THE EXTREME VALUE OF THE GENERALIZED DISTANCES OF THE INDIVIDUAL POINTS IN THE MULTIVARIATE NORMAL SAMPLE

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1. Summary.

The extreme value of the generalized distances, from the origin, of N individual points which may be correlated each other, in the p-variate normal sample is defined and discussed. It contains, as special cases, (i) the extreme deviate from the population mean or the sample mean, (ii) the extreme deviate from the control variate and (iii) the range defined by (2.10) or (2.11) below. The exact sampling distributional theory of this statistic is extremely difficult to find, even its moments. However, the method of obtaining the approximate upper $100\,\alpha$ percentage points for the ordinary significance level α is given. The lower percentage points can be obtained in the similar way if necessary. In connection with the evaluation of the approximate percentage points, the two-dimensional chi-square distribution is discussed and the asymptotic formulas for the joint distribution function of the two generalized distances are given in the special forms for the present aim. The extreme deviate from the sample mean will be explained in some detail and the tables of the approximate upper 5, 2.5 and 1% points are given. For the cases (ii) and (iii) mentioned above the details are omitted and will be discussed in the case of need.

2. Introduction and definition.

Let $\boldsymbol{y}'_\alpha=(y_{1\alpha}, y_{2\alpha}, \cdots, y_{p\alpha})$, $(\alpha=1, 2, \cdots, N)$, be N p-variate vectors with mean vector $\boldsymbol{o}'=(0, 0, \cdots, 0)$ and with covariance matrix $\gamma\Lambda(\gamma>0)$ and let the covariance matrix of $\boldsymbol{y}_\alpha$ and $\boldsymbol{y}_\beta(\alpha\neq\beta)$ be $\delta\Lambda$ where Λ is a symmetric, positive definite matrix and $\gamma>|\delta|$. Karl Pearson [9] defined the generalized distance of the αth variate $\boldsymbol{y}_\alpha$ from the origin by

$$\chi^2_\alpha=\frac{1}{\gamma}\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^{ij}y_{i\alpha}y_{j\alpha}=\frac{1}{\gamma}\boldsymbol{y}'_\alpha\Lambda^{-1}\boldsymbol{y}_\alpha\ ,$$

where $\Lambda^{-1}=(\lambda^{ij})$ is the inverse of Λ. However, when Λ is not known a *priori*, we must use the estimate of Λ. Let $\boldsymbol{L}=(l_{ij})$ be the unbiasedly estimated matrix of Λ based on the ν degrees of freedom, which is statistically independent of $\boldsymbol{y}_\alpha(\alpha=1, 2, \cdots, N)$ and $\boldsymbol{L}^{-1}=(l^{ij})$ be the inverse of $\boldsymbol{L}$. Putting $\boldsymbol{L}$ instead of Λ in (2.1), we have the so called studentized form, i.e.,

$$T_\alpha^2=\frac{1}{\gamma}\sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}y_{i\alpha}y_{j\alpha}=\frac{1}{\gamma}\boldsymbol{y}_\alpha'\boldsymbol{L}^{-1}\boldsymbol{y}_\alpha . \tag{2.2}$$

In this paper we are concerned with the maximum values of $\gamma\chi_\alpha^2$ and γT_α^2,

$$\hat{\chi}_{\mathrm{MAX}}^2=\max_\alpha\{\gamma\chi_\alpha^2\}=\max_\alpha\{\boldsymbol{y}_\alpha'\Lambda^{-1}\boldsymbol{y}_\alpha\} \tag{2.3}$$

and

$$\hat{T}_{\mathrm{MAX}}^2=\max_\alpha\{\gamma T_\alpha^2\}=\max_\alpha\{\boldsymbol{y}_\alpha'\boldsymbol{L}^{-1}\boldsymbol{y}_\alpha\} . \tag{2.4}$$

For the special cases of these quantities, we shall consider the random sample of size n drawn from a p-variate population. Let $\boldsymbol{x}_\alpha'=(x_{1\alpha}, x_{2\alpha}, \cdots, x_{p\alpha})$, $(\alpha=1, 2, \cdots, n)$, be n observed values of a p-variate vector which has the mean vector $\boldsymbol{m}'=(m_1, m_2, \cdots, m_p)$ and the covariance matrix $\Lambda=(\lambda_{ij})$. Then the maximum deviate from the population mean is defined as the positive square root of

$$\chi_{\mathrm{MAX\cdot D}}^2=\max_\alpha\{(\boldsymbol{x}_\alpha-\boldsymbol{m})'\Lambda^{-1}(\boldsymbol{x}_\alpha-\boldsymbol{m})\} \tag{2.5}$$

or

$$T_{\mathrm{MAX\cdot D}}^2=\max_\alpha\{(\boldsymbol{x}_\alpha-\boldsymbol{m})'\boldsymbol{L}^{-1}(\boldsymbol{x}_\alpha-\boldsymbol{m})\} \tag{2.6}$$

according to the case when Λ is known and the case when Λ is not known, and the maximum deviate from the sample mean, $\bar{\boldsymbol{x}}=(1/n)\sum_{\alpha=1}^n\boldsymbol{x}_\alpha$, by the positive equare root of

$$\hat{\chi}_{\mathrm{MAX\cdot D}}^2=\max_\alpha\{(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})'\Lambda^{-1}(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})\} \tag{2.7}$$

or

$$\hat{T}_{\mathrm{MAX\cdot D}}^2=\max_\alpha\{(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})'\boldsymbol{L}^{-1}(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})\} . \tag{2.8}$$

In the univariate case, $p=1$, $\hat{\chi}_{\mathrm{MAX\cdot D}}^2$ and $\hat{T}_{\mathrm{MAX\cdot D}}^2$ are reduced, respectively, to the forms

$$\hat{\chi}^2_{\text{MAX}\cdot\text{D}}=\max_{\alpha}\{(x_\alpha-\bar{x})^2/\sigma^2\}=(x_{\text{MAX}}-\bar{x})^2/\sigma^2 \quad \text{or} \quad (\bar{x}-x_{\text{MIN}})^2/\sigma^2$$

and

$$\hat{T}^2_{\text{MAX}\cdot\text{D}}=\max_{\alpha}\{(x_\alpha-\bar{x})^2/S^2_\nu\}=(x_{\text{MAX}}-\bar{x})^2/S^2_\nu \quad \text{or} \quad (\bar{x}-x_{\text{MIN}})^2/S^2_\nu$$

where S^2_ν is the unbiased estimate of the population variance σ^2 based on the ν degrees of freedom. Positive square roots of these statistics were studied by K. R. Nair [6, 7] and H. A. David [1, 2].

If $\boldsymbol{x}_0$ is the control variate which has the same distribution with that of $\boldsymbol{x}_\alpha$, then the maximum deviate from $\boldsymbol{x}_0$ is defined by the positive square root of

(2.9)
$$\hat{\chi}^2_{\text{MAX}\cdot\text{C}}=\max_{\alpha}\{(\boldsymbol{x}_\alpha-\boldsymbol{x}_0)'\Lambda^{-1}(\boldsymbol{x}_\alpha-\boldsymbol{x}_0)\}, \text{ or}$$
$$\hat{T}^2_{\text{MAX}\cdot\text{C}}=\max_{\alpha}\{(\boldsymbol{x}_\alpha-\boldsymbol{x}_0)'\boldsymbol{L}^{-1}(\boldsymbol{x}_\alpha-\boldsymbol{x}_0)\}\ .$$

In the same way we define the range of the sample in the multivariate case by the positive square root of the maximum of the squares of the generalized distances between two any observed points, that is, the positive square root of

(2.10)
$$R^2_{\text{MAX}}=\max_{\alpha<\beta}\{R^2_{\alpha\beta}\}=\max_{\alpha<\beta}\{(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)'\Lambda^{-1}(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)\}$$

or

(2.11)
$$\Re^2_{\text{MAX}}=\max_{\alpha<\beta}\{\Re^2_{\alpha\beta}\}=\max_{\alpha<\beta}\{(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)'\boldsymbol{L}^{-1}(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)\}\ .$$

When $p=1$, we have the square of the usual range in the univariate case.

It is easily seen that the various statistics defined in (2.5)–(2.11) are the special cases of (2.3) and (2.4) and that the values of γ and δ of the respective cases are as in the following table:

Statistics	$\boldsymbol{y}_\alpha$ $(\boldsymbol{y}_{\alpha\beta})$	γ	δ	N
$\chi^2_{\text{MAX}\cdot\text{D}}$ and $T^2_{\text{MAX}\cdot\text{D}}$	$\boldsymbol{x}_\alpha-\boldsymbol{m}$	1	0	n
$\hat{\chi}^2_{\text{MAX}\cdot\text{D}}$ and $\hat{T}^2_{\text{MAX}\cdot\text{D}}$	$\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}}$	$\frac{n-1}{n}$	$-\frac{1}{n}$	n
$\hat{\chi}^2_{\text{MAX}\cdot\text{C}}$ and $\hat{T}^2_{\text{MAX}\cdot\text{C}}$	$\boldsymbol{x}_\alpha-\boldsymbol{x}_0$	2	1	n
R^2_{MAM} and $\Re^2_{\text{MAX}}$	$\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta$	2	0 or 1	$\frac{1}{2}n(n-1)$

First we shall deal with the general statistics defined in (2.3) and (2.4) and then with their special cases. To find the sampling distributions of $\hat{\chi}^2_{\text{MAX}}$ and $\hat{T}^2_{\text{MAX}}$ exactly is extremely difficult, but the values of the upper (or the lower) percentage points of these statistics can be approximately evaluated for the ordinary significance levels with the accuracy sufficient for the most practical applications. Though we treat, in this paper, the case of the upper percentage point, we can evaluate the lower percentage point by the analogous way if necessary.

In order to evaluate the upper percentage points of $\hat{\chi}^2_{\text{MAX}}$ and $\hat{T}^2_{\text{MAX}}$, we shall use the following well-known formula:

The probability P of the realization of at least one among the N events $E_1, E_2, \cdots, E_N$ is given by

$$P = S_1 - S_2 + S_3 - \cdots \pm S_N\,, \tag{2.12}$$

where S_r is the sum of the $\binom{N}{r}$ joint probabilities of $r(\leqq N)$ events with r different subscripts.
In our cases E_α corresponds to the event $\hat{\chi}^2_\alpha \equiv \gamma\chi^2_\alpha > a^2$ or $\hat{T}^2_\alpha \equiv \gamma T^2_\alpha > b^2$ and P corresponds to $P_r(\hat{\chi}^2_{\text{MAX}} > a^2)$ or $P_r(\hat{T}^2_{\text{MAX}} > b^2)$ where a^2 and b^2 are some constants. If the joint distribution of the E's is a symmetric function of the E's, then (2.12) becomes

$$P = NP_r(E_1) - \frac{1}{2}N(N-1)P_r(E_1, E_2) + \cdots \pm P_r(E_1, E_2, \cdots, E_N)\,. \tag{2.13}$$

The maximum deviates, (2.5)-(2.9) are of this case.

3. First approximate upper percentage points of $\hat{\chi}^2_{\text{MAX}}$ and $\hat{T}^2_{\text{MAX}}$.

For our cases of $\hat{\chi}^2_{\text{MAX}}$ and $\hat{T}^2_{\text{MAX}}$, (2.12) can be written as

$$P_r\{\hat{\chi}^2_{\text{MAX}} > a^2\} = NP_r\{\hat{\chi}^2_1 > a^2\} - S_2 + \cdots \pm S_N \tag{3.1}$$

and

$$P_r\{\hat{T}^2_{\text{MAX}} > b^2\} = NP_r\{\hat{T}^2_1 > b^2\} - S'_2 + \cdots \pm S'_N\,, \tag{3.2}$$

respectively. For reasonably large values of a^2 and b^2, the good first approximations to the left-hand sides of (3.1) and (3.2) are provided by $NP_r\{\hat{\chi}^2_1 > a^2\}$ and $NP_r\{\hat{T}^2_1 > b^2\}$.

Now let us assume that the distribution of $\boldsymbol{y}_\alpha$ is normal and that the unbiased estimate $\boldsymbol{L}$ of $\boldsymbol{\Lambda}$ has the Wishart distribution with ν degrees of freedom. Then $\chi_\alpha^2=(1/\gamma)\boldsymbol{y}_\alpha'\Lambda^{-1}\boldsymbol{y}_\alpha$ is distributed according to the chi-square distribution with p degrees of freedom and $T_\alpha^2=(1/\gamma)\boldsymbol{y}_\alpha'\boldsymbol{L}^{-1}\boldsymbol{y}_\alpha$ has the Hotelling's T^2-distribution with ν degrees of freedom, i.e.,

$$(3.3)\qquad dF\left(\frac{T_\alpha^2}{\nu}\right) = \frac{\Gamma[(\nu+1)/2]}{\Gamma(p/2)\Gamma[(\nu+1-p)/2]}\left(\frac{T_\alpha^2}{\nu}\right)^{p/2-1}\left(1+\frac{T_\alpha^2}{\nu}\right)^{-(\nu+1)/2} d\left(\frac{T_\alpha^2}{\nu}\right).$$

Consequently we have as the first approximations

$$(3.4)\qquad P_r\{\hat{\chi}_{\mathrm{MAX}}^2>a^2\}\doteqdot NP_r\{\hat{\chi}_1^2>a^2\}=NP_r\{\chi_1^2>a^2/\gamma\} = N\int_{(1/\gamma)a^2}^{\infty}\frac{t^{p/2-1}e^{-t/2}}{2^{p/2}\Gamma(p/2)}dt$$

and

$$P_r\{\hat{T}_{\mathrm{MAX}}^2>b^2\}\doteqdot NP_r\{\hat{T}_1^2>b^2\}=NP_r\left\{\frac{T_1^2}{\nu\gamma}>\frac{b^2}{\nu\gamma}\right\}$$
$$=\frac{N\Gamma[(\nu+1)/2]}{\Gamma(p/2)\Gamma[(\nu+1-p)/2]}\int_{\frac{b^2}{\nu\gamma}}^{\infty}u^{p/2-1}(1+u)^{-(\nu+1)/2}du$$
$$=\frac{N}{B[(\nu+1-p)/2,\, p/2]}\int_0^{\frac{\nu\gamma}{\nu\gamma+b^2}}v^{(\nu+1-p)/2-1}(1-v)^{p/2-1}dv$$
$$=NI_{\frac{\nu\gamma}{\nu\gamma+b^2}}\left(\frac{1}{2}(\nu+1-p),\,\frac{1}{2}p\right),$$

where $I_x(c, d)$ is the K. Pearson's incomplete beta function [10]. If we denote the upper $100\alpha^*\%$ point of the chi-square distribution with p degrees of freedom by $\chi^2(\alpha^*;p)$ and the lower $100\alpha^*\%$ point of the beta-distribution with parameters $(\nu+1-p)/2$ and $p/2$ by $C(\alpha^*;(\nu+1-p)/2, p/2)$, then $A_1^2(\alpha;p,N)$, the first approximation to the upper $100\alpha\%$ point of $\hat{\chi}_{\mathrm{MAX}}^2$, is calculated from

$$(3.6)\qquad A_1^2(\alpha;p,N)=\gamma\chi^2(\alpha/N;p)$$

and $B_1^2(\alpha;p,N,\nu)$, the first approximation to upper $100\alpha\%$ point of $\hat{T}_{\mathrm{MAX}}^2$ is given by

$$(3.7)\qquad B_1^2(\alpha;p,N,\nu)=\nu\gamma\left\{\frac{1}{C(\alpha/N;(\nu+1-p)/2,\, p/2)}-1\right\}.$$

Next it is necessary to examine whether $A_1^2(\alpha;p,N)$ and $B_1^2(\alpha;p,N,\nu)$ have the accuracy sufficient for most practical applications.

By Bonferroni's inequalities, we have, for the significance level α,

$$\alpha-\beta(\alpha\,;p,N)<P_r\{\hat{\chi}^2_{\mathrm{MAX}}>A_1^2(\alpha\,;p,N)\}<\alpha \tag{3.8}$$

and

$$\alpha-\beta^*(\alpha\,;p,N,\nu)<P_r\{\hat{T}^2_{\mathrm{MAX}}>B_1^2(\alpha\,;p,N,\nu)\}<\alpha\,, \tag{3.9}$$

where $\beta(\alpha\,;p,N)$ is S_2 for $a^2=A_1^2(\alpha\,;p,N)$ in (3.1) and $\beta^*(\alpha\,;p,N,\nu)$ is S_2' for $B_1^2(\alpha\,;p,N,\nu)$ in (3.2). For the symmetric cases, $\beta(\alpha\,;p,N)$ and $\beta^*(\alpha\,;p,N,\nu)$ can be written respectively

$$\beta(\alpha\,;p,N)=\frac{1}{2}N(N-1)P_r\{\hat{\chi}_1^2>A_1^2(\alpha\,;p,N),\ \hat{\chi}_2^2>A_1^2(\alpha\,;p,N)\} \tag{3.10}$$

and

$$\begin{aligned}\beta^*(\alpha\,;p,N,\nu)\\ =\frac{1}{2}N(N-1)P_r\{\hat{T}_1^2>B_1^2(\alpha\,;p,N,\nu),\ \hat{T}_2^2>B_1^2(\alpha\,;p,N,\nu)\}\,.\end{aligned} \tag{3.11}$$

In order to evaluate β and β^* for fixed α, p and N, we shall find, in the following two sections, individual terms $P_r\{\hat{\chi}_\alpha^2>A_1^2(\alpha;p,N),\ \hat{\chi}_\beta^2>A_1^2(\alpha;p,N)\}$ and $P_r\{\hat{T}_\alpha^2>B_1^2(\alpha\,;p,N,\nu),\ \hat{T}_\beta^2>B_1^2(\alpha\,;p,N,\nu)\}$ $(\alpha\neq\beta)$ of S_2 and S_2', respectively.

4. Two-dimensional chi-square distribution.

Let $(x_1,y_1),(x_2,y_2),\cdots,(x_m,y_m)$ be m independent random variables, each of which is distributed according to a bivariate standard normal distribution with correlation coefficient ρ, that is,

$$f(x,y)dxdy=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}(x^2-2\rho xy+y^2)\right]dxdy\,. \tag{4.1}$$

Then it is easy to find the joint distribution of two variables

$$U=\textstyle\sum_{\alpha=1}^m x_\alpha^2 \quad\text{and}\quad V=\sum_{\alpha=1}^m y_\alpha^2\,. \tag{4.2}$$

Since the joint characteristic function of x^2 and y^2 is

$$\varphi(t_1,t_2)=\mathscr{E}\{\exp(it_1x^2+it_2y^2)\}=\{(1-2it_1)(1-2it_2)+4\rho^2t_1t_2\}^{-1/2}$$

and (x_α^2,y_α^2), $(\alpha=1,2,\cdots,m)$, are independent, the characteristic function of U and V is, by the multiplication theorem,

$$\Phi_m(t_1,t_2)=\{\varphi(t_1,t_2)\}^m=\{(1-2it_1)(1-2it_2)+4\rho^2t_1t_2\}^{-m/2} \tag{4.3a}$$

which is also expressible as

(4.3b) $\Phi_m(t_1, t_2)$

$$=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\sum_{j=0}^{\infty}\frac{\Gamma(m/2+j)}{j\,!}\frac{\rho^{2j}}{[\{1-2(1-\rho^2)it_1\}\{1-2(1-\rho^2)it_2\}]^{(m/2)+j}}\,.$$

By the inversion theorem we have the joint probability element of U and V in the form

(4.4) $f(U, V)dUdV$

$$=\frac{1}{2^m\Gamma(m/2)(1-\rho^2)^{m/2}}\sum_{j=0}^{\infty}\frac{1}{j\,!\,\Gamma(m/2+j)}\left(\frac{\rho}{2(1-\rho^2)}\right)^{2j}(UV)^{m/2+j-1}$$
$$\times e^{-(1/[2(1-\rho^2)])(U+V)}dUdV$$

and hence the joint density function of

$$\chi_1^2=U/(1-\rho^2) \quad\text{and}\quad \chi_2^2=V/(1-\rho^2) \tag{4.5}$$

is written in the form

$$f_m(\chi_1^2, \chi_2^2)=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\sum_{j=0}^{\infty}\frac{\Gamma(m/2+j)\rho^{2j}}{j\,!}g^*_{m+2j}(\chi_1^2)g^*_{m+2j}(\chi_2^2)\,, \tag{4.6}$$

where $g^*_\nu(z)=2^{-\nu/2}[\Gamma(\nu/2)]^{-1}z^{(\nu/2)-1}e^{-z/2}$, the density function of the chi-square distribution with ν degrees of freedom.

DEFINITION. The distribution defined by (4.6) is called the two-dimensional (or bivariate) chi-square distribution with m degrees of freedom and with the parameter ρ and the combined variate (χ_1^2, χ_2^2) which has the density (4.6) is called the two-dimensional (or bivariate) chi-square variate with m degrees of freedom and with the parameter ρ.

We obtain easily the following theorem from (4.3a) and (4.5) or directly from (4.6):

THEOREM 4.1. *The characteristic function of the two-dimensional chi-square distribution with m degrees of freedom and with the parameter ρ is*

$$\Psi_m(t_1, t_2)=\left\{\left(1-\frac{2it_1}{1-\rho^2}\right)\left(1-\frac{2it_2}{1-\rho^2}\right)+4\left(\frac{\rho}{1-\rho^2}\right)^2 t_1t_2\right\}^{-m/2} \tag{4.7}$$

It is obvious that, when $\rho=0$, χ_1^2 and χ_2^2 are independent (one-dimensional) chi-square variates and that the distribution of the sum of two independent bivariate chi-square variates with the common parameter ρ and with m_1 and m_2 degrees of freedom respectively is also a bivariate chi-square distribution with m_1+m_2 degrees of freedom and with the parameter ρ.

Now we shall prove the following

THEOREM 4.2. Let $(\boldsymbol{y}_1', \boldsymbol{y}_2')$ be a $(p+p)$-dimensional normal variate with mean vector (o, o) and with covariance matrix $\boldsymbol{\Sigma}=\begin{bmatrix}\gamma & \delta \\ \delta & \gamma\end{bmatrix}\times\Lambda$ (direct product) where Λ $(p\times p)$ is a symmetric, positive definite matrix and $\gamma>|\delta|$. Then the combined variate of $\hat{\chi}_1^2=\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1$ and $\hat{\chi}_2^2=\boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2$ is distributed as $[(\gamma^2-\delta^2)/\gamma]\cdot(\chi_1^2, \chi_2^2)$, where (χ_1^2, χ_2^2), is the two-dimensional chi-square variate with p degrees of freedom and with the parameter $\rho=\delta/\gamma$.

PROOF.

$$f(\boldsymbol{y}_1, \boldsymbol{y}_2)=(2\pi)^{-p}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{y}_1'\,\boldsymbol{y}_2')\boldsymbol{\Sigma}^{-1}\begin{pmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2\end{pmatrix}\right\}$$
$$=(2\pi)^{-p}|\Lambda|^{-1}(\gamma^2-\delta^2)^{-p/2}$$
$$\times\exp\left\{-\frac{1}{2}[\gamma/(\gamma^2-\delta^2)]\cdot[\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1-2(\delta/\gamma)\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_2+\boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2]\right\}.$$

Let us consider the non-singular linear transformation

$$\boldsymbol{y}_1'=\sqrt{\gamma}\,\boldsymbol{z}_1'\boldsymbol{C}' \quad\text{and}\quad \boldsymbol{y}_2'=\sqrt{\gamma}\,\boldsymbol{z}_2'\boldsymbol{C}'$$

such that $\boldsymbol{C}'\Lambda^{-1}\boldsymbol{C}=\boldsymbol{I}$, where $\boldsymbol{I}$ is the unit matrix of order p. Under these transformations $(\boldsymbol{z}_1', \boldsymbol{z}_2')$ has the density

$$f(\boldsymbol{z}_1, \boldsymbol{z}_2)=\left(\frac{1}{2\pi\sqrt{1-\delta^2/\gamma^2}}\right)^p\exp\left\{-\frac{1}{2}\frac{1}{(1-\delta^2/\gamma^2)}\sum_{\alpha=1}^{p}\left(z_{1\alpha}^2-2\frac{\delta}{\gamma}z_{1\alpha}z_{2\alpha}+z_{2\alpha}^2\right)\right\},$$

from which it is seen that $(z_{11}, z_{21}), (z_{12}, z_{22}), \cdots, (z_{1p}, z_{2p})$ are considered as p independent combined variables, each of which has the bivariate normal distribution with mean $(0, 0)$, with unit variances and the correlation coefficient $\rho=\delta/\gamma$. Then $\chi_1^2=\sum_{\alpha=1}^{p}z_{1\alpha}^2/(1-\rho^2)$ and $\chi_2^2=\sum_{\alpha=1}^{p}z_{2\alpha}^2/(1-\rho^2)$ have a two-dimensional chi-square distribution with p degrees of freedom and with $\rho=\delta/\gamma$. Since $\chi_1^2=\boldsymbol{z}_1'\boldsymbol{z}_1/(1-\delta^2/\gamma^2)=[\gamma/(\gamma^2-\delta^2)]\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1=[\gamma/(\gamma^2-\delta^2)]\hat{\chi}_1^2$ and $\chi_2^2=[\gamma/(\gamma^2-\delta^2)]\hat{\chi}_2^2$, our theorem is proved.

From this theorem and (4.6), we have

$$(4.8)\quad P_r\{\hat{\chi}_1^2>a^2, \hat{\chi}_2^2>a^2\}=P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>a^2, \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>a^2\}$$
$$=[(\gamma^2-\delta^2)/\gamma^2]^{p/2}[\Gamma(p/2)]^{-1}\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}(\delta/\gamma)^{2j}G_{p/2+j}^2(\eta)$$

where $\eta=(1/2)[\gamma/(\gamma^2-\delta^2)]a^2$ and $G_m(\eta)=[\Gamma(m)]^{-1}\int_\eta^\infty t^{m-1}e^{-t}dt=\int_{2\eta}^\infty g_{2m}^*(\chi^2)d\chi^2$.

5. Evaluation of $P_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1>B_1^2(\alpha; p, N, \nu), \boldsymbol{y}_2'\boldsymbol{L}^{-1}\boldsymbol{y}_2>B_1^2(\alpha; p, N, \nu)\}$. Let $(\boldsymbol{y}_1', \boldsymbol{y}_2')$ be the $(p+p)$-dimensional normal variate defined in Theorem 4.2 and let $\boldsymbol{L}$ be the unbiased estimate of Λ, which has the Wishart distribution with ν degrees of freedom, (5.1) $W_p(\boldsymbol{L}\,;\Lambda,\nu)$

$$=\frac{1}{\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{\nu-i+1}{2}\right)}\left(\frac{\nu^p}{2^p|\Lambda|}\right)^{\nu/2}|\boldsymbol{L}|^{(\nu-p-1)/2}\exp\left\{-\frac{\nu}{2}tr\Lambda^{-1}\boldsymbol{L}\right\}.$$

and is independent of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. In order to evaluate $\beta^*(\alpha\,;p,N,\nu)$ in (3.9), we shall consider, in this section, $P_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1>B_1^2(\alpha\,;p,N,\nu),\ \boldsymbol{y}_2'\boldsymbol{L}^{-1}\boldsymbol{y}_2>B_1^2(\alpha;p,N,\nu)\}$, where $B_1^2(\alpha\,;p,N,\nu)$ is the first approximation to the upper $100\alpha\%$ point of $\hat{T}^2_{\text{MAX}}$, that is, the solution of the equation

$$\alpha=NP_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1>B_1^2(\alpha\,;p,N,\nu)\}\,. \tag{5.2}$$

However the exact evaluation of

$$P_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1>B_1^2(\alpha\,;p,N,\nu),\ \boldsymbol{y}_2'\boldsymbol{L}^{-1}\boldsymbol{y}_2>B_1^2(\alpha\,;p,N,\nu)\}$$

is very difficult and hence we shall try to obtain the asymptotic formula for this probability with respect to ν^{-1}.*)

To do this we shall preliminarily prove several lemmas.

LEMMA 5.1. $B_1^2(\alpha\,;p,N,\nu)$ has the following asymptotic expression up to the order ν^{-2},

(5.3) $B_1^2(\alpha\,;p,N,\nu)$

$$=\gamma\chi^2+\frac{1}{2\nu}\gamma\chi^2(\chi^2+p)+\frac{1}{24\nu^2}\gamma\chi^2\{4\chi^4+(13p-2)\chi^2+7p^2-4\}+O(\nu^{-3}),$$

where $\chi^2\equiv\chi^2(\alpha/N\,;p)$ the upper $100\alpha/N\%$ point of the chi-square distribution with p degrees of freedom.

PROOF. This is the direct consequence from the fact that $(1/\gamma)B_1^2(\alpha\,;p,N,\nu)$ is the upper $100\alpha/N\%$ point of the Hotelling's T^2-distribution with ν degrees of freedom and from the Hotelling and Frankel's formula [3]

$$T^2(\alpha/N\,;p,\nu)=\chi^2+\frac{1}{2\nu}\chi^2(\chi^2+p)+\frac{1}{24\nu^2}\chi^2\{4\chi^4+(13p-2)\chi^2+7p^2-4\}+O(\nu^{-3})\,,$$

where $T^2(\alpha/N\,;p,\nu)$ is the $100\alpha/N\%$ point of T^2-distribution.

LEMMA 5.2. (The integral of Dirichlet's type)

$$\int_{\infty>\sum_{j=1}^{p}z_j<\eta}\cdots\int\prod_{j=1}^{p}z_j^{l_j-1}\exp\left\{-\sum_{j=1}^{p}z_j\right\}\prod_{j=1}^{p}dz_j=\prod_{j=1}^{p}\Gamma(l_j)]\cdot\left[\Gamma\left(\sum_{j=1}^{p}l_j\right)\right]^{-1}\int_{\eta}^{\infty}u^{\Sigma l_j-1}e^{-u}du \tag{5.4}$$

*) If necessary, we can obtain the asymptotic formulas for $P_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1<b_1^2,\ \boldsymbol{y}_2'\boldsymbol{L}^{-1}\boldsymbol{y}_2<b_2^2\}$ and $P_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1<B_1^2(\alpha;p,N,\nu),\ \boldsymbol{y}_2'\boldsymbol{L}^{-1}\boldsymbol{y}_2<B_1^2(\alpha;p,N,\nu)\}$ by the obvious modification of the method in this section.

where $z_j>0,\ j=1, 2, \cdots, p$.

The next lemma concerns the simple expression of the multiple integral

$$(5.5)\qquad J=\frac{|I-\zeta|}{\pi^p}\left(\frac{\gamma^2-\delta^2}{\gamma^2}\right)^{p/2}\int\cdots\int_{\mathfrak{D}}$$
$$\times\exp\{-w_1'(I-\zeta)w_1+2(\delta/\gamma)w_1'(I-\zeta)w_2-w_2'(I-\zeta)w_2\}dw_1dw_2$$

where $w_i(i=1, 2)$ are p-dimensional column vector variates, ζ is a diagonal matrix, $\mathrm{diag}\{\zeta_1, \zeta_2, \cdots, \zeta_p\}$, with $|\zeta_i|<1,\ i=1, 2, \cdots, p,\ \gamma>|\delta|,\ \eta>0$ and $\mathfrak{D}$ is the domain defined by $w_1'w_1>\eta$ and $w_2'w_2>\eta$.

LEMMA 5.3. *J can be expressed in the form*

$$(5.6)\qquad J=[1-(\delta/\gamma)^2]^{p/2}|(I-\zeta)^{-1}(I-\zeta E_1)(I-\zeta)^{-1}(I-\zeta E_2)-(\delta/\gamma)^2E_1E_2|^{-1/2}\times$$
$$\times G_{p/2}^{(1)}(\eta)G_{p/2}^{(2)}(\eta)\ .$$

where $E_i(i=1, 2)$ are operators such that $E_i^mG_\gamma^{(i)}(\eta)=G_{\gamma+m}^{(i)}(\eta)$ for any integer $m\geqq 0$ and $G_\gamma^{(i)}(\eta)\equiv G(\eta)=[\Gamma(\gamma)]^{-1}\int_\eta^\infty t^{\gamma-1}e^{-t}dt$, ($i$) being attached for convenience of the operation of E_i.

PROOF. This lemma is proved by expanding the exponential function in the integrand of (5.5), by integrating term by term (this is allowable) and by using the integral formula of Dirichlet's type. From (5.5)

$$J=|I-\zeta|\,\pi^{-p}[1-(\delta/\gamma)^2]^{p/2}\sum_{\substack{\nu_{1\alpha},\nu_{2\alpha},\mu_\alpha=0\\ \alpha=1,\cdots,p}}^{\infty}(2\delta/\gamma)^{\Sigma\mu_\alpha}\frac{\prod_{\alpha=1}^p\zeta_\alpha^{\nu_{1\alpha}+\nu_{2\alpha}}(1-\zeta_\alpha)^{\mu_\alpha}}{\prod_{\alpha=1}^p\nu_{1\alpha}!\ \nu_{2\alpha}!\ \mu_\alpha!}\times$$
$$\times\int\cdots\int_{\Sigma w_{1\alpha}^2>\eta}\prod_{\alpha=1}^p w_{1\alpha}^{2\nu_{1\alpha}+\mu_\alpha}\exp\left\{-\sum_{\alpha=1}^p w_{1\alpha}^2\right\}\prod_{\alpha=1}^p dw_{1\alpha}\times$$
$$\times\int\cdots\int_{\Sigma w_{2\alpha}^2>\eta}\prod_{\alpha=1}^p w_{1\alpha}^{2\nu_{2\alpha}+\mu_\alpha}\exp\left\{-\sum_{\alpha=1}^p w_{2\alpha}^2\right\}\prod_{\alpha=1}^p dw_{2\alpha}$$

$$=|I-\zeta|\pi^{-p}[1-(\delta/\gamma)^2]^{p/2}\sum_{\substack{\nu_{1\alpha},\nu_{2\alpha},\mu_\alpha=0\\ \alpha=1,\cdots,p}}^{\infty}(2\delta/\gamma)^{2\Sigma\mu_\alpha}\frac{\prod_{\alpha=1}^p\zeta_\alpha^{\nu_{1\alpha}+\nu_{2\alpha}}(1-\zeta_\alpha)^{2\mu_\alpha}}{\prod_{\alpha=1}^p\nu_{1\alpha}!\ \nu_{2\alpha}!\ (2\mu_\alpha)!}\times$$
$$\times\int\cdots\int_{\Sigma z_{1\alpha}>\eta}\prod_{\alpha=1}^p z_{1\alpha}^{\nu_{1\alpha}+\mu_\alpha-(1/2)}\exp\left\{-\sum_{\alpha=1}^p z_{1\alpha}\right\}\prod_{\alpha=1}^p dz_{1\alpha}\times$$
$$\times\int\cdots\int_{\Sigma z_{2\alpha}>\eta}\prod_{\alpha=1}^p z_{2\alpha}^{\nu_{2\alpha}+\mu_\alpha-(1/2)}\exp\left\{-\sum_{\alpha=1}^p z_{2\alpha}\right\}\prod_{\alpha=1}^p dz_{2\alpha}\ .$$

The integrals are of Dirichlet's type and from Lemma (5.2)

$$J=|I-\zeta|\pi^{-p}[1-(\delta/\gamma)^2]^{p/2}\sum_{\substack{\nu_{1\alpha},\nu_{2\alpha},\mu_\alpha=0\\ \alpha=1,\cdots,p}}(2\delta/\gamma)^{2\Sigma\mu_\alpha}\frac{\prod_{\alpha=1}^{p}\zeta_\alpha^{\nu_{1\alpha}+\nu_{2\alpha}}(1-\zeta_\alpha)^{2\mu_\alpha}}{\prod_{\alpha=1}^{p}\nu_{1\alpha}!\,\nu_{2\alpha}!\,(2\mu_\alpha)!}$$

$$\times\prod_{\alpha=1}^{p}\Gamma\Big(\nu_{1\alpha}+\mu_\alpha+\frac{1}{2}\Big)\Gamma\Big(\nu_{2\alpha}+\mu_\alpha+\frac{1}{2}\Big)G_{\Sigma(\nu_{1\alpha}+\mu_\alpha)+p/2}(\eta)G_{\Sigma(\nu_{2\alpha}+\mu_\alpha)+p/2}(\eta)\,.$$

Using the operators $E_i (i=1, 2)$ and the superscripts (i) for convenience defined in the lemma, we can have the expression

$$\prod_{\alpha=1}^{p}\sum_{\nu_{i\alpha}=0}^{\infty}\frac{\zeta_\alpha^{\nu_{i\alpha}}}{\nu_{i\alpha}!}\frac{\Gamma(\nu_{i\alpha}+\mu_\alpha+1/2)}{\Gamma(\mu_\alpha+1/2)}G^{(i)}_{\Sigma(\nu_{i\alpha}+\mu_\alpha)+p/2}(\eta)$$
$$=|I-\zeta E_i|^{-1/2}\prod_{\alpha=1}^{p}(1-\zeta_\alpha E_i)^{-\mu_\alpha}G^{(i)}_{\Sigma\mu_\alpha+p/2}(\eta),\quad (i=1, 2)\,.$$

Hence we can write J as

$$J=\left\{\frac{|I-\zeta E_1||I-\zeta E_2|}{|I-\zeta|^2}\right\}^{-1/2}\pi^{-p/2}[1-(\delta/\gamma)^2]^{p/2}$$
$$\times\sum_{\mu_1,\cdots,\mu_p=0}^{\infty}\prod_{\alpha=1}^{p}\frac{\Gamma(\mu_\alpha+1/2)}{\mu_\alpha!}\left\{\frac{(1-\zeta_\alpha)^2}{(1-\zeta_\alpha E_1)(1-\zeta_\alpha E_2)}\Big(\frac{\delta}{\gamma}\Big)^2\right\}^{\mu_\alpha}G^{(1)}_{\Sigma\mu_\alpha+p/2}(\eta)G^{(2)}_{\Sigma\mu_\alpha+p/2}(\eta)$$
$$=[1-(\delta/\gamma)^2]^{p/2}|(I-\zeta)^{-1}(I-\zeta E_1)(I-\zeta)^{-1}(I-\zeta E_2)|^{-1/2}$$
$$\times\left|I-(I-\zeta E_2)^{-1}(I-\zeta)(I-\zeta E_1)^{-1}(I-\zeta)\Big(\frac{\delta}{\gamma}\Big)^2E_1E_2\right|^{-1/2}G^{(1)}_{p/2}(\eta)G^{(2)}_{p/2}(\eta)$$
$$=[1-(\delta/\gamma)^2]^{p/2}\left|(I-\zeta)^{-1}(I-\zeta E_1)(I-\zeta)^{-1}(I-\zeta E_2)-\Big(\frac{\delta}{\gamma}\Big)^2E_1E_2\right|^{-1/2}$$
$$\times G^{(1)}_{p/2}(\eta)G^{(2)}_{p/2}(\eta)\,,$$

which proves the lemma.

LEMMA 5.4.

(5.7) $$DP_r\{y_1'\Lambda^{-1}y_1>a^2,\ y_2'\Lambda^{-1}y_2>a^2\}=-[\gamma/(\gamma^2-\delta^2)][1-(\delta/\gamma)^2]^{p/2}\Big[\Gamma\Big(\frac{1}{2}p\Big)\Big]^{-1}$$
$$\times\sum_{j=1}^{\infty}\frac{\Gamma(p/2+j)}{j!}\Big(\frac{\delta}{\gamma}\Big)^{2j}g_{p/2+j}(\eta)G_{p/2+j}(\eta)$$

(5.8) $$D^2P_r\{y_1'\Lambda^{-1}y_1>a^2,\ y_2'\Lambda^{-1}y_2>a^2\}=\frac{1}{2}[\gamma/(\gamma^2-\delta^2)]^2[1-(\delta/\gamma)^2]^{p/2}\Big[\Gamma\Big(\frac{1}{2}p\Big)\Big]^{-1}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}\Big(\frac{\delta}{\gamma}\Big)^{2j}[\{g_{p/2+j}(\eta)-g_{p/2+j-1}(\eta)\}G_{p/2+j}(\eta)+g^2_{p/2+j}(\eta)]\,,$$

where $g_m(\eta)=[\Gamma(m)]^{-1}\eta^{m-1}e^{-\eta} (m>0)$ with the following exceptions,

$$g_{p/2+j-1}(\eta)=\begin{cases}0 & \text{for } p=2,\ j=0\,,\\ \dfrac{-1}{2\sqrt{\pi}}\eta^{-3/2}e^{-\eta} & \text{for } p=1,\ j=0\,,\end{cases}$$

and $\eta=(1/2)\{\gamma/(\gamma^2-\delta^2)\}a^2$ and $D\equiv\partial/\partial a^2$.

PROOF of this lemma is easily obtained by differentiating (4.8) directly with respect to a^2.

Now we shall consider the evaluation of the derivatives of $P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>a^2, \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>a^2\}$ with respect to λ_{rs} which is an element of Λ. To do this, we consider

$$(5.9)\quad J\equiv P_r\{\boldsymbol{y}_1'(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{y}_1>a^2, \boldsymbol{y}_2'(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{y}_2>a^2\}$$
$$=\left\{1+\sum_{rs}\varepsilon_{rs}\partial_{rs}+\frac{1}{2}\sum_{rstu}\varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu}+\cdots\right\}P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>a^2, \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>a^2\} ,$$

where $\partial_{rs}\equiv(1/2)(1+\delta_{rs})\partial/\partial\lambda_{rs}$ (δ_{rs} is Kronecker's delta), and $\boldsymbol{\varepsilon}$ is a $p\times p$ symmetric matrix consisting of small increments ε_{ij} to λ_{ij} $(i, j, =1, 2, \cdots, p)$. On the other hand we can also express J in the form

$$J=\frac{1}{(2\pi)^p|\Lambda|(\gamma^2-\delta^2)^{p/2}}\int\cdots\int_{\mathfrak{D}^*}$$
$$\times\exp\left\{-\frac{1}{2}[\gamma/(\gamma^2-\delta^2)](\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1-2(\delta/\gamma)\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_2+\boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2)\right\}d\boldsymbol{y}_1d\boldsymbol{y}_2 ,$$
$$\mathfrak{D}^* : \boldsymbol{y}_1'(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{y}_1>a^2, \boldsymbol{y}_2'(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{y}_2>a^2 .$$

Making the non-singular linear transformations

$$\boldsymbol{y}_1=\sqrt{(\gamma^2-\delta^2)/\gamma}\,\boldsymbol{C}\boldsymbol{w}_1 \quad\text{and}\quad \boldsymbol{y}_2=\sqrt{(\gamma^2-\delta^2)/\gamma}\,\boldsymbol{C}\boldsymbol{w}_2$$

such that

$$\frac{1}{2}\boldsymbol{C}'(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{C}=\boldsymbol{I} \quad\text{and}\quad \frac{1}{2}\boldsymbol{C}'\Lambda^{-1}\boldsymbol{C}=\boldsymbol{I}-\boldsymbol{\zeta}$$

where $\boldsymbol{I}$ is the unit matrix and $\boldsymbol{\zeta}$ is a diagonal matrix, diag $\{\zeta_1, \zeta_2, \cdots, \zeta_p\}$, with $|\zeta_i|<1$ for all i, J becomes

$$J=|\boldsymbol{I}-\boldsymbol{\zeta}|\pi^{-p}[1-(\delta/\gamma)^2]^{p/2}\int\cdots\int_{\mathfrak{D}}$$
$$\times\exp\{-\boldsymbol{w}_1'(\boldsymbol{I}-\boldsymbol{\zeta})\boldsymbol{w}_1+2(\delta/\gamma)\boldsymbol{w}_1'(\boldsymbol{I}-\boldsymbol{\zeta})\boldsymbol{w}_2-\boldsymbol{w}_2'(\boldsymbol{I}-\boldsymbol{\zeta})\boldsymbol{w}_2\}d\boldsymbol{w}_1d\boldsymbol{w}_2$$
$$\mathfrak{D} : \boldsymbol{w}_1'\boldsymbol{w}_1>\eta, \boldsymbol{w}_2'\boldsymbol{w}_2>\eta ,$$

where $2\eta=[\gamma/(\gamma^2-\delta^2)]a^2$. This is the form of (5.5) and hence, from Lemma 5.3, J can be expressed in the form (5.6). Putting $\Delta_i=E_i-1$, $(i=1, 2)$, $H=[1-(\delta/\gamma)^2E_1E_2]^{-1}$ and $\boldsymbol{X}=(\Lambda+\boldsymbol{\varepsilon})^{-1}\Lambda-\boldsymbol{I}$, we have

$$(5.10)\quad J=[1-(\delta/\gamma)^2]^{p/2}|\boldsymbol{I}-\{\boldsymbol{X}(\Delta_1+\Delta_2)-\boldsymbol{X}^2\Delta_1\Delta_2\}H|^{-1/2}H^{p/2}G_{p/2}^{(1)}(\eta)G_{p/2}^{(2)}(\eta) .$$

In order to obtain the derivatives of $P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>a^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>a^2\}$ with respect to λ_{rs}, we expand (5.10) in power series of ε's and compare the resultant with (5.9). Since, for fixed p and for small ε's, we have

$$|\boldsymbol{I}-\{\boldsymbol{X}(\Delta_1+\Delta_2)-\boldsymbol{X}^2\Delta_1\Delta_2\}H|^{-1/2}=\exp\Big[-\frac{1}{2}\log|\boldsymbol{I}-\{\boldsymbol{X}(\Delta_1+\Delta_2)-\boldsymbol{X}^2\Delta_1\Delta_2\}H|\Big]$$
$$=\exp\Big[\frac{1}{2}tr\{\boldsymbol{X}(\Delta_1+\Delta_2)-\boldsymbol{X}^2\Delta_1\Delta_2\}H+\frac{1}{4}tr\{\boldsymbol{X}(\Delta_1+\Delta_2)-\boldsymbol{X}^2\Delta_1\Delta_2\}^2H^2+\cdots\Big]$$
$$=1+\frac{1}{2}tr\boldsymbol{X}(\Delta_1+\Delta_2)H$$
$$+\frac{1}{4}\Big\{\mathrm{tr}\boldsymbol{X}^2+\frac{1}{2}(\mathrm{tr}\boldsymbol{X})^2\Big\}(\Delta_1+\Delta_2)^2H^2-\frac{1}{2}(\mathrm{tr}\boldsymbol{X}^2)\Delta_1\Delta_2H$$
$$+\frac{1}{6}\Big\{\mathrm{tr}\boldsymbol{X}^3+\frac{3}{4}(\mathrm{tr}\boldsymbol{X}^2)(\mathrm{tr}\boldsymbol{X})+\frac{1}{8}(\mathrm{tr}\boldsymbol{X})^3\Big\}(\Delta_1+\Delta_2)^3H^3$$
$$-\frac{1}{2}\Big\{\mathrm{tr}\boldsymbol{X}^3+\frac{1}{2}(\mathrm{tr}\boldsymbol{X}^2)(\mathrm{tr}\boldsymbol{X})\Big\}(\Delta_1+\Delta_2)\Delta_1\Delta_2H^2$$
$$+\frac{1}{8}\Big\{\mathrm{tr}\boldsymbol{X}^4+\frac{2}{3}(\mathrm{tr}\boldsymbol{X}^3)(\mathrm{tr}\boldsymbol{X})+\frac{1}{4}(\mathrm{tr}\boldsymbol{X}^2)^2$$
$$+\frac{1}{4}(\mathrm{tr}\boldsymbol{X}^2)(\mathrm{tr}\boldsymbol{X})^2+\frac{1}{48}(\mathrm{tr}\boldsymbol{X})^4\Big\}(\Delta_1+\Delta_2)^4H^4$$
$$-\frac{1}{2}\Big\{\mathrm{tr}\boldsymbol{X}^4+\frac{1}{2}(\mathrm{tr}\boldsymbol{X}^3)(\mathrm{tr}\boldsymbol{X})+\frac{1}{4}(\mathrm{tr}\boldsymbol{X}^2)^2+\frac{1}{8}(\mathrm{tr}\boldsymbol{X}^2)(\mathrm{tr}\boldsymbol{X})^2\Big\}(\Delta_1+\Delta_2)^2\Delta_1\Delta_2H^3$$
$$+\frac{1}{4}\Big\{\mathrm{tr}\boldsymbol{X}^4+\frac{1}{2}(\mathrm{tr}\boldsymbol{X}^2)^2\Big\}\Delta_1^2\Delta_2^2H^2$$
$$+\cdots,$$

and $\boldsymbol{X}$ has the expansion, with the notation $\Lambda_{rs}=\partial_{rs}\Lambda$,

$$\boldsymbol{X}=(\Lambda+\boldsymbol{\varepsilon})^{-1}\Lambda-\boldsymbol{I}=(\boldsymbol{I}+\Sigma\varepsilon_{rs}\Lambda^{-1}\Lambda_{rs})^{-1}-\boldsymbol{I}$$
$$=-\sum\varepsilon_{rs}\Lambda^{-1}\Lambda_{rs}+\sum\varepsilon_{rs}\varepsilon_{tu}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}-\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\Lambda_{vw}+\cdots,$$

J can be expanded in the following form

$$J=[1-(\delta/\gamma)^2]^{p/2}[1-\sum\varepsilon_{rs}(M_1)+\sum\varepsilon_{rs}\varepsilon_{tu}(M_2)-\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}(M_3)$$
$$+\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}(M_4)-\cdots]H^{p/2}G^{(1)}_{p/2}(\gamma)G^{(2)}_{p/2}, \tag{5.11}$$

where

$$(M_1)=\frac{1}{2}[rs](\Delta_1+\Delta_2)H,$$
$$(M_2)=\frac{1}{4}\{[rs|tu]+\frac{1}{2}[rs][tu]\}(\Delta_1+\Delta_2)^2H^2+\frac{1}{2}[rs|tu](\Delta_1+\Delta_2-\Delta_1\Delta_2)H,$$
$$(M_3)=\frac{1}{6}\Big\{[rs|tu|vw]+\frac{3}{4}[rs][tu|vw]+\frac{1}{8}[rs][tu][vw]\Big\}(\Delta_1+\Delta_2)^3H^3$$

$$+\frac{1}{2}\Big\{[rs|tu|vw]+\frac{1}{2}[rs][tu|vw]\Big\}(\Delta_1+\Delta_2)(\Delta_1+\Delta_2-\Delta_1\Delta_2)H^2$$

$$+\frac{1}{2}[rs|tu|vw](\Delta_1+\Delta_2-2\Delta_1\Delta_2)H\ ,$$

$$(M_4)=\frac{1}{8}\Big\{[rs|tu|vw|xy]+\frac{2}{3}[rs][tu|vw|xy]+\frac{1}{4}[rs|tu][vw|xy]$$

$$+\frac{1}{4}[rs][tu][vw|xy]+\frac{1}{48}[rs][tu][vw][xy]\Big\}(\Delta_1+\Delta_2)^4H^4$$

$$+\frac{1}{2}\Big\{[rs|tu|vw|xy]+\frac{1}{2}[rs][tu|vw|xy]+\frac{1}{4}[rs|tu][vw|xy]$$

$$+\frac{1}{8}[rs][tu][vw|xy]\Big\}(\Delta_1+\Delta_2)^2(\Delta_1+\Delta_2-\Delta_1\Delta_2)H^3$$

$$+\frac{1}{4}\Big\{3[rs|tu|vw|xy]+[rs][tu|vw|xy]$$

$$+\frac{1}{2}[rs|tu][vw|xy]\Big\}(\Delta_1+\Delta_2)(\Delta_1+\Delta_2-2\Delta_1\Delta_2)H^2$$

$$+\frac{1}{4}\Big\{[rs|tu|vw|xy]+\frac{1}{2}[rs|tu][vw|xy]\Big\}\Delta_1^2\Delta_2^2H^2$$

$$+\frac{1}{2}[rs|tu|vw|xy](\Delta_1+\Delta_2-3\Delta_1\Delta_2)H\ ,$$

and here we have used the abbreviated notations (see [4] and [11]) $[rs]=\mathrm{tr}\Lambda^{-1}\Lambda_{rs}=\lambda^{rs}$, $[rs|tu]=\mathrm{tr}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}=(1/2)(\lambda^{ur}\lambda^{st}+\lambda^{us}\lambda^{rt})$, and so on. Comparing (5.11) with (5.9), we have

$$(5.12)\quad \partial_{rs}P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>a^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>a^2\}$$
$$=-\frac{1}{2}[rs](\Delta_1+\Delta_2)H^{p/2+1}[1-(\delta/\gamma)^2]^{p/2}G^{(1)}_{p/2}(\eta)G^{(2)}_{p/2}G(\eta)\ ,$$

$$(5.13)\quad \partial_{rs}\partial_{tu}P_r\{\cdots\}=\frac{1}{2}\Big[\Big\{[rs|tu]+\frac{1}{2}[rs][tu]\Big\}(\Delta_1+\Delta_2)^2H^{p/2+2}$$
$$+2[rs|tu](\Delta_1+\Delta_2-\Delta_1\Delta_2)H^{p/2+1}\Big][1-(\delta/\gamma)^2]^{p/2}G^{(1)}_{p/2}(\eta)G^{(2)}_{p/2}(\eta).$$

$$(5.14)\quad \partial_{rs}\partial_{tu}\partial_{vw}P_r\{\cdots\}=-\Big[\Big\{[rs|tu|vw]+\frac{3}{4}[rs][tu|vw]$$
$$+\frac{1}{8}[rs][tu][vw]\Big\}(\Delta_1+\Delta_2)^3H^{p/2+3}$$
$$+3\Big\{[rs|tu|vw]+\frac{1}{2}[rs][tu|vw]\Big\}(\Delta_1+\Delta_2)(\Delta_1+\Delta_2-\Delta_1\Delta_2)H^{p/2+2}$$
$$+3[rs|tu|vw](\Delta_1+\Delta_2-2\Delta_1\Delta_2)H^{p/2+1}\Big]$$
$$\times[1-(\delta/\gamma)^2]^{p/2}G^{(1)}_{p/2}(\eta)G^{(2)}_{p/2}(\eta)\ ,$$

$$
\begin{aligned}
(5.15)\quad \partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}P_r\{\cdots\} = \Big[\Big\{ & 2[rs\,|\,tu\,|\,vw\,|\,xy] + [rs\,|\,vw\,|\,tu\,|\,xy] \\
& + 2[rs][tu\,|\,vw\,|\,xy] + \frac{1}{4}[rs\,|\,tu][vw\,|\,xy] + \frac{1}{2}[rs\,|\,vw][tu\,|\,xy] \\
& + \frac{1}{4}[rs][tu][vw\,|\,xy] + \frac{1}{2}[rs][vw][tu\,|\,xy] \\
& + \frac{1}{16}[rs][tu][vw][xy] \Big\} (\Delta_1+\Delta_2)^4 H^{p/2+4} \\
+ \Big\{ & 8[rs\,|\,tu\,|\,vw\,|\,xy] + 4[rs\,|\,vw\,|\,tu\,|\,xy] + 6[rs][tu\,|\,vw\,|\,xy] \\
& + [rs\,|\,tu][vw\,|\,xy] + 2[rs\,|\,vw][tu\,|\,xy] + \frac{1}{2}[rs][tu][vw\,|\,xy] \\
& + [rs][vw][tu\,|\,xy] \Big\} (\Delta_1+\Delta_2)^2(\Delta_1+\Delta_2-\Delta_1\Delta_2) H^{p/2+3} \\
+ \Big\{ & 12[rs\,|\,tu\,|\,vw\,|\,xy] + 6[rs\,|\,vw\,|\,tu\,|\,xy] + 6[rs][tu\,|\,vw\,|\,xy] \\
& + [rs|tu][vw|xy] \\
& + 2[rs|vw][tu|xy] \Big\} (\Delta_1+\Delta_2)(\Delta_1+\Delta_2-2\Delta_1\Delta_2) H^{p/2+2} \\
+ \Big\{ & 4[rs\,|\,tu\,|\,vw\,|\,xy] + 2[rs\,|\,vw\,|\,tu\,|\,xy] + [rs\,|\,tu][vw\,|\,xy] \\
& + 2[rs\,|\,vw][tu\,|\,xy] \Big\} \Delta_1^2\Delta_2^2 H^{p/2+2} \\
+ \Big\{ & 8[rs\,|\,tu\,|\,vw\,|\,xy] + 4[rs\,|\,vw\,|\,tu\,|\,xy] \Big\} (\Delta_1+\Delta_2-3\Delta_1\Delta_2) H^{p/2+1} \Big] \\
& \times [1-(\delta/\gamma)^2]^{p/2} G^{(1)}_{p/2}(\gamma) G^{(2)}_{p/2}(\gamma) \,.
\end{aligned}
$$

Lemma 5.5. Let $(\boldsymbol{y}_1', \boldsymbol{y}_2')$ be $(p+p)$-dimensional normal vector variate defined in Theorem 4.2. Then the first, second, third and fourth order derivatives of $P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1 < a^2, \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2 > a^2\}$ with respect to the elements of Λ are given by (5.12), (5.13), (5.14) and (5.15) respectively.

On the basis of the preliminary works discussed above, we can now obtain the asymptotic formula for

$$P_r\{\boldsymbol{y}_1'L^{-1}\boldsymbol{y}_1 < B_1^2(\alpha; p, N, \nu),\ \boldsymbol{y}_2'L^{-1}\boldsymbol{y}_2 > B_1^2(\alpha; p, N, \nu)\}$$

up to the term of order ν^{-2}. According to James' formula [(5.16) of 4], we have

$$
\begin{aligned}
(5.16)\quad & P_r\{\boldsymbol{y}_1'L^{-1}\boldsymbol{y}_1 > B_1^2(a; p, N, \nu),\ \boldsymbol{y}_2'L^{-1}\boldsymbol{y}_2 > B_1^2(\alpha; p, N, \nu)\} \\
& = \Big[1 + \frac{1}{\nu} \sum_{rstu} \lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu}
\end{aligned}
$$

$$+\frac{1}{\nu^2}\left\{\frac{4}{3}\sum_{rstuvw}\lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw}+\frac{1}{2}\sum_{rstuvwxy}\lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\right\}$$
$$\left.+O\left(\frac{1}{\nu^3}\right)\right]\cdot P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>B_1^2(\alpha;p,N,\nu),\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>B_1^2(\alpha;p,N,\nu)\}.$$

Since for fixed α, p and N, $B_1^2(\alpha; p, N, \nu)$ is a function of ν and when ν is large, $B_1^2(\alpha; p, N, \nu)$ will approach $A_1^2(\alpha; p, N)$, the first approximation to the upper $100\alpha\%$ point of $\hat{\chi}^2_{\text{MAX}}$, we need to expand

$$P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>B_1^2(\alpha;p,N,\nu),\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>B_1^2(\alpha;p,N,\nu)\}$$

in Taylor's series. From Lemma 5.1, we can represent $B_1^2(\alpha; p, N, \nu)$ as

$$B_1^2(\alpha;p,N,\nu)=A_1^2(\alpha;p,N)+h_1(\alpha)+h_2(\alpha)+\cdots$$
$$A_1^2(\alpha;p,N)\equiv A_1^2=\gamma\chi^2(\alpha/N;p)\ ,$$
$$h_1(\alpha)=\frac{1}{2\nu}\gamma\chi^2(\alpha/N;p)\{\chi^2(\alpha/N;p)+p\}\ .$$
$$h_2(\alpha)=\frac{1}{24\nu^2}\gamma\chi^2(\alpha/N;p)\{4\chi^4(\alpha/N;p)+(13p-2)\chi^2(\alpha/N;p)+7p^2-4\}\ .$$

Then by Taylor's expansion

$$P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>B_1^2(\alpha;p,N,\nu),\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>B_1^2(\alpha;p,N,\nu)\}$$
$$=\left[1+\{h_1(\alpha)+h_2(\alpha)+\cdots\}D+\frac{1}{2}\{h_1(\alpha)+h_2(\alpha)+\cdots\}^2D^2+\cdots\right]$$
$$\times P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>A_1^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>A_1^2\}\ ,$$

where $D\equiv\partial/\partial A_1^2$. Substituting this into (5.16), we obtain

$$(5.17)\quad P_r\{\boldsymbol{y}_1'L^{-1}\boldsymbol{y}_1>B_1^2(\alpha;p,N,\nu),\ \boldsymbol{y}_2'L^{-1}\boldsymbol{y}_2>B_1^2(\alpha;p,N,\nu)\}$$
$$=\left[1+\left\{h_1(\alpha)D+\frac{1}{\nu}\sum_{rstu}\lambda_{ur}\lambda_{st}\partial_{rstu}\right\}\right.$$
$$+\left\{h_2(\alpha)D+\frac{1}{2}h_1^2(\alpha)D^2+\frac{1}{\nu}h_1(\alpha)\sum_{rstu}\lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu}D\right.$$
$$+\frac{4}{3}\frac{1}{\nu^2}\sum_{rstuvw}\lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw}$$
$$\left.+\frac{1}{2}\frac{1}{\nu^2}\sum_{rstuvwxy}\lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\right\}$$
$$\left.+O\left(\frac{1}{\nu^3}\right)\right]\cdot P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1>A_1^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2>A_1^2\}\ .$$

From this formula we can evaluate the desired probability with the aid of Lemmas 5.4, 5.5 and (5.3).

(a) *Term of order zero,* $O_0(\alpha, p, N)$.

$$(5.18)\quad O_0(\alpha, p, N) \equiv P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1 > A_1^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2 > A_1^2\}$$
$$= [1-(\delta/\gamma)^2]^{p/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j}G_{p/2+j}^2(\eta)$$

where $2\eta = [\gamma/(\gamma^2-\delta^2)]\cdot A_1^2(\alpha;\,p,N) = [1-(\delta/\gamma)^2]^{-1}\chi^2(\alpha/N;\,p)$.

(b) *Term of order* ν^{-1}, $O_1(\alpha, p, N) = O_{11}(\alpha, p, N) + O_{12}(\alpha, p, N)$.

$$(5.19)\quad O_{11}(\alpha, p, N) \equiv h_1(\alpha)\cdot DP_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1 > A_1^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2 > A_1^2\}$$
$$= -\frac{1}{\nu}\frac{\chi^2(\chi^2+p)}{2}[1-(\delta/\gamma)^2]^{(p-2)/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j}g_{p/2+j}(\eta)G_{p/2+j}(\eta)$$

where $\chi^2 \equiv \chi^2(\alpha/N;\,p)$.

$$O_{12}(\alpha, p, N) \equiv \frac{1}{\nu}\sum_{rstu}\lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu}P_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1 > A_1^2,\ \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2 > A_1^2\}$$
$$= \frac{1}{\nu}[1-(\delta/\gamma)]^{p/2}\sum_{rstu}\lambda_{ur}\lambda_{st}\left[\left\{\frac{1}{2}[rs\,|\,tu] + \frac{1}{4}[rs][tu]\right\}(\Delta_1+\Delta_2)^2H^{p/2+2}\right.$$
$$\left. + [rs\,|\,tu](\Delta_1+\Delta_2-\Delta_1\Delta_2)H^{p/2+1}\right]G_{p/2}^{(1)}(\eta)G_{p/2}^{(2)}(\eta)$$
$$= \frac{1}{\nu}[1-(\delta/\gamma)^2]^{p/2}\left\{\frac{1}{4}p(p+2)(\Delta_1+\Delta_2)^2H^{p/2+2}\right.$$
$$\left. + \frac{1}{2}p(p+1)(\Delta_1+\Delta_2-\Delta_1\Delta_2)H^{p/2+1}\right\}G_{p/2}^{(1)}(\eta)G_{p/2}^{(2)}(\eta)\,.$$

Noting that

$$H^{p/2+m}G_{p/2}^{(1)}(\eta)G_{p/2}^{(2)}(\eta) = \{1-(\delta/\gamma)^2E_1E_2\}^{-(p/2+m)}G_{p/2}^{(1)}(\eta)G_{p/2}^{(2)}(\eta)$$
$$= \frac{1}{\Gamma(p/2+m)}\sum_{j=0}^{\infty}\frac{\Gamma(p/2+m+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j}G_{p/2+j}^{(1)}(\eta)G_{p/2+j}^{(2)}(\eta)\,,\qquad (m=0, 1, 2, \cdots)$$

and

$$\Delta_iG_{p/2+j}^{(i)}(\eta) = (E_i-1)G_{p/2+j}^{(i)}(\eta) = \left[\Gamma\left(\frac{p}{2}+1+j\right)\right]^{-1}\eta^{p/2+j}e^{-\eta} \equiv g_{p/2+j+1}^{(i)}(\eta)\,,$$
$$\Delta_i^2G_{p/2+j}^{(i)}(\eta) = (E_i^2-2E_i+1)G_{j/2+j}^{(i)}(\eta) = g_{p/2+j+2}^{(i)}(\eta) - g_{p/2+j+1}^{(i)}(\eta)\,,$$
$$\cdots$$

we obtain after some arrangement

$$(5.20)\quad O_{12}(\alpha, p, N) = \frac{1}{\nu}[1-(\delta/\gamma)^2]^{p/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}\sum_{j=0}^{\infty}\frac{\Gamma(p/2+1+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j}$$
$$\times\{(2\eta+p-2j)g_{p/2+j+1}(\eta)G_{p/2+j}(\eta) + (2j+1)g_{p/2+j+1}^2(\eta)\}\,,$$

where we put $G^{(i)}_{p/2+j}(\eta)\equiv G_{p/2+j}(\eta)$ and $g^{(i)}_{p/2+j+m}(\eta)\equiv g_{p/2+j+m}(\eta)$ after operation by E_i is carried out.

In the analogous way, though a great deal of algebra is needed, the term of order ν^{-2} can be obtained in the following form.

(c) *Term of order* ν^{-2}, $O_2(\alpha, p, N) = O_{21}(\alpha, p, N) + O_{22}(\alpha, p, N) + O_{23}(\alpha, p, N) + O_{24}(\alpha, p, N) + O_{25}(\alpha, p, N)$.

(5.21) $$O_{21}(\alpha; p, N)\equiv h_2(\alpha)\cdot DP_r\{\boldsymbol{y}_1'\Lambda^{-1}\boldsymbol{y}_1 > A_1^2, \boldsymbol{y}_2'\Lambda^{-1}\boldsymbol{y}_2 > A_1^2\}$$
$$= -\frac{1}{24\nu^2}[1-(\delta/\gamma)^2]^{(p-2)/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1} \times\chi^2\{4\chi^4+(13p-2)\chi^2+7p^2-4\} \times\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j} g_{p/2+j}(\eta)G_{p/2+j}(\eta)\ ,$$

(5.22) $$O_{22}(\alpha, p, N)\equiv\frac{1}{2}h_1^2(\alpha)D^2P_r\{\cdots\}$$
$$=\frac{1}{16\nu^2}[1-(\delta/\gamma)^2]^{p/2-2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}\chi^4(\chi^2+p)^2 \times\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j} \times[\{g_{p/2+j}(\eta)-g_{p/2+j-1}(\eta)\}G_{p/2+j}(\eta)+g^2_{p/2+j}(\eta)]$$

(5.23) $$O_{23}(\alpha, p, N)$$
$$\equiv h_1(\alpha)\frac{1}{\nu}\sum_{rstu}\lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu}DP_r\{\cdots\}$$
$$=-\frac{1}{2\nu^2}[1-(\delta/\gamma)^2]^{(p-2)/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}\chi^2(\chi^2+p)\sum_{j=0}^{\infty}\frac{\Gamma(p/2+1+j)}{j!} \times\Big[\left(\eta+\frac{p}{2}-j\right)\{(g_{p/2+j+1}(\eta)-g_{p/2+j}(\eta))G_{p/2+j}(\eta)+g_{p/2+j+1}(\eta)g_{p/2+j}(\eta)\} +(2j+1)\{g_{p/2+j+1}(\eta)-g_{p/2+j}(\eta)\}g_{p/2+j+1}(\eta)-g_{p/2+j+1}(\eta)G_{p/2+j}(\eta)\Big]\ ,$$

(5.24) $$O_{24}(\alpha, p, N)\equiv\frac{4}{3}\frac{1}{\nu^2}\sum_{rstuvw}\lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw}P_r\{\cdots\}$$
$$=\frac{1}{\nu^2}[1-(\delta/\gamma)^2]^{p/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\delta}{\gamma}\right)^{2j}\left\{\frac{4}{3}H_j(\eta)\right\}\ ,$$

where

(5.25) $$-\frac{4}{3}H_j(\eta)=\frac{8}{3}\Gamma\left(\frac{p}{2}+3+j\right)[\{g_{p/2+3+j}(\eta)-2g_{p/2+2+j}(\eta) +g_{p/2+1+j}(\eta)\}G_{p/2+j}(\eta)+3\{g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}g_{p/2+1+j}(\eta)] +4(p+3)\Gamma\left(\frac{p}{2}+2+j\right)[\{g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}$$

$$\times \{G_{p/2+j}(\eta)-g^2_{p/2+1+j}(\eta)\}+g^2_{p/2+1+j}(\eta)]$$
$$+2(p^2+3p+4)\Gamma\left(\frac{p}{2}+1+j\right)\{G_{p/2+j}(\eta)-g_{p/2+1+j}(\eta)\}g_{p/2+1+j}(\eta) .$$

(5.26) $$O_{25}(\alpha, p, N)\equiv\frac{1}{2}\frac{1}{\nu^2}\sum_{rstuvwxy}\lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}P_r\{\cdots\}$$
$$=\frac{1}{\nu^2}[1-(\delta/\gamma)^2]^{p/2}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\delta}{\gamma}\right)^{2j}\left\{\frac{1}{2}K_j(\eta)\right\} ,$$

where

(5.27) $$\frac{1}{2}K_j(\eta)=\Gamma\left(\frac{p}{2}+4+j\right)[\{g_{p/2+4+j}(\eta)-3g_{p/2+3+j}(\eta)$$
$$+3g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}G_{p/2+j}(\eta)+3\{g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}^2$$
$$+4\{g_{p/2+3+j}(\eta)-2g_{p/2+2+j}(\eta)+g_{p/2+1+j}(\eta)\}g_{p/2+1+j}(\eta)]$$
$$+2(p+5)\Gamma\left(\frac{p}{2}+3+j\right)[\{g_{p/2+3+j}(\eta)-2g_{p/2+2+j}(\eta)$$
$$+g_{p/2+j+1}(\eta)\}G_{p/2+j}(\eta)-\{g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}^2$$
$$-\{g_{p/2+3+j}(\eta)-5g_{p/2+2+j}(\eta)+4g_{p/2+1+j}(\eta)\}g_{p/2+1+j}(\eta)]$$
$$+(p^2+12p+23)\Gamma\left(\frac{p}{2}+2+j\right)[\{g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}$$
$$\times\{G_{p/2+j}(\eta)-2g_{p/2+1+j}(\eta)\}+g^2_{p/2+1+j}(\eta)]$$
$$+\frac{1}{2}(p^2+4p+7)\Gamma\left(\frac{p}{2}+2+j\right)\{g_{p/2+2+j}(\eta)-g_{p/2+1+j}(\eta)\}^2$$
$$+(2p^2+5p+5)\Gamma\left(\frac{p}{2}+1+j\right)$$
$$\times[2\{G_{p/2+j}(\eta)-g_{p/2+1+j}(\eta)\}g_{p/2+1+j}(\eta)-g^2_{p/2+1+j}(\eta)] .$$

The result are summarized in

THEOREM 5.1. Let $(\boldsymbol{y}_1', \boldsymbol{y}_2')$ be the $(p+p)$-dimensional normal variate defined in Theorem 4.2 and let $\boldsymbol{L}$ be the unbiased estimate of Λ, distributed independently of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ according to the Wishart distribution (5.1). Then, for the solution $B_1^2(\alpha; p, N, \nu)$ of (5.2), the asymptotic formula for $P_r\{\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1>B_1^2(\alpha; p, N, \nu), \boldsymbol{y}_2'\boldsymbol{L}^{-1}\boldsymbol{y}_2>B_1^2(\alpha; p, N, \nu)\}$ up to the term of order ν^{-2} is given by (5.18), (5.19), (5.20), (5.21), (5.22), (5.23), (5.24) and (5.26).

6. The maximum deviates from the sample mean, $\hat{\chi}^2_{\text{MAX.D}}$ and $\hat{T}^2_{\text{MAX.D}}$. As an example of evaluating the $\beta(\alpha; p, N)$ and the $\beta^*(\alpha; p, N, \nu)$ by application of Theorem 5.1, we shall discuss the maximum deviates from the sample mean defined in (2.7) and (2.8), i.e.,

$$\hat{\chi}^2_{\text{MAX}\cdot\text{D}}=\max_{\alpha}\{(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})'\Lambda^{-1}(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})\} \quad \text{and} \tag{6.1}$$

$$\hat{T}^2_{\text{MAX}\cdot\text{D}}=\max_{\alpha}\{(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})'\boldsymbol{L}^{-1}(\boldsymbol{x}_\alpha-\bar{\boldsymbol{x}})\} . \tag{6.2}$$

The statistical procedure based on $\chi^2_{\text{MAX}\cdot\text{D}}$ has an optimum property as a multiple decision procedure. This is the straight-forward generalization of the slippage problem treated by E. Paulson [8] in the univariate case and in the case of unknown variance.

Let $\boldsymbol{z}_\alpha, (\alpha=1, \cdots, n)$ be normal vector variates with mean vectors $\boldsymbol{m}_\alpha, (\alpha=1, 2, \cdots, n)$, respectively and with a common covariance matrix Λ which is assumed to be known. Then we consider the null hypothesis H_0 that $\boldsymbol{m}_1=\boldsymbol{m}_2=\cdots=\boldsymbol{m}_n$ and n alternative hypotheses $H_\alpha(\Delta\boldsymbol{m})$, $(\alpha=1, 2, \cdots, n)$, that $\boldsymbol{m}_1=\cdots=\boldsymbol{m}_{\alpha-1}=\boldsymbol{m}_\alpha-\Delta\boldsymbol{m}=\boldsymbol{m}_{\gamma+1}=\cdots=\boldsymbol{m}_n$ where $\Delta\boldsymbol{m}$ is a non-zero vector and $(n+1)$ decisions $D\,(\alpha=0, 1, 2, \cdots, n)$ to accept $H_\alpha(\alpha=0, 1, 2, \cdots, n)$. A. Kudo [5] has shown that the decision procedure for selecting one of the $(n+1)$ decisions $D_0, D_1, \cdots, D_n$ which, under some restrictions [(i), (ii) and (iii) of 5], maximizes the probability of making the correct decision when one of the hypotheses $H_0, H_1, \cdots, H_n$ is true, is given by the following:

$$\text{if } (\boldsymbol{z}_M-\bar{\boldsymbol{z}})'\Lambda^{-1}(\boldsymbol{z}_M-\bar{\boldsymbol{z}})>L_\varepsilon, \text{ select } D_M \text{ and}$$
$$\text{if } (\boldsymbol{z}_M-\bar{\boldsymbol{z}})'\Lambda^{-1}(\boldsymbol{z}_M-\bar{\boldsymbol{z}})\leqq L_\varepsilon, \text{ select } D_0$$

where $\bar{\boldsymbol{z}}=\sum_{\alpha=1}^{n}\boldsymbol{z}_\alpha/n$, M is defined by

$$(\boldsymbol{z}_M-\bar{\boldsymbol{z}})'\Lambda^{-1}(\boldsymbol{z}_M-\bar{\boldsymbol{z}})=\max_{\alpha}\{(\boldsymbol{z}_\alpha-\bar{\boldsymbol{z}})'\Lambda^{-1}(\boldsymbol{z}_M-\bar{\boldsymbol{z}})\}$$

and L_ε is the upper $100\varepsilon\%$ point of $\max_{\alpha}\{(\boldsymbol{z}_\alpha-\bar{\boldsymbol{z}})'\Lambda^{-1}(\boldsymbol{z}_\alpha-\bar{\boldsymbol{z}})\}$ under H_0.

When Λ is not known to us, it is nearly certain that the procedure based on $\hat{T}^2_{\text{MAX}\cdot\text{D}}$ has the analogous optimum property though the rigorous proof has not been given.

Now we return to the problem of the evaluation of the upper percentage point of $\hat{\chi}^2_{\text{MAX}\cdot\text{D}}$ and $\hat{T}^2_{\text{MAX}\cdot\text{D}}$.

6.1 *When* Λ *is kown*—$\hat{\chi}^2_{\text{MAX}\cdot\text{D}}$— Noting that $\gamma=(n-1)/n$, $\delta=-1/n$ and $N=n$, we have from (3.6), (3.10) and (4.8)

$$A_1^2(\alpha; p, n)=[(n-1)/n]\chi^2(\alpha/n; p) \tag{6.3}$$

and

$$(6.4)\quad \beta(\alpha;\, p,\, n)=\frac{1}{2}n(n-1)\frac{[n(n-2)/(n-1)^2]^{p/2}}{\Gamma(p/2)}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(p/2+j)}{j!}\frac{1}{(n-1)^{2j}}\left(\int_{\frac{n-1}{n-2}A_1^2(\alpha;\, p,\, n)}^{\infty} g_{p+2j}^{*}(\chi^2)d\chi^2\right)^2 .$$

Table 6.1 shows the values of $A_1^2(\alpha; p, n)$ and the lower bounds, $\alpha-\beta(\alpha; p, n)$, of $P_r\{\hat{\chi}^2_{\text{MAX.D}}>A_1^2(\alpha; p, n)\}$ for $\alpha=0.05, 0.01$; $p=2, 3, 4$; $n=3, 5, 10, 20$.

TABLE 6.1.

Values of $A_1^2(\alpha; p, n)$ and the lower bounds, $\alpha-\beta(\alpha;p, n)$, for $\alpha=0.05$ and 0.01.

		$\alpha=0.05$				$\alpha=0.01$			
p \ n		3	5	10	20	3	5	10	20
2	A_1^2	5.459	7.368	9.537	11.38	7.605	9.943	12.43	14.44
	$\alpha-\beta$	0.0447	0.0475	0.0485	0.0487	0.00945	0.00984	0.00993	0.00995
3	A_1^2	6.825	9.076	11.55	13.61	9.138	11.84	14.64	16.85
	$\alpha-\beta$	0.0450	0.0477	0.0485	0.0487	0.00950	0.00986	0.00993	0.00995
4	A_1^2	8.063	10.62	13.37	15.61	10.52	13.54	16.62	19.00
	$\alpha-\beta$	0.0453	0.0478	0.0486	0.0488	0.00954	0.00987	0.00993	0.00995

From Table 6.1, it can be seen that for the usual values of α, $A_1^2(\alpha; p, n)$ has the accuracy sufficient for most practical application except for $n<5$, though it slightly overestimates the true value. However, the more accurate approximation to the upper $100\alpha\%$ point of $\hat{\chi}^2_{\text{MAX.D}}$ can be obtained by calculating the modified second approximation, $A_2^2(\alpha; p, n)$ such that

$$(6.5)\qquad nP_r\{\hat{\chi}^2_{1.D}=(\boldsymbol{x}_1-\bar{\boldsymbol{x}})'\Lambda^{-1}(\boldsymbol{x}_1-\bar{\boldsymbol{x}})>A_2^2(\alpha; p, n)\}=\alpha+\beta(\alpha; p, n)$$

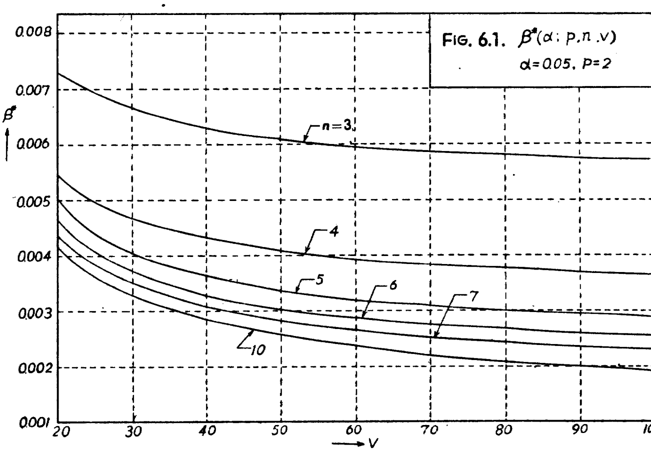
for $\beta(\alpha; p, n)$ calculated for $A_1^2(\alpha; p, n)$, which are given in Table I at the end of this paper.

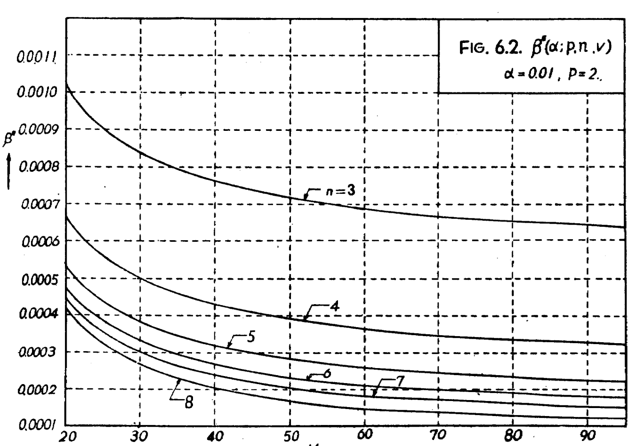
6.2. *When Λ is not known—Studentized form $\hat{T}^2_{\text{MAX.D}}$*— The first approximation $B_1^2(\alpha; p, n, \nu)$ to the upper $100\alpha\%$ point of $\hat{T}^2_{\text{MAX.D}}$ is from (3.7) for $\gamma=(n-1)/n$, $\delta=-1/n$ and $N=n$,

$$(6.6)\quad B_1^2(\alpha; p, n, \nu)=[(n-1)\nu/n]\left\{\frac{1}{C(\alpha/n;\, (\nu+1-p)/2,\, p/2)}-1\right\},$$

which is calculated with the aid of Tables of the Beta-Function [10]: $\beta^*(\alpha; p, n, \nu)$ can be obtained from Theorem 5.1 by putting $\gamma=(n-1)/n$,

$\delta=-1/n$ and $N=n$. Though the approximation by the terms up to order ν^{-2} are not so accurate to evaluate the probability $P_r\{\hat{T}^2_{1\cdot D}=(\boldsymbol{x}_1-\bar{\boldsymbol{x}})'\boldsymbol{L}^{-1}(\boldsymbol{x}_1-\bar{\boldsymbol{x}})>B^2_1(\alpha;\ p,\ n,\ \nu),\ \hat{T}^2_{2\cdot D}=(\boldsymbol{x}_2-\bar{\boldsymbol{x}})'\boldsymbol{L}^{-1}(\boldsymbol{x}_2-\bar{\boldsymbol{x}})>B^2_1(\alpha;\ p,\ n,\ \nu)\}$ itself, we can conclude from the numerical computation based on the formulas in Theorem 5.1 that $\beta^*(\alpha; p, n, \nu)$ is lower order than α for moderately large ν. Figures 6.1 and 6.2 show the curves of $\beta^*(\alpha;\ p,\ n,\ \nu)$ for $\alpha=0.05$ and $\alpha=0.01$, respectively and for $p=2$, which are calculated by FACOM 128, the automatic relay computor, in our institute.





From these figures it is seen that the effect of $\beta^*(\alpha;\ p,\ n,\ \nu)$ is significant for the moderate magnitude of n and so we need to consider the second approximation. But in the same way with the evaluation of $A^2_2(\alpha;\ p,\ n)$, we use the modified second approximation procedure, that is, we calculate $B^2_2(\alpha;\ p,\ n,\ \nu)$ such that

(6.7) $\alpha+\beta^*(\alpha; p, n, \nu)=nP_r\{(\boldsymbol{x}_1-\overline{\boldsymbol{x}})'L^{-1}(\boldsymbol{x}_1-\overline{\boldsymbol{x}})>B_2^2(\alpha; p, n, \nu)\}$

for $\beta^*(\alpha; p, n, \nu)$ calculated for $B_1^2(\alpha; p, n, \nu)$, which are contained in Table II.

7. **The squares of ranges, R^2_{MAX} and $\mathfrak{R}^2_{\text{MAX}}$.** We shall give some comments on the evaluation of the upper percentage points of the squares of the ranges in the multivariate case, R^2_{MAX} and $\mathfrak{R}^2_{\text{MAX}}$:

(7.1) $$R^2_{\text{MAX}}=\max_{i<j} R^2_{ij}=\max_{i<j}\{(\boldsymbol{x}_i-\boldsymbol{x}_j)'\Lambda^{-1}(\boldsymbol{x}_i-\boldsymbol{x}_j)\} ,$$

(7.2) $$\mathfrak{R}^2_{\text{MAX}}=\max_{i<j} \mathfrak{R}^2_{ij}=\max_{i<j}\{(\boldsymbol{x}_i-\boldsymbol{x}_j)'L^{-1}(\boldsymbol{x}_i-\boldsymbol{x}_j)\} .$$

We arrange $(1/2)n(n-1)$ R_{ij}'s in a row in a certain way and rename them $R_1^2, R_2^2, \cdots, R_N^2$, where $N=(1/2)n(n-1)$. Using the formula (2.12) for N events $R_1^2>r^2, R_2^2>r^2, \cdots, R_N^2>r^2$, we have

(7.3) $$P_r\{R^2_{\text{MAX}}>r^2\}=NP_r\{R_1^2>r^2\}-\sum_{i<j}P_r\{R_i^2>r^2, R_j^2>r^2\}+\cdots .$$

It is easily seen that, though $P_r\{R_i^2>r^2, R_j^2>r^2\}, (i<j=1, 2, \cdots, N)$, are not symmetric functions of R^2's as a whole, there are following two groups G_1 and G_2 in each of which $P_r\{R_i^2>r^2, R_j^2>r^2\}$ has the same value:

G_1: group of $P_r\{R_j^2>r^2, R_j^2>r^2\}$'s such that R_i^2 and R_j^2 are independent; for example, $R_i^2=(\boldsymbol{x}_1-\boldsymbol{x}_2)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)$ and $R_j^2=(\boldsymbol{x}_3-\boldsymbol{x}_4)'\Lambda^{-1}(\boldsymbol{x}_3-\boldsymbol{x}_4)$.

G_2: group of $P_r\{R_{i'}^2>r^2, R_{j'}^2>r^2\}$'s such that $R_{i'}^2$ and $R_{j'}^2$ have a common $\boldsymbol{x}$; for example, $R_{i'}^2=(\boldsymbol{x}_1-\boldsymbol{x}_2)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)$ and $R_{j'}^2=(\boldsymbol{x}_1-\boldsymbol{x}_3)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_3)$.

Thus we have, for the first approximation $r_1^2(\alpha; p, N)$ such that

(7.4) $$\alpha=NP_r\{(\boldsymbol{x}_1-\boldsymbol{x}_2)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)>r_1^2(\alpha; p, N)\} ,$$

(7.5) $$\beta(\alpha; p, N)=\sum_{i<j}P_r\{R_i^2>r_1^2(\alpha; p, N), R_j^2>r_1^2(\alpha; p, N)\}$$
$$=M_1P_r\{(\boldsymbol{x}_1-\boldsymbol{x}_2)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)>r_1^2(\alpha; p, N), (\boldsymbol{x}_3-\boldsymbol{x}_4)'\Lambda^{-1}(\boldsymbol{x}_3-\boldsymbol{x}_4)>r_1^2(\alpha; p, N\}$$
$$+M_2P_r\{(\boldsymbol{x}_1-\boldsymbol{x}_2)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)>r_1^2(\alpha; p, N), (\boldsymbol{x}_1-\boldsymbol{x}_3)'\Lambda^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_3)>r_1^2(\alpha; p, N)\},$$

where M_1, M_2 are the numbers of the elements of G_1, G_2, respectively and

(7.5) $$M_1=\binom{(n-1)(n-2)/2}{2} \quad \text{and} \quad M_2=\frac{1}{2}n(n-1)(n-2) ,$$

which are shown in [12].

Analogously, in the case of unknown Λ, that is, for $\mathfrak{R}^2_{\text{MAX}}$, we have

(7.7) $$\alpha=NP_r\{(\boldsymbol{x}_1-\boldsymbol{x}_2)'L^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)>\mathfrak{r}_1^2(\alpha; p, N, \nu)\}$$

and

$$(7.8)\quad \beta^*(\alpha;\, p,\, N,\, \nu)$$
$$=M_1P_r\{(\boldsymbol{x}_1-\boldsymbol{x}_2)'\boldsymbol{L}^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)>\mathfrak{r}_1^2(\alpha;p,N,\nu),(\boldsymbol{x}_3-\boldsymbol{x}_4)'\boldsymbol{L}^{-1}(\boldsymbol{x}_3-\boldsymbol{x}_4)>\mathfrak{r}_1^2(\alpha;p,N,\nu)\}$$
$$+M_2P_r\{(\boldsymbol{x}_1-\boldsymbol{x}_2)'\boldsymbol{L}^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2)>\mathfrak{r}_1^2(\alpha;p,N,\nu),(\boldsymbol{x}_1-\boldsymbol{x}_3)\boldsymbol{L}^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_3)>\mathfrak{r}_1^2(\alpha;p,N,\nu)\}.$$

In the author's seperate paper [12], he has discussed the range in some details and examined the order of magnitude of $\beta(\alpha;\, p,\, N)$ and $\beta^*(\alpha;\, p,\, N,\, \nu)$ numerically by using the formulas obtained in this paper. According to this examination, β and β^* are, in the multivariate range case, significantly large and so the first approximations $r_1^2(\alpha;\, p,\, N)$ and $\mathfrak{r}_1^2(\alpha;\, p,\, N,\, \nu)$ are not accurate but the modified second approximations give values with the sufficient accuracy for practical application.

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TABLE I

Upper percentage points of the extreme deviate from the sample mean

$$\hat{\chi}^2_{\mathrm{MAX}\cdot\mathrm{D}} = \max_i \{(\boldsymbol{x}_i - \bar{\boldsymbol{x}})'\Lambda^{-1}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})\}$$

p \ n	3	4	5	6	7	8	9	10	12	14	16	18	20	25	30
	$\alpha=0.05$														
2	5.32	6.48	7.29	7.91	8.41	8.82	9.18	9.48	9.99	10.40	10.77	11.06	11.32	11.88	12.31
3	6.69	8.05	9.00	9.72	10.28	10.74	11.15	11.49	12.05	12.53	12.93	13.26	13.55	14.15	14.63
4	7.92	9.47	10.54	11.34	11.97	12.49	12.93	13.31	13.94	14.45	14.87	15.23	15.55	16.19	16.70
	$\alpha=0.025$														
2	6.28	7.55	8.43	9.09	9.62	10.06	10.44	10.76	11.30	11.73	12.09	12.41	12.68	13.24	13.69
3	7.72	9.20	10.22	10.98	11.58	12.08	12.50	12.86	13.45	13.93	14.34	14.68	14.98	15.59	16.08
4	9.00	10.70	11.84	12.68	13.35	13.90	14.36	14.75	15.40	15.91	16.35	16.71	17.04	17.71	18.23
	$\alpha=0.01$														
2	7.53	8.95	9.92	10.64	11.21	11.68	12.08	12.42	12.98	13.44	13.88	14.13	14.42	15.02	15.49
3	9.07	10.70	11.81	12.63	13.28	13.80	14.24	14.62	15.26	15.76	16.18	16.53	16.84	17.47	17.96
4	10.45	12.28	13.51	14.41	15.12	15.70	16.19	16.61	17.29	17.83	18.28	18.66	18.99	19.67	20.21

TABLE II

Upper percentage points of the studentized extreme deviate from the sample mean

$$\hat{T}^2_{\mathrm{MAX}\cdot\mathrm{D}} = \max_i \{(\boldsymbol{x}_i - \bar{\boldsymbol{x}})'\boldsymbol{L}^{-1}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})\}$$

ν \ n	3	4	5	6	7	8	9	10	11	12	14
	$\alpha=0.05$										
20	6.88	8.53	9.72	10.64	11.44	12.10	12.67	13.18	13.56	14.04	14.76
22	6.72	8.30	9.45	10.36	11.10	11.73	12.28	12.76	13.19	13.58	14.26
24	6.58	8.13	9.24	10.12	10.83	11.44	11.97	12.43	12.84	13.21	13.86
26	6.47	7.98	9.06	9.92	10.61	11.20	11.71	12.16	12.55	12.91	13.54
28	6.37	7.86	8.92	9.75	10.42	11.00	11.49	11.93	12.31	12.66	13.28
30	6.29	7.75	8.79	9.61	10.27	10.83	11.31	11.74	12.11	12.45	13.05
32	6.23	7.66	8.69	9.49	10.14	10.69	11.16	11.57	11.94	12.27	12.86
34	6.17	7.58	8.60	9.38	10.02	10.56	11.02	11.43	11.79	12.12	12.69
36	6.12	7.51	8.51	9.29	9.92	10.45	10.91	11.31	11.67	11.98	12.55
38	6.07	7.45	8.44	9.21	9.83	10.35	10.81	11.20	11.55	11.87	12.42
40	6.03	7.40	8.38	9.14	9·75	10.27	10.71	11.10	11.45	11.76	12.30
45	5.94	7.29	8.25	8.99	9.59	10.09	10.53	10.91	11.24	11.54	12.07
50	5.88	7.20	8.14	8.87	9.46	9.95	10.38	10.75	11.08	11.37	11.89
55	5.82	7.13	8.06	8.78	9.36	9.84	10.26	10.63	10.95	11.24	11.74
60	5.78	7.07	7.99	8.70	9.27	9.75	10.16	10.52	10.84	11.13	11.62
100	5.59	6.82	7.70	8.37	8.91	9.36	9.75	10.09	10.39	10.65	11.12
150	5.50	6.71	7.56	8.21	8.74	9.18	9.55	9.88	10.17	10.43	10.88
200	5.45	6.65	7.49	8.13	8.64	9.09	9.46	9.78	10.06	10.32	10.76

TABLE II

ν \ n	3	4	5	6	7	8	9	10	11	12	14
						$\alpha=0.025$					
20	8.46	10.37	11.73	12.80	13.67	14.41	15.04	15.61	16.11	16.56	17.36
22	8.22	10.05	11.36	12.38	13.21	13.91	14.51	15.05	15.52	15.95	16.70
24	8.03	9.80	11.06	12.05	12.84	13.51	14.09	14.60	15.05	15.46	16.18
26	7.87	9.60	10.82	11.78	12.55	13.19	13.75	14.24	14.68	15.07	15.75
28	7.74	9.42	10.62	11.55	12.30	12.93	13.47	13.94	14·36	14.74	15.40
30	7.63	9.28	10.45	11.36	12.09	12.70	13.23	13.69	14.10	14.47	15.11
32	7.53	9.16	10.30	11.19	11.91	12.51	13.02	13.47	13.87	14.23	14.86
34	7.45	9.05	10.18	11.05	11.75	12.34	12.85	13.29	13.68	14.03	14.65
36	7.37	8.95	10.07	10.93	11.62	12.20	12.69	13.13	13.51	13.86	14.46
38	7.31	8.87	9.97	10.82	11.50	12.07	12.56	12.99	13.36	13.70	14.29
40	7.25	8.80	9.88	10.72	11.39	11.96	12.44	12.86	13.23	13.57	14.15
45	7.13	8.64	9.71	10.52	11.18	11.72	12.19	12.60	12.96	13.28	13.84
50	7.04	8.52	9.57	10.36	11.01	11.54	12.00	12.39	12.75	13.06	13.61
55	6.97	8.43	9.45	10.24	10.87	11.39	11.84	12.23	12.58	12.89	13.42
60	6.51	8.35	9.36	10.14	10.76	11.27	11.71	12.10	12.44	12.74	13.27
100	6.65	8.01	8.97	9.70	10.28	10.76	11.18	11.53	11.85	12.13	12.62
150	6.52	7.85	8.78	9.49	10.06	10.52	10.92	11.27	11.57	11.84	12.31
200	6.46	7.78	8.69	9.39	9.93	10.40	10.79	11.13	11.43	11.70	12.16

ν \ n	3	4	5	6	7	8	9	10	11	12	14
						$\alpha=0.01$					
20	10.72	12.99	14.61	15.85	16.86	17.71	18.44	19.08	19.66	20.17	21.07
22	10.36	12.53	14.07	15.24	16.20	17.00	17.69	18.29	18.83	19.31	20.16
24	10.07	12.16	13.63	14·76	15.68	16.44	17.10	17.67	18.18	18.64	19.44
26	9.84	11.86	13.28	14.37	15.25	15.98	16.62	17.16	17.66	18.09	18.86
28	9.64	11.63	12.99	14.05	14.90	15.62	16.22	16.75	17.22	17.64	18.37
30	9.47	11.40	12.74	13.77	14.60	15.30	15.88	16.40	16.85	17.26	17.97
32	9.33	11.22	12.54	13.54	14.35	15.02	15.60	16.10	16.54	16.94	17.63
34	9.21	11.06	12.36	13.34	14.13	14.79	15.35	15.84	16.28	16.66	17.34
36	9.10	10.93	12.20	13.17	13.95	14.59	15.14	15.62	16.04	16.42	17.08
38	9.01	10.81	12.06	13.01	13.78	14.41	14.95	15.42	15.84	16.21	16.86
40	8.93	10.70	11.94	12.88	13.63	14.26	14.79	15.25	15.66	16.03	16.66
45	8.75	10.48	11.69	12.60	13.33	13.93	14.45	14.89	15.29	15.64	16.25
50	8.62	10.31	11.49	12.38	13.09	13.68	14.18	14.62	15.00	15.34	15.93
55	8.51	10.18	11.33	12.21	12.90	13.48	13.97	14.39	14.77	15.10	15.68
60	8.42	10.07	11.20	12.06	12.75	13.32	13.80	14.21	14.58	14.91	15.48
100	8.05	9.59	10.66	11.46	12.10	12.63	13.07	13.45	13.79	14.09	14.61
150	7.87	9.37	10.40	11.18	11.79	12.30	12.73	13.10	13.42	13.71	14.21
200	7.79	9.26	10.28	11.04	11.62	12.14	12.57	12.92	13.24	13.52	14.01

Errata

Annals, Vol. X, No. 1.

Page	Line	Read	Instead of
48	13	$V_{(6)}=15$	$V_{(6)}=16$
"	14	(8 17)	(8 7)

Vol. X, No. 3.

Page	Line	Read	Instead of
200	17—18	$\cdots \sum_{j=0}^{\infty} \frac{\Gamma(p/2+1+j)}{j!}\left(\frac{\delta}{\gamma}\right)^{2j} \times\left[\left(\eta+\frac{p}{2}-j\right)\cdots\right.$	$\cdots \sum_{j=0}^{\infty} \frac{\Gamma(p/2+1+j)}{j!} \times\left[\left(\eta+\frac{p}{2}-j\right)\cdots\right.$

Vol. XI, No. 2.

Page	Line	Read	Instead of
108	16	$[X]_n$	$[X_n]$
112	11	while X_{ii} are defined by···	X_{ij} defined by···