ON SAMPLING INSPECTION PLANS

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1. Introduction.

Since the work by Dodge and Romig appeared (1929), various sampling inspection plans have been elaborated by many authors. However, many sampling inspection plans of them seem not to be satisfactory, because, first, they are based on the assumption that the distribution which the number of defectives in lot follows is unknown, and, second, in them, the analysis of costs related to the activity of sampling inspections and its result is not completely considered.

F. J. Anscombe [1] pointed out the second point and treated continuous sampling inspection plans taking into account various costs.

In this paper, we shall discuss about the single sampling inspection plans taking into account the a priori distribution of the number of defectives in lot. First, we state the main differences between Dodge-Romig's sampling inspection plan (D.R.S.I.P.) and ours.

1) In D.R.S.I.P., the average fraction defective of lot is only one available information about the manufacturing process. However, in our inspection plan, we treat the problem under the assumption that the a priori distribution of the fraction defectives of lots is known. This treatment is based on the possibility and validity of construction of the a priori distribution from the past data.

2) In D.R.S.I.P., they take into account only the cost of sampling and inspection, which is assumed to be proportional to the number of items inspected. However, we, further, consider two loss functions that represent losses incurred by the acceptance or rejection of the lot of a given fraction defective.

3) In D.R.S.I.P., the optimal sampling inspection plan is decided to be one which minimizes the average sampling number required for the lot of the same fraction defective as the average one of the manufacturing process. The minimization here is considered under the restriction that the consumer's risk or AOQL must not be larger than the previously specified value. But, in our case, the optimal sampling

inspection plan is determined as one which minimizes the sum of the decision-risk and inspection cost, where the decision-risk is given taking expectation of sum of the above mentioned loss functions using the a priori distribution.

Further analysis of various costs attached to sampling inspection will be made in the near future by the present author. In this paper, Bayes procedure is introduced into inspection scheme and examples are treated where the a priori distribution is of Beta type.

2. Bayes procedure.

Our inspection procedure is as follows:

1. Take a random sample of size n from each lot.
2. Inspect all items involved in the sample.
3. Sentence the lot to be accepted if the number of defectives in the sample is not larger than the specified number c.
4. Sentence the lot to be rejected if the number of defectives is larger than c.

Therefore, our sampling inspection scheme is specified by (n, c) just like the Dodge-Romig's single sampling inspection plan. However, we want to select an optimal sampling inspection scheme using the a priori distribution of fraction defective of lot, two loss functions and inspection cost.

Let us assume that the lot size is N and the a priori distribution function is denoted by $F(p)$, where p shows the fraction defective of lot. Further, let us denote the losses due to the acceptance and rejection of the lot of fraction defective p by $L_1(p)$ and $L_2(p)$, respectively, where $L_2(p)$ is monotone decreasing, $L_1(p)$ is monotone increasing and $L_1(0)=L_2(1)=0$.

Now, the probability of the acceptance of the lot of fraction defective p by the sampling scheme (n, c) is

$$(2.1) \qquad \sum_{i=0}^{c} \binom{Np}{i}\binom{N-Np}{n-i} \Big/ \binom{N}{n}$$

and the probability of the rejection is

$$(2.2) \qquad 1-\sum_{i=0}^{c} \binom{Np}{i}\binom{N-Np}{n-i} \Big/ \binom{N}{n}$$

Therefore, the expected loss of the inspection procedure (n, c) for the lot of fraction defective p is

$$(2.3)\qquad r(n,\ c \mid p)=\sum_{i=0}^{c} L_1(p)\binom{Np}{i}\binom{N-Np}{n-i}\Big/\binom{N}{n} + L_2(p)\left\{1-\sum_{i=0}^{c}\binom{Np}{i}\binom{N-Np}{n-i}\Big/\binom{N}{n}\right\}$$

Further, the decision-risk attached to our inspection procedure $(n,\ c)$ is obtained by taking the expectation of $r(n,\ c \mid p)$ with respect to $F(p)$; that is

$$(2.4)\qquad R(n,\ c)=\int_0^1 r(n,\ c \mid p)dF(p) = \int_0^1 L_2(p)dF(p)+\sum_{i=0}^{c}\int_0^1 \{L_1(p)-L_2(p)\}\frac{\binom{Np}{i}\binom{N-Np}{n-i}}{\binom{N}{n}}dF(p)\ .$$

Let the inspection cost for a sample of size n be $S(n)$, where $S(n)$ is monotone increasing and $\lim_{n\to\infty} S(n)=\infty$. Thus, the total risk suffered by the sampling inspection plan $(n,\ c)$ is the sum of the decision-risk and the inspection cost, if both are evaluated by means of a common unit, that is

$$(2.5)\qquad T(n,\ c)=R(n,\ c)+S(n)\ .$$

Then, our problem is to obtain the $(n,\ c)$ which minimizes the total risk. This problem is very difficult to solve analytically. But, the following lemma will be useful.

LEMMA. *If for some n and i,*

$$(2.6)\qquad \int_0^1 \{L_1(p)-L_2(p)\}\binom{Np}{i}\binom{N-Np}{n-i}dF(p)\leqq 0$$

holds, then

$$(2.7)\qquad \int_0^1 \{L_1(p)-L_2(p)\}\binom{Np}{i-1}\binom{N-Np}{n-i+1}dF(p)\leqq 0$$

and the strict inequality of (2.6) *results in the strict inequality of* (2.7).

PROOF. Let p_0 be such a point that $L_1(p)=L_2(p)$. Then (2.6) means the following inequality,

$$(2.8)\qquad \int_0^{p_0} \{L_2(p)-L_1(p)\}\binom{Np}{i}\binom{N-Np}{n-i}dF(p) \geqq \int_{p_0}^1 \{L_1(p)-L_2(p)\}\binom{Np}{i}\binom{N-Np}{n-i}dF(p)$$

Since

$$\binom{Np}{i}\binom{N-Np}{n-i}=\frac{Np-i+1}{i}\cdot\frac{n-i+1}{N-Np-n+i}\binom{Np}{i-1}\binom{N-Np}{n-i+1}$$

and

$$\frac{Np-i+1}{N-Np-n+i}\,, \qquad (i\leqq Np)$$

is monotone increasing function of p, we obtain

$$\text{(2.9)}\quad \frac{Np_0-i+1}{i}\cdot\frac{n-i+1}{N-Np_0-n+i}\int_0^{p'}\{L_2(p)-L_1(p)\}\binom{Np}{i-1}\binom{N-Np}{n-i+1}dF(p)$$
$$\geqq\int_0^{p_0}\{L_2(p)-L_1(p)\}\binom{Np}{i}\binom{N-Np}{n-i+1}dF(p)$$

and

$$\text{(2.10)}\quad \int_{p_0}^1\{L_1(p)-L_2(p)\}\binom{Np}{i}\binom{N-Np}{n-i}dF(p)$$
$$\geqq\frac{Np_0-i+1}{i}\cdot\frac{n-i+1}{N-Np_0-n+i}\int_{p_0}^1\{L_1(p)-L_2(p)\}\binom{Np}{i}\binom{N-Np}{n-i+1}dF(p)\,.$$

From (2.9) and (2.10), we obtain

$$\text{(2.11)}\quad \int_0^{p_0}\{L_2(p)-L_1(p)\}\binom{Np}{i-1}\binom{N-Np}{n-i+1}dF(p)$$
$$\geqq\int_{p_0}^1\{L_1(p)-L_2(p)\}\binom{Np}{i-1}\binom{N-Np}{n-i+1}dF(p)$$

which proves the lemma.

This lemma teaches us how to select the value of c in order to minimize $R(n,\ c)$ for any fixed sample size. It assures the existence of a integral valued function $i_0(n)$ such that $0\leqq i_0(n)\leqq n$ and

$$\text{(2.12)}\quad \int_0^1\{L_1(p)-L_2(p)\}\binom{Np}{i}\binom{N-Np}{n-i}dF(p)\leqq 0\,,\quad \text{for any } i;\ i\leqq i_0(n)$$

and

$$\int_0^1\{L_1(p)-L_2(p)\}\binom{Np}{i}\binom{N-Np}{n-i}dF(p)>0\,,\quad \text{for any } i;\ i_0(n)<i\leqq n\,.$$

Therefore, to minimize $R(n,\ c)$ for a fixed n, it suffices to determine the value of c as $i_0(n)$. Thus, for a fixed sample size n, the optimal sampling inspection plan is specified as $(n,\ i_0(n))$. Further, the optimal sample size can be determined taking into account the inspection cost $S(n)$. That is, the optimal sample size n_0 can be determined as the value of n which minimizes $T(n, i_0(n))$. The existence of n_0 is obvious, because $R(n, i_0(n))$ is monotone decreasing function of n and, on the contrary, $S(n)$ is monotone increasing and $S(n)\to\infty$ when $n\to\infty$.

Now, let us consider the case where the lot size is large enough to be able to replace the hypergeometric distribution by the binomial distribution and introduce a lemma which shows the method of selection of c for each fixed sample size n. In the binomial case, the decision-risk function is

$$R_b(n, c)=\int_0^1 L_2(p)dF(p)+\sum_{i=0}^{c}\binom{n}{i}\int_0^1\{L_1(p)-L_2(p)\}p^i(1-p)^{n-i}dF(p) . \tag{2.13}$$

and the total risk is

$$T(n, c)=R_b(n, c)+S(n) .$$

Before we consider the minimization of $T(n, c)$, we can obtain the value of c which minimizes $R_b(n, c)$ for each fixed n. To obtain such values of c, the following lemma will be useful.

LEMMA. *If*

$$\int_0^1\{L_1(p)-L_2(p)\}p^i(1-p)^{n-i}dF(p)\leqq 0 \quad \textit{for some } i,\ (1\leqq i\leqq n) , \tag{2.14}$$

then

$$\int_0^1\{L_1(p)-L_2(p)\}p^{i-1}(1-p)^{n-i+1}dF(p)\leqq 0 . \tag{2.15}$$

and strict inequality of (2.14) *results in the strict inequality of* (2.15).

PROOF. Let p_0 be such a point that $L_1(p_0)=L_2(p_0)$. From (2.14), we obtain

$$\int_0^{p_0}\{L_2(p)-L_1(p)\}p^i(1-p)^{n-i}dF(p)\geqq\int_{p_0}^1\{L_1(p)-L_2(p)\}p^i(1-p)^{n-i}dF(p)$$

or

$$\begin{aligned}&\int_0^{p_0}\{L_2(p)-L_1(p)\}\frac{p}{1-p}p^{i-1}(1-p)^{n-i+1}dF(d)\\ &\geqq\int_{p_0}^1\{L_1(p)-L_2(p)\}\frac{p}{1-p}p^{i-1}(1-p)^{n-i+1}dF(p) .\end{aligned} \tag{2.16}$$

But, $\frac{p}{1-p}$ is a monotone increasing when $0\leqq p<1$. If $0\leqq p\leqq p_0$, then $\frac{p}{1-p}\leqq\frac{p_0}{1-p_0}$ and if $p_0\leqq p<1$ then $\frac{p_0}{1-p_0}\leqq\frac{p}{1-p}$. Therefore,

$$\frac{p_0}{1-p_0}\int_0^{p_0}\{L_2(p)-L_1(p)\}p^{i-1}(1-p)^{n-i+1}dF(p)$$

$$\geqq\frac{p_0}{1-p_0}\int_{p_0}^1\{L_1(p)-L_2(p)\}p^{i-1}(1-p)\}p^{n-i+1}dF(p)\,.$$

This means (2.9). The final statement of the lemma is easily verified.

By the above lemma, we can assure the existence of the function $i_0(n)$*) such that $0\leqq i_0(n)\leqq n$ and

$$\int_0^1\{L_1(p)-L_2(p)\}p^i(1-p)^{n-i}dF(p)\leqq 0 \qquad \text{for any } i;\ 0\leqq i\leqq i_0(n) \tag{2.17}$$

and

$$\int_0^1\{L_1(p)-L_2(p)\}p^i(1-p)^{n-i}dF(p)>0 \quad \text{for any } i;\ i_0(n)<i\leqq n\,.$$

From this fact and (2.6), we can conclude that, for fixed sample size n, the value of c which minimizes $R_b(n, c)$ is to be $i_0(n)$. In other words, when the sample size is fixed to be n, then the optimal sampling inspection plan is $(n, i_0(n))$.

The optimal size of sample is determined as the value of n which minimizes $T(n, i_0(n))$. Obviously such a value exists, because $R_b(n, i_0(n))$ is monotone decreasing, $S(n)$ is monotone increasing and $S(n)\to\infty$, when $n\to\infty$.

3. Example

Let us assume that the a priori distribution of the fraction defective is a Beta distribution, that is;

$$dF(p)=f(p)dp=\frac{1}{B(\lambda,\ \mu)}p^{\lambda-1}(1-p)^{\mu-1}dp\,. \qquad (\lambda\geqq 0,\ \mu\geqq 0) \tag{3.1}$$

Further, we assume that

$$\begin{aligned}L_1(p)&=\alpha p^u\\ L_2(p)&=\beta(1-p)^v \qquad (u,\ v>0)\end{aligned} \tag{3.2}$$

Then, by (2.6), we obtain

$$R_b(n,\ c)=\beta\int_0^1(1-p)^v\frac{1}{B(\lambda,\ \mu)}p^{\lambda-1}(1-p)^{\mu-1}dp \tag{3.3}$$

* The existence of $i_0(n)$ may be reduced to the monotone likelihood ratio property of the binomial distribution, which studied by S. Karlin and H. Rubin [2].

$$+\sum_{i=0}^{c}\binom{n}{i}\int_0^1\{\alpha p^u-\beta(1-p)^v\}p^i(1-p)^{n-i}\frac{1}{B(\lambda,\ \mu)}p^{\lambda-1}(1-p)^{\mu-1}dp$$

$$=\frac{\beta}{B(\lambda,\ \mu)}\int_0^1 p^{\lambda-1}(1-p)^{v+\mu-1}dp$$

$$+\frac{1}{B(\lambda,\ \mu)}\sum_{i=0}^{c}\binom{n}{i}\Big\{\alpha\int_0^1 p^{u+i+\lambda-1}(1-p)^{n-i+\mu-1}dp$$

$$-B\int_0^1 p^{i+\lambda-1}(1-p)^{v+n-i+\mu-1}dp\Big\}=\frac{\beta B(\lambda,\ v+\mu)}{B(\lambda,\ \mu)}$$

$$+\frac{1}{B(\lambda+\mu)}\sum_{i=0}^{c}\binom{n}{i}\{\alpha B(u+i+\lambda,\ n-i+\mu)-\beta B(i+\lambda,\ v+n-i+\mu)\}$$

In this case, the function $i_0(n)$ is obtained by the above lemma, that is, $i_0(n)$ is the maximum integral value of i's which satisfy the following inequality,

$$\alpha B(u+i+\lambda,\ n-i+\mu)-\beta B(i+\lambda,\ v+n-i+\mu)\leqq 0 \tag{3.4}$$

Especially, when $u=v=1$, (3.4) can be reduced to the following simple formula;

$$\alpha(i+\lambda)-\beta(n-i+\mu)\leqq 0 \tag{3.5}$$

or

$$\frac{\beta(n+\mu)-\alpha\lambda}{\alpha+\beta}\geqq i$$

Therefore, in this case,

$$i_0(n)=\left[\frac{\beta(n+\mu)-\alpha\lambda}{\alpha+\beta}\right] \tag{3.6}$$

where [ ] denotes the Gauss symbol. From (3.6), it is easily seen that $i_0(n+1)-i_0(n)=0$ or 1.

Now, we must obtain the optimal sample size which minimizes $T(n,\ i_0(n))$. Such a sample size can be obtained as the minimum integral value of n such that

$$T(n+1,\ i_0(n+1))-T(n,\ i_0(n))\geqq 0 \tag{3.7}$$

that is,

$$\sum_{i=0}^{i_0(n+1)}\binom{n+1}{i}\{\alpha B(i+\lambda+1,\ n+1-i+\mu)-\beta B(i+\lambda,\ n+1-i+\mu+1)\} \tag{3.8}$$

$$-\sum_{i=0}^{i_0(n)}\binom{n}{i}\{\alpha B(i+\lambda+1,\ n-i+\mu)-\beta B(i+\lambda,\ n-i+\mu+1)\}$$

$$+S(n+1)-S(n)\geqq 0.$$

(3.8) can be modified as follows.

$$
\begin{aligned}
&T(n+1,\ i_0(n+1))-T(n,\ i_0(n)) \\
&=\frac{1}{(n+\lambda+\mu)(n+\lambda+\mu+1)}\sum_{i=0}^{i_0(n)}\Big[\alpha\Big\{(n-i+\mu)\binom{n+1}{i} \\
&\quad-(n+\lambda+\mu+1)\binom{n}{i}\Big\}(i+\lambda)-\beta\Big\{(n-i+\mu+1)\binom{n+1}{i} \\
&\quad-(n+\lambda+\mu+1)\binom{n}{i}\Big\}(n-i+\mu)\Big]B(i+\lambda,\ n-i+\mu) \\
&\quad+Z+S(n+1)-S(n)\geqq 0
\end{aligned}
$$

where,

$$
(3.9)\qquad Z=\begin{cases}\binom{n+1}{i_0(n+1)}\{\alpha B(i_0(n+1)+\lambda+1,\ n+1-i_0(n+1)+\mu) \\ \quad -\beta B(i_0(n+1)+\lambda,\ n+1-i_0(n+1)+\mu+1)\} \\ \qquad\qquad ,\ \text{if}\ i_0(n+1)=i_0(n)+1 \\ 0 \qquad\qquad ,\ \text{if}\ i_0(n+1)=i_0(n)\end{cases}
$$

Thus the optimal sample size n_0 can be obtained by means of numerical computation.

4. Conclusion.

In D.R.S.I.P., the sampling inspection plan adopted has no relation with the state of the production process except its average fraction defective. We think this fact shows an important weak-point of Dodge-Remng's plan. In this paper, we have shown another sampling scheme, where we use more information about the production process, given in the form of the a priori distribution. Therefore, in actual case, it is important to estimate the a priori distribution as exactly as possible. The role of the a prior distribution in sampling inspection plan was emphasized by G. A. Barnard [3].

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