SOME EVALUATIONS FOR CONTINUOUS MONTE CARLO METHOD BY USING BROWNIAN HITTING PROCESS

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1. Introduction.

In this paper, we want to give an estimator of the solution of Dirichlet's problem by Monte Carlo method (which is essentially the same as in Muller's paper [1]), and to evaluate the mean square error of the estimator and the mean number of steps needed in the Monte Carlo method. These mean values are closely connected to the mean sojourn times of certain Brownian hitting motions. Therefore we shall proceed with the evaluation by using them.

2. Definitions.

Let D be a bounded domain in the m-dimensional Euclidian space R^m, whose boundary B is sufficiently smooth. Following Muller [1] we define a spherical process as follows. Let $K(x)$ be a maximum sphere contained in $\overline{D}=D+B$ with a fixed centre x $(x \in \overline{D})$ and of radius $\rho(x)$, and $k(x)$ its boundary.

Definition of spherical process $S(x)=\{x_0, x_1, \cdots, x_n\}$ $(x \in D)$

1) $x_0 \equiv x$

2) If the sequence of random variables $x_0, \cdots, x_n$ is already defined, then we define x_{n+1} as a random variable which is uniformly distributed on $k(x_n)$ under the condition $x_0, x_1, \cdots, x_n$.

Let $N=\inf\{n : \rho(x_n) \leqq \delta\}$. The process $S(x, \delta)=\{x_0, \cdots, x_N\}$ is called δ-truncated spherical process, which is given by stopping $S(x)$ after N-steps. The fact that N is almost surely finite is easily proved, but we postpone the proof up to section 4.

Now let f be a bounded continuous function on B and n be its Dirichlet's solution with respect to D, i.e.,

$$\Delta u=0 \quad \text{in} \quad D\,,$$
$$u=f \quad \text{on} \quad B\,.$$

For the value of $u(x)(x \in D)$, we construct its estimator $U(x)$ as follows:

1) We construct the process $S(x, \delta)$

2) and set $U(x)=f(\bar{x}_N)$
where $\bar{x}_N$ is any point on $k(x_N)\cap B$.

3. Evaluation of the mean square error of $U(x)$.

In this section we take the following assumption A for u.

A. $u(x)$ satisfies Lipschitz' condition uniformly in $\bar{D}$;

$$|u(x)-u(y)|\leqq c\,\|x-y\| \qquad \text{for all } x, y\in \bar{D},^{(1)}$$

where $\|x-y\|$ represents a Euclidian distance between x and y.

Setting

$$M=\sup_{x\in B}|f(x)|=\sup_{x\in D}|u(x)|$$
$$L=\sup_{x,y\in D}\|x-y\|$$

we define the intermediate estimator $U^*(x)$ by

$$U^*(x)=u(x_N)\ .$$

LEMMA 1. $E(U^*(x)\mid x_0, \cdots, x_k, N\geqq k)=u(x_k)$[2]
especially $E(U^*(x))=u(x)$

PROOF. For the process $S(x)=\{x_0, \cdots, x_n, \cdots\}$ we have

$$E\{u(x_{n+1})\mid x_1, \cdots, x_n\}=E\{u(x_{n+1})\mid x_n\}=u(x_n)$$

by the mean value theorem of the harmonic function u. Therefore, the process $\{u(x_0), \cdots, u(x_n), \cdots\}$ is a martingale with respect to $x_1, \cdots, x_n, \cdots$. As $\{u(x_0), \cdots, u(x_N)\}$ is a stopped process constructed from $\{u(x_0), \cdots, u(x_n), \cdots\}$, $\{u(x_0), \cdots, u(x_N)\}$ itself is a martingale with respect to $x_1, \cdots, x_n$, (see Doob [2] chap. 7). Especially $E(U^*(x)\mid x_1, \cdots, x_k, N\geqq k)=E(u(x_N)\mid x_k, N\geqq k)=u(x_k)$

LEMMA 2. $\bar{\sigma}^2=E\{U^*(x)-u(x)\}^2\leqq c^2L^2$

PROOF.
$$\begin{aligned}\bar{\sigma}^2&=E(u(x_N)-u(x_0))^2\\&=E\{E\{(u(x_N)-u(x_1))\chi(N>1)\\&\quad+(u(x_1)-u(x_0))\chi(N>0)\mid x_0, x_1\}\}^{2(3)}\end{aligned}$$

$$(1)\qquad =E\{(u(x_N)-u(x_1))^2\chi(N>1)\}+E\{(u(x_1)-u(x_0))^2\chi(N>0)\}$$

by Lemma 1

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1) For sufficiently smooth B and $f(x)$, this condition A is always satisfied. It is desirable to give the simple evaluation of c from B and f.

For example, if D is a 2-dimensional circle with radius L and centre origin and if the boundary function $\bar{f}(\theta)=f(re^{i\theta})$ is twice differentiable, then $c\leqq\sqrt{\frac{2}{\pi}K}\cdot\frac{1}{L}$ where $K=\int(f''(\theta))^2d\theta$.

2) $E(X\mid Y)$ represents a conditional expectation of X under Y.

3) $\chi(A)$ is a characteristic function of the event A.

$$= \sum_{k=0}^{\infty} E\{(u(x_{k+1}) - u(x_k))^2 \chi(N>k)\} \qquad \text{for } N<\infty \text{ a.s.}$$

$$= \sum_{k=0}^{\infty} E\{c^2 \rho(x_k)^2 \chi(N>k)\} ,$$

for $x_{k+1} \in k(x_k)$ $\quad |u(x_{k+1}) - u(x_k)| \leqq c \| x_{k+1} - x_k \| = c\rho(x_k)$.

On the other hand, we can construct a Brownian motion $\tilde{x}(t)$ starting from $\tilde{x}(0) \equiv x \equiv x_0$ and put

$$\sigma_1 = \inf\{t : \tilde{x}(t) \notin K(x_0)\} \text{ and } \tilde{x}_1 = \tilde{x}(\sigma_1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sigma_n = \inf\{t : t > \sigma_{n-1}, \tilde{x}(t) \notin K\} \text{ and } \tilde{x}_n = \tilde{x}(\sigma_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tilde{N} = \inf\{n : \rho(\tilde{x}_n) \leqq \delta\}$$

Then, processes $\{\tilde{x}_1, \cdots, \tilde{x}_n, \cdots, \tilde{N}\}$ and $\{x_1, \cdots, x_n, \cdots, N\}$ have the same joint distribution.

Setting $\tilde{\sigma}_B = \inf\{t : \tilde{x}(t) \in B\}$, we have

$$E(\tilde{\sigma}_B) \geqq \sum_{k=0}^{\infty} E(\sigma_{k+1} - \sigma_k)$$

$$\geqq E\left\{ \sum_{k=0}^{\infty} (\chi(\tilde{N} \geqq k)(\sigma_{k+1} - \sigma_k) \mid \tilde{x}_k \right\} \tag{2}$$

$$= \sum_{k=0}^{\infty} E\{\chi(\tilde{N}>k) a \rho(x_k)^2\}$$

where a is a constant $(a>0)$ such that

$$E\{\tilde{\sigma}(\rho) \mid x(t) = x\} = a\rho^2$$

where $\quad \tilde{\sigma}(\rho) = \inf\{s - t : s > t \| \tilde{x}(s) - \tilde{x}(t) \| > \rho\}$.

But $D \subset K(x, L) = \{y : \| x - y \| \leq L\}$, therefore,

$$E(\tilde{\sigma}_B) \leqq E(\tilde{\sigma}(L) \mid x(0) = x) = aL^2 \tag{3}$$

From (1), (2) and (3), we get

$$\bar{\sigma}^2 \leqq c^2 L^2 .$$

THEOREM *We have*

$$\sigma(x)^2 = E(U(x) - u(x))^2 \leq c^2\delta^2 + 4Mc\delta + c^2L^2 . \tag{4}$$

When $U_1(x), \cdots, U_n(x)$ *are* n*-independent estimators of* $u(x)$, *we have*

$$\sigma_n(x)^2 = E\left(\frac{1}{n}\sum_{k=1}^{n} U_k(x) - u(x)\right)^2 \leqq c^2\delta^2 + \frac{1}{n}(4Mc\delta + c^2L^2) . \tag{5}$$

PROOF. We shall prove only (5). ((4) is a special case of (5).)

$$\sigma_n(x)^2=\frac{1}{n^2}\{E(\sum(U_k(x)-U_k^*(x))+\sum(U_k^*(x)-U_k(x))^2\} .$$

Since
$$| U_k(x)-U_k^*(x) |=| u(x_N^{-(k)})-u(x_N^{(k)}) |\leqq c\rho(x_N^{(k)})\leqq c\delta$$
$$| U_k^*(x)-u_k(x) |<2M , \qquad E(U_k^*(x))=u(x) ,$$

and $\{U_k(x), U_k^*(x)\}$ and $\{U_k(x), U_l^*(x)\}$ are independent $(k\neq l)$, we get

$$\sigma_n(x)^2\leqq\frac{1}{n^2}[c^2\delta^2n^2+2Mc\delta n+\bar{\sigma}^2(x)n\} ,$$

which proves (5).

Example. Let D be a circle with redius L_1 and centre origin, and $\bar{f}(\theta)=f(Le^{i\theta})=c_1L_1\sin\theta$. Then $c_1\geqq c$ $L_1=\frac{L}{2}$ and $\sigma(o)^2=\frac{1}{2}c_1^2L_1^2\geqq\frac{1}{8}c^2L^2$.

4. Evaluation of the mean number of steps, $E(N)$.

In this section we assume that D is a convex domain (which may be unbounded). On it we can construct a process $S(x, \delta)$ and $\rho=\rho(x)=\inf_{y\in D}\| y-x \|$.

Let D^* be an upper half domain,

$$D^*=\{x: x=(x^1, \cdots, x^m) , \quad x^1>0\}$$

and set
$$x^*=(\rho, 0, \cdots, 0) ,$$

We construct a process $S^*(x^*, \delta^*)$. Then, by Muller [1] we have $E(N)\leqq E(N^*)$. Therefore, to evaluate the upper bound of $E(N)$, we can assume that $D\equiv D^*$ and $x=x^*=(\rho, 0, \cdots 0)$. Now, we consider the Brownian hitting process $\{\tilde{x}(t)\}$ starting from x whose generator is given by $(x^1)^2\left(\sum_{i=1}^m\frac{\partial}{\partial x^{i2}}\right)$ in D[4]. Then this process has the following properties ([3]).

(p. 1) $\tilde{x}(t)$ is a continuous (a.s.) strong Markov process in D.

(p. 2) Let G be a bounded domain in D and $M\subset D$. Then

$$E\left(\int_0^{\sigma_G}\chi_M(\tilde{x}(s))ds \mid x(o)=x\right)=\int_M g_D(x, y)\frac{1}{(y^1)^2}dy^1, \cdots, dy^m .$$

[4] This process is given by the random change of time from the Brownian motion, that is,

$$\tilde{x}(t)=\tilde{x}^*(S^{-1}(t)) , \qquad S(u)=\int_0^u\frac{dv}{\tilde{x}^{*1}(v))^2}$$

where $\tilde{x}^*$ is a Brownian process.

where $\chi_M(\cdot)$ is an indicator function of set M, $g_D(x, y)$ is a Green's function of domain D, and $\sigma_G=\inf\{t: x(t)\notin G\}$.

(p. 3) If $\sigma_1,\cdots,\sigma_n,\cdots,\tilde{x}_1,\cdots,\tilde{x}_n,\cdots,\tilde{N}$ are defined as in the proof of lemma 2 (in section 2), then $\{\tilde{x}_1,\cdots,\tilde{x}_n,\cdots,\tilde{N}\}$ and $\{x_1,\cdots,x_n,\cdots,N\}$ of $S(x)$ have the same distribution.

As D is assumed to be the upper half domain, we get

$$K(x)=\{y: \|y-x\|<\rho(x)=x^1\}.$$

Let $\bar{K}(x)$ be an upper half of sphere $K(x)$, i.e.,

$$\bar{K}(x)=\{y: y=(y^1,\cdots,y^N),\ y^1>x^1,\ y\in K(x)\}$$

LEMMA 3.

$$E\left(\int_{\sigma_k}^{\sigma_{k+1}}\chi_{\bar{K}(x_k)}(\tilde{x}(t))dt \mid x_k\right)\geqq\frac{1}{16m} \tag{6}$$

PROOF. For $m\geqq 3$,[5] the left hand side of (6) is equal to

$$E\left(\int_0^{\sigma}\chi_{\bar{K}(x_k)}(\tilde{x}(t))dt \mid x(o)=x_k\right)\quad(\sigma=\inf t: \tilde{x}(t)\notin K(x)).$$

By using the explicit form of Green's function for the sphere in (p. 2),

$$=\int_0^{\rho}\frac{1}{\omega_m(m-2)}p^{m-1}\left(\frac{1}{p^{m-2}}-\frac{1}{p^{m-2}}\right)dp\int_{\bar{\Omega}}\frac{d\omega}{(\rho+p\cos\theta)^2}\quad\left(\begin{array}{c}(\rho=\rho(x_k))\\ \cos\theta=\dfrac{y^1-x^1}{\rho(x_k)}\end{array}\right)$$

where ω_m is the volume of the surface of unit m-sphere, $d\omega$ is a volume element, and $\bar{\Omega}$ is an upper half part of it. Then,

$$=\int_0^1\frac{1}{\omega_m(m-2)}(p-p^{m-1})dp\int_{\bar{\Omega}}\frac{d\omega}{(1+p\cos\theta)^2}$$

$$\geqq\frac{1}{8}\int_0^1\frac{1}{m-2}(p-p^{N-1})dp\geqq\frac{1}{16m}$$

THEOREM $$E_x(N)<\left(1+\log\frac{\rho(x)}{\delta}\right)16m\qquad(\rho(x)>\delta).$$

PROOF. Let D_1 be a domain $D_1=\{y: y^1>\delta\}\subset D$. Then, for $x>\delta$, $\bar{K}(x)\subset D_1$. Therefore,

$$E\left\{\int_0^{\infty}\chi_{D_1}(\tilde{x}(t))dt\right\}\geqq\sum_{k=0}^{\infty}E\left\{\left(\left(\int_{\sigma_k}^{\sigma_{k+1}}\chi_{\bar{K}(x_k)}(\tilde{x}(t))dt\right)\chi(\tilde{x}_k^1>\delta)\right\}\right.$$

$$=\sum_{k=0}^{\infty}E\left\{E\left(\int_{\sigma_k}^{\sigma_{k+1}}\chi_{\bar{K}(x_k)}(\tilde{x}(t))dt\right)\chi(\tilde{x}_k^1>\delta)\mid x_k\right)\right\}$$

$$\geqq\sum_{k=0}^{\infty}\frac{1}{16\,m}E(\chi(\tilde{x}_k^1>\delta))\geqq\frac{1}{16\,m}E(N)$$

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5) For $m=2$, the proof is the same, except the explicit form for Green's function.

On the other hand, setting $\rho=\rho(x)$, by using the explicit form of Green's function for upper half domain, we have

$$E\left\{\int_0^\infty \chi_{D_1}(\tilde{x}(t))dt\right\}=\int_\delta^\rho x^1\cdot\frac{dx^2}{x^1}+\rho\int_\rho^\infty\frac{dx^1}{(x^1)^2}=1+\log\frac{\rho}{\delta},$$

which proves the theorem.

Remark: Using the above auxiliary process, we can also show

$$E(N)>C_1\left(\log\frac{\rho(x)}{\delta}\right)m$$

for certain constant C_1 independent of δ, $\rho(x)$ and m ($\rho(x)>\delta$), when $D=\{y:y^1>0\}$.

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REFERENCES

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[3] Ito, K and McKean, H. P., Lecture note.