A NOTE ON THE INTERVAL ESTIMATION RELATED TO THE REGRESSION MATRIX

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Let us consider the $(p+q)$ dimensional vector $\boldsymbol{x}'(1\times\overline{p+q})=[\boldsymbol{x}_1'(1\times p), \boldsymbol{x}_2'(1\times q)]$, where $\boldsymbol{x}_1$ has a non-singular p-variate normal distribution $N(\boldsymbol{\mu}_1, \boldsymbol{\Lambda}_{11})$ with linear regression on $\boldsymbol{x}_2$ but $\boldsymbol{x}_2$ may have any distribution or may be a non-random vector variate. The regression matrices in the population and in the sample are respectively denoted by $\boldsymbol{\beta}=\boldsymbol{\Lambda}_{12}\boldsymbol{\Lambda}_{22}^{-1}$ and $\boldsymbol{B}=\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}$, where $\boldsymbol{\Lambda}_{22}$ (positive definite) is the population dispersion matrix of $\boldsymbol{x}_2$, $\boldsymbol{\Lambda}_{12}$ is the population covariance matrix between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and $\boldsymbol{S}_{22}$, $\boldsymbol{S}_{12}$ are the corresponding matrices in the sample.

S. N. Roy [(14.11.13) in [1]: (3.2) in [2]] has constructed the simultaneous confidence bounds on $\boldsymbol{d}_1'\boldsymbol{\beta}\boldsymbol{d}_2$ for all unit vectors $\boldsymbol{d}_1(p\times 1)$ and $\boldsymbol{d}_2(q\times 1)$ with a simultaneous confidence coefficient $\geqq 1-\alpha$ $(0<\alpha<1)$ in the form

$$(1)\qquad \boldsymbol{d}_1'\boldsymbol{B}\boldsymbol{d}_2-\sqrt{E}\leqq \boldsymbol{d}_1'\boldsymbol{\beta}\boldsymbol{d}_2\leqq \boldsymbol{d}_1'\boldsymbol{B}\boldsymbol{d}_2+\sqrt{E}\ ,$$
$$E=\{c_\alpha/(1-c_\alpha)\}C_{\max}(\boldsymbol{S}_{11}-\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{12}')C_{\max}(\boldsymbol{S}_{22}^{-1})\ ,$$

where $C_{\max}(\boldsymbol{M})$ means the largest characteristic root of a square matrix $\boldsymbol{M}$ and c_α is the upper 100α per cent point of the distribution of $C_{\max}(\boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{12}')$ in the null case, i.e., in the case $\boldsymbol{\Lambda}_{12}=\boldsymbol{0}$. However, these bounds are of undesirable form, because while the parametric function varies according to the choice of $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, the width of the interval keeps constant. This is the point which we are concerned with in this short note.

Now we shall give another set of the simultaneous confidence bounds on $\boldsymbol{\beta}$ which are more appropriate than those of (1). Let us start from Roy's inequality [(14.11.14.2) in [1] or (3.4) in [2]]

$$(2)\qquad \frac{\boldsymbol{a}'(\boldsymbol{B}-\boldsymbol{\beta})\boldsymbol{S}_{22}(\boldsymbol{B}-\boldsymbol{\beta})'\boldsymbol{a}}{\boldsymbol{a}'(\boldsymbol{S}_{11}-\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{12}')\boldsymbol{a}}\leqq\frac{c_\alpha}{1-c_\alpha}$$

for all non-null $\boldsymbol{a}(p\times 1)$, which is a confidence statement with confidence coefficient $1-\alpha$. From (2), we have, with the same $1-\alpha$,

$$\frac{a'(B-\beta)S_{22}(B-\beta)'a}{a'a} \leqq \frac{c_\alpha}{1-c_\alpha} \frac{a'(S_{11}-S_{12}S_{22}^{-1}S'_{12})a}{a'a} \tag{3}$$

for all non-null a. Obviously this implies

$$\frac{a'(B-\beta)S_{22}(B-\beta)'a}{a'a} \leqq \frac{c_\alpha}{1-c_\alpha} C_{\max}(S_{11}-S_{12}S_{22}^{-1}S'_{12})$$

or equivalently

$$\text{all } C((B-\beta)S_{22}(B-\beta)') \leqq \frac{c_\alpha}{1-c_\alpha} C_{\max}(S_{11}-S_{12}S_{22}^{-1}S'_{12}) \text{ ,} \tag{4}$$

now with a joint confidence coefficient $\geqq 1-\alpha$, where 'all $C(M)$' means all characteristic roots of the square matrix M. Since

$$\text{all non-zero } C((B-\beta)S_{22}(B-\beta)') = \text{all non-zero } C(S_{22}(B-\beta)'(B-\beta)) \text{ ,}$$

we can replace (4) by

$$\frac{d'(B-\beta)'(B-\beta)d}{d'S_{22}^{-1}d} \leqq \frac{c_\alpha}{1-c_\alpha} C_{\max}(S_{11}-S_{12}S_{22}^{-1}S'_{12})$$

for all non-null $d(q\times 1)$, or

$$d'_2(B-\beta)'(B-\beta)d_2 \leqq \frac{c_\alpha}{1-c_\alpha} C_{\max}(S_{11}-S_{12}S_{22}^{-1}S'_{12}) d'_2 S_{22}^{-1} d_2 \tag{5}$$

for all unit vector $d_2 = d/\sqrt{d'd}$. Applying the well-known result,

$$y'(1\times p)y(p\times 1) \leqq h(>0) \Longleftrightarrow |\, y'd_1 \,| \leqq \sqrt{h}$$

for all arbitrary unit vectors $d_1(p\times 1)$, to (5), we have, with a joint confidence coefficient $\geqq 1-\alpha$,

$$|\, d'_2(B-\beta)'d_1 \,| \leqq \left[\frac{c_\alpha}{1-c_\alpha} C_{\max}(S_{11}-S_{12}S_{22}^{-1}S'_{12}) d'_2 S_{22}^{-1} d_2 \right]^{1/2}$$

or finally

$$\begin{aligned} &d'_1 B d_2 - \sqrt{E^*} \leqq d'_1 \beta d_2 \leqq d'_1 B d_2 + \sqrt{E^*} \\ &E^* = [c_\alpha/(1-c_\alpha)] C_{\max}(S_{11}-S_{12}S_{22}^{-1}S'_{12}) d'_2 S_{22}^{-1} d_2 \end{aligned} \tag{6}$$

for all unit vectors d_1 and d_2. These bounds are obviously narrower than those of (1), since $d'_2 S_{22}^{-1} d_2 \leqq C_{\max}(S_{22}^{-1})$.

Roy's comments for the case of truncated matrices of $\boldsymbol{\beta}$, $\boldsymbol{B}$, $\boldsymbol{S}_{22}$ and $\boldsymbol{S}_{11}-\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}'_{12}$ can be also applied to this case.

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REFERENCES

[1] S. N. Roy, *Some Aspects of Multivariate Analysis*, 1957, John Wiley & Sons.
[2] S. N. Roy and R. Gnanadesikan, "Further contributions to multivariate confidence bounds," *Biometrika*, Vol. 44, (1957), pp. 399-410.