ON A MIN-MAX THEOREM AND SOME OF ITS APPLICATIONS

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Summary

A theorem is obtained which enables us to get the arrangement which gives the minimum of the maximum diagonal element in all the possible rearrangements of a matrix of which elements are arranged in the decreasing order of magnitude in each row.

Some implications of the theorem with the rational economic behavior of consumers and with the preparation for the Seidel's process of successive approximation are discussed.

1. A min-max theorem

We consider an $n\times n$ matrix A of which (i, j) elements a_{ij} satisfy the inequalities

$$a_{ij}\geqq a_{ij+1} \qquad i, j=1, 2, \cdots, n .$$

We shall denote by B one of $n!$ $n\times n$ matrices which are obtained by all the possible rearrangements of the rows of the original A and by b_{ij} the (i, j) element of B. Now we represent by $R(B)$ the maximum of the diagonal elements of B, i.e., we define

$$R(B)\equiv\max_i b_{ii} .$$

Then our problem is how to obtain the B which gives the minimum value of $R(B)$ in the $n!$ B's. The solution of this problem is quite simple and we have the following

THEOREM. *The B with minimum $R(B)$ is obtained by the following procedure.*

a. the first row of the B is determined as the row of which first element is the smallest in the first column of A,

b. when the jth row of the B is determined the $(j+1)$st row of the B is determined as the row of which $(j+1)$st element is the smallest in the $(j+1)$st column of the remaining rows of A,

c. if in the $(j+1)$st stage $(j=0, 1, 2, \cdots, n-1)$, there are more *than one rows left of which $(j+1)$st elements are equal, we leave the $(j+1)$st row undetermined and proceed to the determination of the $(j+2)$nd row in the same way as is for the determination of the $(j+1)$st row,*

d. we proceed further until there is left only one of the rows which are the candidates for the $(j+1)$st row, and we adopt this last one row as the $(j+1)$st row of the B. In this case, if there are left more than one candidates for the nth row, any one of them may be taken as the nth row of the B and the determination of the remaining rows proceeds likewise.

PROOF: We shall denote by C the B which is obtained by the above stated procedure and by c_{ij} its (i, j) element. We suppose

$$c_{kk}=\max_i c_{ii}=R(C) .$$

Take an arbitrary B matrix. If the kth row of C forms the $(k-j)$th row $(j\geqq 0)$ of B, then we obviously have $R(B)=\max b_{ii}\geqq b_{k-j,k-j}=c_{k,k-j}\geqq c_{k,k}=R(C)$.

If the kth row of C forms the $(k+j)$th row $(j=1, 2, \cdots, n-k)$ of B, then there is at least one $(k+\nu)$th row $(\nu=1, 2, \cdots, n-k)$ of C which forms a $(k-\mu)$th row $(\mu=0, 1, 2, \cdots, k-1)$ of B. For such ν and μ we have

$$b_{k-\mu,k-\mu}=c_{k+\nu,k-\mu}\geqq c_{k+\nu,k}\geqq c_{k,k} ,$$

and we get

$$R(B)=\max b_{ii}\geqq b_{k-\mu,k-\mu}\geqq c_{k,k}=R(C) .$$

This completes the proof.

Note: If we restrict our attension to the case where the elements are mutually different in each column of A, then in the present procedure of determination of C we do not need any information of the values in the $(k+j)$th columns $(j\geqq 1)$ to make the kth decision. This is a salient feature of this decision procedure.

By replacing the signs $\geqq$, max, and min by $\leqq$, min, and max, respectively, we can get the dual of the above stated theorem. Applying this dual theorem to the matrix which is obtained by inversely ordering the columns of the original A, we can easily find a rearrangement of rows of A which gives the maximum of minimum b_{ii} in the B's.

2. Applications of the theorem

In this section we shall show two examples of the application of the theorem. The first is concerned with the rational economic behavior of consumers, and the second with the preparations for the numerical solution of a linear simultaneous equation by using Seidel's successive approximation procedure.

1. Suppose that there are n commodities which are almost of the same utility to a consumer, and that the prices of these commodities are gradually going down year by year. Then what is the rational behavior of a consumer who intends to buy these n commodities within n successive years, one in each year? In economic life of an ordinary citizen it will be fairly natural to imagine that the consumer tries to keep minimum the maximum expense of the year. If he wants to behave himself according to this minimax rule, our present theorem assures that he has only to purchase in each year the commodity which is the cheapest of those yet to be purchased. Of course, some condition like the one stated in the note to the theorem must be satisfied to make decision in each year, and the restriction of the time unit of one year is not essential. Recently S. Nisihira obtained a survey result which shows a very high scalability of the possession of electric household equipments[*]. The result shows that the consumers usually purchase the electric equipments in order of their prices, for example, in order of the electric iron, radio receiver, electric washer, TV set, electric refrigerator.

Many explanations will be possible for such scalabilities, and we imagine that our present minimax rule may be adapted for explaining some of such phenomena.

2. We shall here consider the numerical solution of a simultaneous linear equation of the form

$$(I-D)x=b ,$$

where I denotes the $n \times n$ identity matrix and D an $n \times n$ matrix with elements d_{ij} $(i, j=1, 2, \cdots, n)$, satisfying the conditions

$$\sum_{j=1}^{n} | d_{ij} | < 1 \qquad (i=1, 2, \cdots, n)$$

and b and x are n-dimensional column vectors of given constants and

[*] This was orally communicated by Mr. Nisihira to the authors.

unknowns, respectively. Then the Seidel's process of successive approximation is defined by

$$x_i^{(k+1)}=b_i+\sum_{j=1}^{i-1} d_{ij}x_j^{(k+1)}+\sum_{j=i}^{n} d_{ij}x_j^{(k)} \qquad (i=1, 2, \cdots, n)$$

where $x_i^{(k)}$ denotes the ith component of the kth approximation $x^{(k)}$ to x and b_i denotes the ith component of b. Set

$$\varepsilon_i^{(k)}=x_i^{(k)}-x_i$$

where x_i is the ith component of the solution vector x.

We have for this $\varepsilon_i^{(k)}$ the following relation

$$\varepsilon_i^{(k+1)}=\sum_{j=1}^{i-1} d_{ij}\varepsilon_j^{(k+1)}+\sum_{j=i}^{n} d_{ij}\varepsilon_j^{(k)} .$$

Therefore, if we define

$$\| \varepsilon^{(k)} \| \equiv \max_i | \varepsilon_i^{(k)} | ,$$

we have

$$| \varepsilon_i^{(k+1)} | \leqq \sum_{j=1}^{i-1} | d_{ij} | \cdot \| \varepsilon^{(k+1)} \| + \sum_{j=i}^{n} | d_{ij} | \cdot \| \varepsilon^{(k)} \| .$$

If $\| \varepsilon^{(k+1)} \| = | \varepsilon_\nu^{(k+1)} |$, we have

$$\| \varepsilon^{(k+1)} \| \leqq \sum_{j=1}^{\nu-1} | d_{\nu j} | \cdot \| \varepsilon^{(k+1)} \| + \sum_{j=\nu}^{n} | d_{\nu j} | \cdot \| \varepsilon^{(k)} \|$$

and

$$\| \varepsilon^{(k+1)} \| \leqq \max_i \left(\frac{\sum_{j=i}^{n} | d_{ij} |}{1-\sum_{j=1}^{i-1} | d_{ij} |} \right) \| \varepsilon^{(k)} \| .$$

When we define $r(D)$ by

$$r(D)=\max_i \left(\frac{\sum_{j=i}^{n} | d_{ij} |}{1-\sum_{j=1}^{i-1} | d_{ij} |} \right) ,$$

this $r(D)$ may be used as an index of the speed of convergence of the present Seidel's approximation procedure, and arrangement of rows of D which gives the minimum of $r(D)$ will be appropriate for this approximation procedure [1. p. 131]. For this purpose we apply the procedure of our theorem to the matrix A with elements a_{ij} defined by

$$a_{ij}=\frac{\sum_{k=j}^{n}|d_{ik}|}{1-\sum_{k=1}^{j-1}|d_{ik}|}.$$

The desired rearrangement is easily performed. To determine the kth row we have only to know the values of those a_{ik}'s which are in the rows still left in D, and usually values of $(n+2)(n-1)/2$ out of n^2 of a_{ij}'s are necessary to get the arrangement. Of course, more precise analysis of the rate of convergence of Seidel's process is desirable, but by using the present theorem we can get the minimum of $r(D)$ with the corresponding rearrangement of D, and we can use it as an index of the speed of convergence of the Seidel's process in practical computations.

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REFERENCE

[1] V. N. Faddeeva, *Computational Methods of Linear Algebra.* Dover Publications, Inc., (1959).