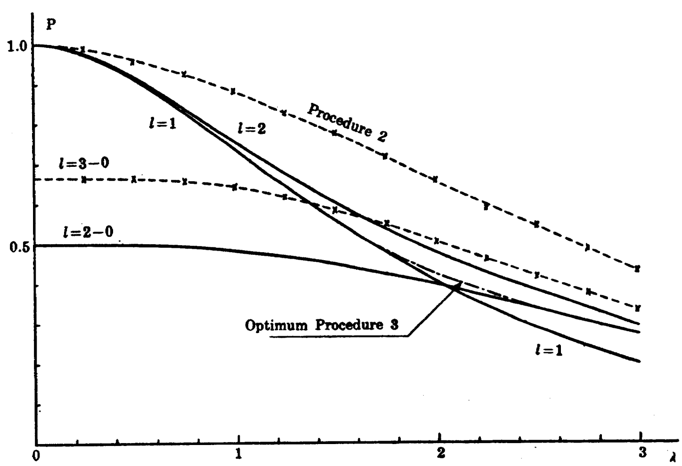

Fig. 2. The yield P for $c=1$

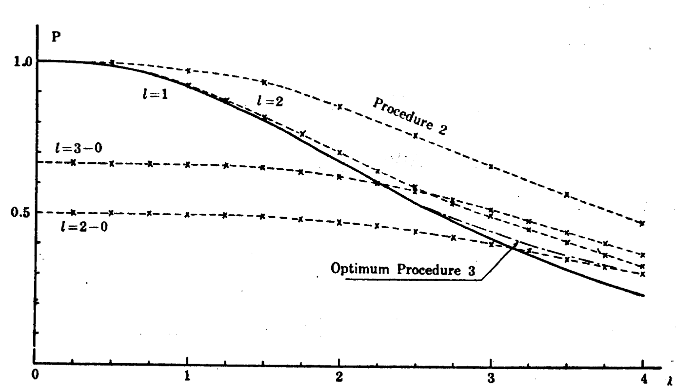

Fig. 3. The yield P for $c=2$

As the simultaneous distribution of (W_i, W_{i+1}) is the same for all $i=0, 1, \cdots, m-2$, we consider only that of (W_0, W_1) for $m\geqq 3$.

Integrating the probability density function of (U_1, U_2, U_3),

$$f(u_1, u_2, u_3)=m^{(3)}(1-u_3)^{m-3}, \quad 0\leqq u_1\leqq u_2\leqq u_3\leqq 1 , \tag{28}$$

with respect to u_3 over the range $\max(W_0, W_1)<u_3<\min(1, w_0+w_1)$, we find for the p.d.f. of (W_0, W_1)

$$f(w_0, w_1)=m^{(2)}[\{\min(1-w_0, 1-w_1)\}^{m-2}-\{\max(0, 1-w_0-w_1)\}^{m-2}] , \tag{29}$$

and from this

$$\begin{cases}\Pr\{W_i\geqq 1/l,\ W_{i+1}\geqq 1/l\}=2(1-1/l)^m , \\ \Pr\{W_i\geqq 1/l\}=(1-1/l)^{m-1}\{1+(m-1)/l\} , \end{cases} \quad 1\leqq l\leqq 2. \tag{30}$$

These formulas are also valid for $m=2$.

Substituting in (27)

$$\Pr\{\max W_i\geqq 1/l\}=(1-1/l)^{m-1}\{(m+2)(m-1)/l-(m-2)\} , \tag{31}$$

and from (10) the yield becomes

$$P_{\text{III}}=(1/l)[\{\lambda^2(l-1)+\lambda(2-l)+2\}e^{-\lambda}-e^{-\lambda l}] . \tag{32}$$

Maximizing the yield with respects l

$$\sup_{1<l<2} P_{\text{III}}=\begin{cases}e^{-\lambda}+\lambda e^{-\lambda}, \text{ if } \lambda<1.618, \text{ for } l=1 , \\ (1/2)[(\lambda^2+2)e^{-\lambda}-e^{-2\lambda}], \text{ if } \lambda>2.593, \text{ for } l=2 , \end{cases} \tag{33}$$

where 1.618 is the approximate value of $(\sqrt{5}+1)/2$ and 2.593 is the solution of the equation

$$(1+2\lambda)e^{-\lambda}=2+2\lambda-\lambda^2 . \tag{34}$$

In the range $1.618<\lambda<2.593$, the maximum value of P_{II} is attained when l is the root of the equation

$$(1+\lambda l)e^{-\lambda l}=(2+2\lambda-\lambda^2) . \tag{35}$$

The improvement by suitable choice of l, however, is not significant as is seen in Table 1. For the case $l=2$ in Procedure 4, the yield is easily obtained from that for $l=2-0$ in Procedure 3.

TABLE 1. Optimum Procedures 3.

λ	l^*	$\max P_{II}=P_{II}(l^*)$	$P_{II}(l=1)$	$P_{II}(l=2-0)$
1.8	1.14	0.4675	0.4628	0.4194
2.0	1.30	0.4222	0.4060	0.3968
2.2	1.45	0.3833	0.3546	0.3728
2.4	1.65	0.3504	0.3084	0.3479

5. Conclusion

As might have been expected, when the yield of simple Procedure 1 is fairly good, the improvements by the other procedures are not considerable. Other procedures should be adopted, however, when the ordinary yield is poor as is seen in the production of optical glass.

In Procedures 3 and 4, we should take the length l so as to be extreme, that is, when the defects are rare we take l as 1 (Procedure 3) or 2 (Procedure 4), and when they are many we take l as 2 (Procedure 3) or 3 (Procedure 4).

6. Acknowledgement

The author is grateful to Misses A. Maruyama and Y. Urabe for their aids in computation work, and also to the RAND Institute of J.U.S.E. for allowing him to use the electronic computer HIPAC-101B.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCE

[1] S. S. Wilks, "Order statistics," *Bull. Am. Math. Soc.*, Vol. 54 (1948), p. 6.