A NOTE ON THE NORMAL APPROXIMATION TO THE SUM OF INDEPENDENT RANDOM VARIABLES

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1. Introduction

Let $\{X_r\}$, $(r=1, 2, \cdots)$ be a sequence of mutually independent random variables with the mean, variance and finite third order absolute moment such as

$$(1.1) \qquad E(X_r)=0\ , \quad V(X_r)=\sigma_r^2\ , \quad E(|X_r|^3)=\beta_r^3\ , \quad (r=1, 2, \cdots)\ ,$$

and let $\sigma_{(n)}^2=\sum_{r=1}^{n}\sigma_r^2$, $\beta_{(n)}^3=\sum_{r=1}^{n}\beta_r^3$, and $\rho_{(n)}=\beta_{(n)}/\sigma_{(n)}$, then it is well known that, under the Liapunov's conditions

$$(1.2) \qquad \lim_{n\to\infty}\sigma_{(n)}=\infty\ , \qquad \lim_{n\to\infty}\rho_{(n)}=0\ ,$$

the standardized sum $X_{(n)}=\frac{1}{\sigma_{(n)}}\sum_{r=1}^{n}X_r$ converges in law to the standard normal distribution $N(0, 1)$, as $n\to\infty$. If all the X_r's are distributed identically, then (1.1) becomes

$$(1.1)' \qquad E(X_r)=0\ , \quad V(X_r)=\sigma^2\ , \quad E(|X_r|^3)=\beta^3\ , \quad (r=1, 2, \cdots)\ .$$

Therefore the conditions (1.2) are automatically fulfilled since $\sigma_{(n)}^2=n\sigma^2$ and $\rho_{(n)}=\frac{1}{\sqrt[6]{n}}\frac{\beta}{\sigma}$, and the sum $X_{(n)}=\frac{1}{\sqrt{n}\,\sigma}\sum_{r=1}^{n}X_r$ converges in law to $N(0, 1)$. The case of (1.1) with different σ_r's or β_r's is called the case of unequal components and that of (1.1)′ the case of equal components.

Denote by $f_{(n)}(t)$ and $F_{(n)}(x)$ the characteristic function (cf.) and the cumulative distribution function (cdf.) of $X_{(n)}$ respectively, while those for the normal $N(0, 1)$ by $g(t)$ and $G(x)$.

The problem of evaluation of the approximation error

$$(1.3) \qquad D_{(n)}=\sup_{-\infty<x<\infty}|F_{(n)}(x)-G(x)|$$

is called the Liapunov problem, and many works has been done for it, for example, A. Liapunov [1], [2], H. Cramér [3], C. G. Esseen [4], and A. C. Berry [5]. A. C. Berry obtained the following numerical evaluation

(1.4) $$D_{(n)} \leqq 1.88 d_{(n)}, \qquad d_{(n)} = \max_{1 \leqq r \leqq n} \frac{\beta_r^3}{\sigma_{(n)} \sigma_r^2},$$

However, as was pointed out by P. L. Hsu [6], Berry's computation is invalidated by an error, and a corrected result

(1.5) $$D_{(n)} \leqq 2.031 d_{(n)}$$

has been given by K. Takano [7]. In the case of equal components, this becomes

(1.6) $$D_{(n)} \leqq 2.031 \frac{\rho^3}{\sqrt{n}}, \qquad \rho = \frac{\beta}{\sigma}.$$

Fourier analytical method used by these authors seems to be the only one for attacking this problem, though it has an unremovable defect. Perhaps the result (1.5) is the best one obtainable by this method.

Now, as for the difference of cf.'s, $|f_{(n)}(t) - g(t)|$, the following evaluation formula

(1.7) $$|f_{(n)}(t) - g(t)| \leqq c_1 \rho_{(n)} |t|^3 \exp\left(-\frac{t^2}{4}\right), \qquad \left(|t| \leqq \frac{1}{c_2 \rho_{(n)}}\right),$$

is usually used, where c_1 and c_2 are the constants independent of n and of other parameters. A numerical example was given by B. V. Gnedenko-A. N. Kolmogoroff [8] (p. 202), for equal components case

(1.8) $$|f_{(n)}(t) - g(t)| \leqq \frac{7}{6} \frac{\rho^3}{\sqrt{n}} |t|^3 \exp\left(-\frac{t^2}{4}\right), \qquad \left(|t| \leqq \frac{\sqrt{n}}{5\rho^3}\right),$$

which seems to be the most accurate one hitherto published.

In the present paper, the author gives the general formula for evaluating the difference $|f_{(n)}(t) - g(t)|$ with numerical examples, one of which is an improvement of (1.8) above. Evaluation problem of $D_{(n)}$ is also reexamined using Berry's inequality, and it will be shown that Berry-Takano's result is improved in some special cases when the values of $\rho_{(n)}$ are considerably large.

2. Evaluation of $|f_{(n)}(t) - g(t)|$

The following two lemmas offer a general method of evaluation in each of the two cases, unequal components.

LEMMA 2.1 (Unequal component case)

Under the conditions (1.1) *and* (1.2) *it holds that*

(2.1) $$|f_{(n)}(t)-g(t)|\leqq c\rho^3_{(n)}|t|^3\exp\left\{-\left(\frac{1}{2}-\frac{1}{p}\right)t^2\right\},\quad \left(|t|\leqq\frac{1}{pc\rho^3_{(n)}}\right),$$

for any $p>\max\left(2,\frac{6}{5}\frac{b_{(n)}}{\rho^2_{(n)}}\right)$, *where* c *is a constant, and* $b_{(n)}=\frac{\beta^*_{(n)}}{\beta_{(n)}}$, $\beta^*_{(n)}=\max(\beta_1,\beta_2,\cdots,\beta_n)$.

PROOF. The cf. of $X_{(n)}$ is given by

(2.2) $$f_{(n)}(t)=\prod_{r=1}^{n}E\{\exp(itX_r/\sigma_{(n)})\},$$

where each factor of the right hand side is of the form

(2.3) $$E\{\exp(itX_r/\sigma_{(n)})\}=1+\phi_r(t),$$

with

(2.4) $$\phi_r(t)=-\frac{1}{2}\frac{\sigma_r^2}{\sigma^2_{(n)}}t^2+\frac{1}{6}\frac{\beta_r^3}{\sigma_r^3}\theta_r^*t^3,\qquad |\theta_r^*|\leqq 1.$$

Let k be an unspecified positive constant. Then, for the interval of t

(2.5) $$|t|\leqq\frac{1}{k\beta^*_{(n)}/\sigma_{(n)}},$$

it holds, since $\sigma_r\leqq\beta_r$ $(r=1,2,\cdots,n)$, that

(2.6) $$|\phi_r(t)|\leqq c_1(k),\qquad (r=1,2,\cdots,n),$$

where

(2.7) $$c_1(k)=\frac{1}{2k^2}+\frac{1}{6k^3}.$$

Therefore, if the condition $c_1(k)<1$ is satisfied (and this is true if $k\geqq 1$), the following expansion will be obtained by (2.3),

(2.8) $$\log E\{\exp(itX_r/\sigma_{(n)})\}=\phi_r(t)=\frac{1}{2}\theta_r\phi_r^2(t),\qquad |\theta_r|\leqq c_2(k),$$

with

(2.9) $$c_2(k)=\frac{1}{1-c_1(k)},\qquad (r=1,2,\cdots,n).$$

From (2.2) and (2.8) it follows that

(2.10) $$\log f_{(n)}(t)=-\frac{1}{2}t^2+h(t),$$

where

$$h(t)=\frac{1}{6}\sum_{r=1}^{n}\theta_r^*\frac{\beta_r^3}{\sigma_{(n)}^3}t^3+\frac{1}{2}\sum_{r=1}^{n}\theta_r\phi_r^2(t) . \tag{2.11}$$

From (2.4) and (2.5) we obtain

$$\phi_r^2(t)\leqq c_3(k)\left(\frac{\beta_r}{\sigma_{(n)}}\right)^3|t|^3 , \qquad (r=1, 2, \cdots, n) , \tag{2.12}$$

where

$$c_3(k)=\frac{1}{4k}+\frac{1}{6k^2}+\frac{1}{36k^3} , \tag{2.13}$$

hence, it follows from (2.11) that

$$|h(t)|\leqq c(k)\rho_{(n)}^3|t|^3 , \tag{2.14}$$

where

$$c(k)=\frac{1}{6}+\frac{1}{2}c_2(k)c_3(k) . \tag{2.15}$$

Now, from (2.10) we have, since $g(t)=e^{-t^2/2}$,

$$f_{(n)}(t)=g(t)e^{h(t)} \tag{2.16}$$

from which it follows that, for the interval (2.5),

$$|f_{(n)}(t)-g(t)|\leqq g(t)|h(t)|e^{|h(t)|} . \tag{2.17}$$

Let p (>2) be a positive number. Then for the interval of t

$$|t|\leqq\frac{1}{pc(k)\rho_{(n)}^3} , \tag{2.18}$$

it follows from (2.14) and (2.17) that

$$|f_{(n)}(t)-g(t)|\leqq c(k)\rho_{(n)}^3|t|^3\exp\left\{-\left(\frac{1}{2}-\frac{1}{p}\right)\right\}t^2 . \tag{2.19}$$

It must be noted that the interval (2.18) ought to be included by that in (2.5), that is, the following inequality

$$\frac{1}{pc(k)\rho_{(n)}^3}\leqq\frac{1}{k\beta_{(n)}^*/\sigma_{(n)}} , \tag{2.20}$$

or equivalently

$$\frac{k}{c(k)}\leqq p\frac{\rho_{(n)}^2}{b_{(n)}} , \tag{2.20$'$}$$

must hold. The left hand side of the above inequality (2.20)′ is a monotone increasing function of k ($\geqq 1$), and $c(1)=5/6$. Hence, if we take a value of p such as

$$p>\max\left(2, \frac{6}{5}\frac{b_{(n)}}{\rho_{(n)}^2}\right), \tag{2.21}$$

then the equation in k

$$\frac{k}{c(k)}=p\frac{\rho_{(n)}^2}{b_{(n)}} \tag{2.22}$$

has one and only one solution $k_0=k_0(p)$. Putting $c=c(k_0)$, we obtain, from (2.18) and (2.19), the result (2.1) and the proof of the lemma is completed.

The following lemma is a special case of the above, and the proof will be omitted.

LEMMA 2.2 (Equal component case)
Under the condition (1.1)' it holds that

$$|f_{(n)}(t)-g(t)|\leqq c\frac{1}{\sqrt{n}}\rho^3|t|^3\exp\left\{-\left(\frac{1}{2}-\frac{1}{p}\right)\right\}t^2, \quad \left(|t|\leqq\frac{\sqrt{n}}{pc\rho^3}\right), \tag{2.23}$$

for any $p>2$, where c is a constant.

In order to determine the value of c for given p in the equal components case, we must solve the following equation instead of (2.22),

$$\frac{k}{c(k)}=p\rho^2 \tag{2.24}$$

where $c(k)$ is the same as that in (2.15), i.e.,

$$c(k)=\frac{12k^3+9k^2-1}{72k^3-36k-12}. \tag{2.25}$$

The values of $c(k)$ and $k/c(k)$ are tabulated in Table (2.1) below, with step 0.02 for $1.12\leqq k\leqq 3.00$ and with step 0.10 for $3.00\leqq k\leqq 6.00$.

In general, since $\rho\geqq 1$, it is sufficient to use the equation

$$\frac{k}{c(k)}=p, \tag{2.26}$$

instead of (2.24) in the case of equal components, therefore putting $p=4$, we obtain

$$|f_{(n)}(t)-g(t)|\leqq 0.36010\frac{1}{\sqrt{n}}\rho^3|t|^3e^{-t^2/4}, \quad \left(|t|\leqq\frac{\sqrt{n}}{1.44040\rho^3}\right), \tag{2.27}$$

TABLE (2.1)

k	$c(k)$	$k/c(k)$	k	$c(k)$	$k/c(k)$	k	$c(k)$	$k/c(k)$
1.12	0.55592	2.01464	2.02	0.26464	7.63293	2.92	0.22352	13.06365
1.14	0.53093	2.14713	2.04	0.26307	7.75436	2.94	0.22299	13.18389
1.16	0.50902	2.27888	2.06	0.26156	7.87573	2.96	0.22248	13.30412
1.18	0.48963	2.40993	2.08	0.26009	7.99705	2.98	0.22198	13.42434
1.20	0.47237	2.54034	2.10	0.25867	8.11832	3.00	0.22149	13·54455
1.22	0.45690	2.67016	2.12	0.25729	8.23953	3.10	0.21915	14.14547
1.24	0.44257	2.80178	2.14	0.25595	8.36070	3.20	0.21700	14.74620
1·26	0.43037	2.92767	2.16	0.25466	8.48183	3.30	0.21502	15.34676
1.28	0.41878	3.05645	2.18	0.25340	8.60291	3.40	0.21320	15.94719
1.30	0.40825	3.18428	2.20	0.25217	8.72395	3.50	0.21151	16.54749
1.32	0.39858	3.31169	2.22	0.25099	8.84495	3.60	0.20994	17.14769
1.34	0.38968	3.43871	2.24	0.24983	8.96591	3.70	0.20847	17.74781
1.36	0.38144	3.56537	2.26	0.24871	9.08683	3.80	0.20710	18.34785
1.38	0.37381	3.69169	2.28	0.24761	9.20772	3.90	0.20582	18.93862
1.40	0.36671	3.81768	2.30	0.24655	9.32858	4.00	0.20462	19.54774
1.42	0.36009	3.94338	2.32	0.24551	9.44940	4.10	0.20349	20.14762
1.44	0.35391	4.06879	2.34	0.24450	9.57019	4.20	0.20243	20.74745
1.46	0.34812	4.19394	2.36	0.24352	9.69095	4.30	0.20143	21.34725
1.48	0.34268	4.31884	2.38	0.24256	9.81168	4.40	0.20048	21.94702
1.50	0.33757	4.44351	2.40	0.24163	9.93239	4.50	0.19955	22.55054
1.52	0.33257	4.56795	2.42	0.24072	10.05306	4.60	0.19873	23.14649
1.54	0.32820	4.69219	2.44	0.23983	10.17372	4.70	0.19792	23.74620
1.56	0.32390	4.81623	2.46	0.23896	10.29434	4.80	0.19715	24.34589
1.58	0.31983	4.94008	2.48	0.23811	10.41495	4.90	0.19654	24.93025
1.60	0.31597	5.06375	2.50	0.23729	10.53553	5.00	0.19573	25.54524
1.62	0.31230	5.18726	2.52	0.23648	10.65608	5.10	0.19506	26.14490
1.64	0.30881	5.31062	2.54	0.23569	10.77662	5.20	0.19443	26.74455
1.66	0.30549	5.43382	2.56	0.23492	10.89714	5.30	0.19382	27.34420
1.68	0.30232	5.55688	2.58	0.23416	11.01763	5.40	0.19324	27.94384
1.70	0.29930	5.67981	2.60	0.23343	11.13811	5.50	0.19268	28.54349
1.72	0.29641	5.80261	2.62	0.23271	11.25857	5.60	0.19215	29.14312
1.74	0.29365	5.92529	2.64	0.23200	11.37901	5.70	0.19164	29.74276
1.76	0.29101	6.04785	2.66	0.23131	11.49944	5.80	0.19115	30.34240
1.78	0.28847	6.17031	2.68	0.23049	11.62710	5.90	0.19067	30.94204
1.80	0.28604	6.29266	2.70	0.22997	11.74024	6.00	0.19022	31.54168
1.82	0.28371	6.41491	2.72	0.22933	11.86061	∞	0.16666	∞
1.84	0.28147	6.53707	2.74	0.22869	11.98089			
1.86	0.27931	6.65914	2.76	0.22807	12.10132			
1.88	0.27723	6.78112	2.78	0.22746	12.22166			
1.90	0.27524	6.90303	2.80	0.22686	12.34198			
1.92	0.27331	7.02485	2.82	0.22628	12.46229			
1.94	0.27145	7.14660	2.84	0.22570	12.58258			
1.96	0.26966	7.26824	2.86	0.22514	12.70287			
1.98	0.26772	7.39570	2.88	0.22459	12.82314			
2.00	0.26626	7.51145	2.90	0.22405	12.94340			

which is an improvement of the result (1.8) by B. V. Gnedenko and A. N. Kolmogoroff. Giving the different values to p, further examples will be shown. If $p=3$, it holds that

$$(2.28)\qquad |f_{(n)}(t)-g(t)| \leqq 0.43038\frac{1}{\sqrt{n}}\rho^3|t|^3e^{-t^2/6}\,, \quad \left(|t| \leqq \frac{\sqrt{n}}{1.29114\rho^3}\right),$$

if $p=5$,

$$(2.29)\qquad |f_{(n)}(t)-g(t)| \leqq 0.31984\frac{1}{\sqrt{n}}\rho^3|t|^3e^{-3t^2/10}\,, \quad \left(|t| \leqq \frac{\sqrt{n}}{1.59920\rho^3}\right),$$

and so on.

If ρ^2 is known a priori to be greater or equal to m (>1), then we can use the equation

$$\frac{k}{c(k)}=mp\ , \tag{2.30}$$

and the following examples are obtained. Let $p=4$. then if $m=1.5$,

$$|f_{(n)}(t(-g(t)|\leqq 0.29366\frac{1}{\sqrt{n}}\rho^3|t|^3e^{-t^2/4}\ , \quad \left(|t|\leqq\frac{\sqrt{n}}{1.17464\rho^3}\right)\ , \tag{2.31}$$

if $m=3$,

$$|f_{(n)}(t)-g(t)|\leqq 0.22870\frac{1}{\sqrt{n}}\rho^3|t|^3e^{-t^2/4}\ , \quad \left(|t|\leqq\frac{\sqrt{n}}{0.91480\rho^3}\right)\ , \tag{2.32}$$

and so forth.

Finally it will be remarked that, if the value of $\rho^2_{(n)}/b_{(n)}$ is known, then it is possible to obtain the numerical evaluations by the same procedures as above, in the case of unequal components too.

3. Error estimation

Let $H(x)$ be a function non-negative, integrable and symmetric with respect to $x=0$, and let

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(x)e^{-itx}dx\ , \qquad w=\int_0^{\infty}H(x)dx\ . \tag{3.1}$$

Referring to the second section of A. C. Berry [5], we can easily find that the inequality

$$\sqrt{\frac{2}{\pi}}D^*_{(n)}\left(3\int_0^{D^*_{(n)}}H(x)\,dx-2w\right)\leqq\int_{-\infty}^{\infty}|h(t)|\left|\frac{f_{(n)}(t)-g(t)}{t}\right|dt\ , \tag{3.2}$$

holds, where $D^*_{(n)}=\sqrt{\pi/2}D_{(n)}$. If we choose T, $H(x)$ and $h(t)$ such as

$$T=\frac{1}{pc\rho^3_{(n)}}\ , \qquad H(x)=\frac{2(1-\cos Tx)}{x^2}\ , \tag{3.3}$$

$$h(t)=\begin{cases}T-t\ , & (|t|\leqq T)\ ,\\ 0\ , & (|t|>T)\ ,\end{cases}$$

then (3.2) becomes

$$TD^*_{(n)}\left(3\int_0^{TD^*_{(n)}}\frac{1-\cos x}{x^2}dx-\pi\right)\leqq\sqrt{\frac{\pi}{2}}\int_0^{T}(T-t)\left|\frac{f_{(n)}(t)-g(t)}{t}\right|dt\ , \tag{3.4}$$

which is called the Berry's inequality.

Let

(3.5) $$A_j(u)=u\left\{3\left(\frac{u}{2}-\frac{u^3}{3\cdot 4!}+\frac{u^5}{5\cdot 6!}-\cdots-\frac{u^{2j-1}}{(2j-1)(2j)!}\right)-\pi\right\}, \quad (j: \text{even})$$

and, calculating the right-hand member of (3.4) using Lemma 2.1 and let

(3.6) $$B(p)=\frac{\pi}{2}\frac{1}{p-2}\sqrt{\frac{p}{p-2}}.$$

Then, for $u=TD^*_{(n)}$, it follows from (3.4) that

(3.7) $$A_j(u)\leqq B(p).$$

The values of $A_j(u)$ for $j=6$ are listed in the Table (3.1), and the values of $B(p)$ are listed in the Table (3.2).

Now, we state the procedure of estimating the error $D_{(n)}$ of normal approximation. Let u_0 be the minimum value of u satisfying the condition

(3.8) $$B(p)\leqq A_j(u).$$

TABLE (3.1)

u	$A_6(u)$	u	$A_6(u)$	u	$A_6(u)$
2.65	0.41795	2.86	0.90833	3.07	1.41162
2.66	0.44085	2.87	0.93208	3.08	1.43574
2.67	0.46396	2.88	0.95587	3.09	1.45987
2.68	0.48682	2.89	0.97968	3.10	1.48399
2.69	0.50987	2.90	1.00410	3.11	1.50813
2.70	0.53298	2.91	1.02737	3.12	1.53226
2.71	0.55613	2.92	1.05176	3.13	1.55640
2.72	0.57933	2.93	1.07516	3.14	1.58053
2.73	0.60257	2.94	1.09909	3.15	1.60468
2.74	0.62586	2.95	1.12304		
2.75	0.64919	2.96	1.14752	3.20	1.72534
2.76	0.67284	2.97	1.17099	3.25	1.84586
2.77	0.69602	2.98	1.19500	3.30	1.96612
2.78	0.71943	2.99	1.21902	3.35	2.08603
2.79	0.74298	3.00	1.24305	3.40	2.20547
2.80	0.76645	3.01	1.26608	3.50	2.44253
2.81	0.79001	3.02	1.29116	3.60	2.67657
2.82	0.81361	3.03	1.31523	3.70	2.94016
2.83	0.83724	3.04	1.33932	3.80	3.13284
2.84	0.86091	3.05	1.36341	3.90	3.35392
2.85	0.88460	3.06	1.38751	4.00	3.56961

TABLE (3.2)

p	$B(p)$	p	$B(p)$	p	$B(p)$
2.8	3.67336	3.8	1.26795	4.8	0.73451
2.9	3.13296	3.9	1.18446	4.9	0.70407
3.0	2.72069	4.0	1.11072	5.0	0.69341
3.1	2.39724	4.1	1.04516		
3.2	2.13758	4.2	0.98653	5.5	0.56259
3.3	1.92513	4.3	0.93381	6.0	0.48095
3.4	1.74850	4.4	0.88619	6.5	0.41952
3.5	1.59962	4.5	0.84297	7.0	0.37171
3.6	1.47262	4.6	0.80359	7.5	0.33350
3.7	1.36316	4.7	0.76757	8.0	0.30229

Then it holds that

(3.9) $$TD^*_{(n)} \leqq u_0 ,$$

from which we obtain

(3.10) $$D_{(n)} \leqq \sqrt{\frac{2}{\pi}}\, p u_0 c \rho^3_{(n)} .$$

Since the values of u_0 and c are determined if the value of p is given, for the optimum evaluation we must choose the value of p such that the value of pu_0c becomes as small as possible. The following examples are for the case of equal components.

First, since $\rho \geqq 1$, we have, putting $p=4$,

(3.11) $$D_{(n)} \leqq 3.39036 \frac{\rho^3}{\sqrt{n}} ,$$

which is an undesirable result compared with that of (1.6) by Berry-Takano. But, if the value of ρ is large, our method will offer the estimation more accurate than that in (1.6), as will be seen in the following examples.

If $\rho^2 \geqq 4$, then letting $p=3.5$, we obtain

(3.12) $$D_{(n)} \leqq 1.93580 \frac{\rho^3}{\sqrt{n}} ,$$

if $\rho^2 \geqq 5$, letting $p=3.3$,

(3.13) $$D_{(n)} \leqq 1.84076 \frac{\rho^3}{\sqrt{n}} ,$$

and, if $\rho^2 \geqq 6$, letting $p=3.3$,

$$D_{(n)} \leqq 1.77803 \frac{\rho^3}{\sqrt{n}}, \tag{3.14}$$

and so on.

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