ON A LIMITING PROCESS WHICH ASYMPTOTICALLY PRODUCES f^{-2} SPECTRAL DENSITY*

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1. Introduction and summary

In the recent papers in which the results of the spectral analyses of roughnesses of runways or roadways are reported [1, 2, 3,4] the power spectral densities of approximately of the form f^{-2} (f: frequency) are often treated. This fact directed the present author to the investigation of the limiting process which will provide the f^{-2} form under fairly general assumptions. In this paper a very simple model is given which explains a way how the f^{-2} form is obtained asymptotically. Our fundamental model is that the stochastic process, which might be considered to represent the roughness of the runway, is obtained by alternative repetitions of roughening and smoothing. We can easily get the limiting form of the spectrum for this model. Further, by taking into account the physical meaning of roughening and smoothing we can formulate the conditions under which this general result assures that the f^{-2} form will eventually take place.

The derivation of the present result is entirely formal, and it is hoped that further discussion of the model will be given from the standpoint of road engineering as to whether or not it can be adopted as a starting point of the approximation to the above stated f^{-2} phenomena.

While giving the mathematical reasoning of the present model we discuss very briefly about the numerical results reported in the paper by Wilbur E. Thompson to draw the reader's attension to the necessity of a trend elimination procedure in the spectral analysis of runway or roadway elevations. This discussion will be of some help to those who have occasion to check our model in some practical problems in the future.

2. Derivation of the f^{-2} form

We shall consider the sequence of stationary stochastic processes

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$\varepsilon_j(t)$ $(j=1, 2, \cdots)$ where $\varepsilon_j(t)$'s are supposed to be mutually independent and to follow the same finite dimensional distributions. We shall assume that the process $\varepsilon_j(t)$ has zero-mean, finite variance, and spectral density function $p_\varepsilon(f)$. Consider the initial process $x_0(t)$ which may be considered as representing the initial state of roughness. First we smooth this $x_0(t)$ by an integrable smoothing function $s(t)$ and then roughen by adding a stationary stochastic process $\varepsilon_1(t)$ to it. Thus, denoting by $x_1(t)$, the result of one time application of the smoothing and roughening operation, we have

$$x_1(t)=s*x_0(t)+\varepsilon_1(t)$$

where by definition

$$s*x_0(t)=\int_{-\infty}^{\infty} x_0(t-\tau)s(\tau)d\tau \ ,$$
$$s(t)\geqq 0 \ ,$$
$$\int_{-\infty}^{\infty} s(t)dt=1 \ .$$

We shall assume that $x_0(t)$ satisfies some regularity condition which makes the operations $s*s*\cdots*s*x_0(t)$ meaningfull. As such a condition we have the bounded measurability of $x_0(t)$. If we represent by $x_n(t)$ the result of n successive applications of alternative smoothing and roughening operations, we get

$$x_n(t)=s*x_{n-1}(t)+\varepsilon_n(t) \ ,$$

and successively,

$$x_n(t)=\varepsilon_n(t)+s*\varepsilon_{n-1}(t)+s*s*\varepsilon_{n-2}(t)+\cdots+s*s*\cdots*s*\varepsilon_1(t)+s*s*\cdots*s*x_0(t) \ .$$

If we put

$$y_n(t)=\varepsilon_n(t)+s*\varepsilon_{n-1}(t)+s*s*\varepsilon_{n-2}(t)+\cdots+s*s*\cdots*s*\varepsilon_1(t)$$

it can be seen that the process $y_n(t)$ is stationary and has power spectral density $p_n(f)$ which is given as

$$p_n(f)=(1+|\sigma(f)|^2+|\sigma(f)|^{2\times 2}+\cdots+|\sigma(f)|^{2(n-1)})p_\varepsilon(f)$$
$$=\frac{1-|\sigma(f)|^{2n}}{1-|\sigma(f)|^2}p_\varepsilon(f)$$

where by definition

$$\sigma(f)=\int_{-\infty}^{\infty} e^{-2\pi i f t}s(t)dt \ .$$

Thus we get the general limiting formula

$$\lim_{n\to\infty} p_n(f)=\frac{1}{1-|\sigma(f)|^2}p_\varepsilon(f) \qquad \text{for } f\neq 0\ .$$

From this result we can see that if we carefully separate the "trend" or the residual effect $s*s\cdots*s*x_0(t)$, if any, which can not be filtered out by the repeated applications of the smoothing function $s(t)$, and if we restrict our attension to some range $f\geqq f_0$ ($f_0>0$, arbitrarily small but fixed), we can expect that after a sufficient number of repetitions of alternative smoothing and roughening the form of the power spectrum will become stable. Thus, it can be said that our model gives a limiting form of the spectrum. Of course, the restriction of our attension to the range $f\geqq f_0$ ($f_0>0$ arbitrarily small but fixed) is necessary as the power around the zero frequency grows indefinitely as n tends to infinity.

Our limiting form is given to the process $y_n(t)$ and it is obvious that when we try to apply the present result to some practical phenomena care should be taken to eliminate the trend completely before applying the ordinary spectral analysis to observed data. The spectral densities of runway roughness reported in the paper by Thompson [3] are at once seen to be quite-misleading as the trend elimination procedure adopted there was quite insufficient. The effect of such a trend is quite serious at the lower frequencies and the variations of root-mean-square values, σ's, of runway elevations reported therein can almost entirely be attributed to the "trends" of the runways. Taking into consideration this fact we can see that f^{-2} is a fairly good approximation to the results.

Neglecting the term $s*s*\cdots*s*x_0(t)$ we shall hereafter concern ourselves only with $y_n(t)$ and its power spectral density $p_n(f)$. From the general result just obtained we shall now try to deduce the f^{-2} form.

The term roughening means that the power spectrum of $\varepsilon_j(t)$ is widely spread compared with the form of $\sigma(f)$, or the term smoothing means that $|\sigma(f)|^2$ when multiplied with $p_\varepsilon(f)$ cuts the power of $p_\varepsilon(f)$ substantially. Further, in the cases of runways the effective range of smoothing must be short compared with the wave length which is in the range of our concern, i.e., the smoothing is not so sufficient that when it is applied once it substantially suppresses the roughness of the runway or roadway. These physical considerations lead us to the following assumptions.

A.1. $\varepsilon_j(t)$ is a white noise, i.e., $p_\varepsilon(f)=c$ (>0) in the range of our concern.

A.2. $\int_{-\infty}^{\infty} t^2 s(t)dt < +\infty$, i.e., taking into account the relation $\int s(t)dt = 1$, $|\sigma(f)|^2$ permits the expression

$$|\sigma(f)|^2 = 1 - bf^2 + o(f^2) \qquad (|f| \to 0) ,$$

where

$$b = \left(\int_{-\infty}^{\infty} t^2 s(t)dt - \left(\int ts(t)dt\right)^2\right)(2\pi)^2 .$$

A.3. $|\sigma(f)|^2 - (1 - bf^2)$ is sufficiently small in magnitude in the range of f of our concern.

Then we get the following

THEOREM. *Under the above stated assumptions* A.1 *and* 2 *we have*

$$\lim_{n\to\infty} p_n(f) = \frac{c}{bf^2} + o(f^2) \qquad (|f| \to 0) .$$

Thus if the assumption A.3 holds and if we neglect the very low frequency component we observe that the f^{-2} law is valid. We can interpret the result of the theorem as saying that when the intensity of roughening, c, is large or the range of smoothing, b, is small the power of roughness increases, and in the opposite case it decreases. If we plot the spectral density on the (log, log) paper the line representing the spectral density of slope -2 goes upward for intensive roughening and poor smoothing and goes downward in the opposite case. At least qualitatively this fact seems to be in agreement with the observations reported in the paper by Walls and others [4]. If we further consider the approximation of the form

$$|\sigma(f)|^2 = 1 - bf^2 + df^4 + o(f^4) \qquad (d > 0)$$

we have

$$\lim_{n\to\infty} p_n(f) = \frac{c}{bf^2(1 - df^2/b)} + o(f^4) \qquad (|f| \to 0) .$$

Such an approximation may become necessary if the observed spectral density shows concavity upwards at higher frequency besides the effects of holding and other noises in the course of computation which have tendencies to whiten the spectrum. It seems that in practical cases $s(t)$ may be taken nearly as Gaussian. This is supported by the reasoning that in some cases $s(t)$ itself may be taken to be a result of repeated convolutions of smoothing functions and thus the central limit theorem applies. In any case $|\sigma(f)|^2$ may be considered to be the characteristic function of the difference of two random variables which are mutually independent and follow one and the same distribution and thus will be close to the Gaussian error function. These considerations will be of some use to those who want to use our present result for the understanding of some practical f^{-2} phenomena.

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