THE EXTREME VALUE OF THE GENERALIZED DISTANCES OF THE INDIVIDUAL POINTS IN THE MULTIVARIATE NORMAL SAMPLE

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1. Summary.

The extreme value of the generalized distances, from the origin, of \( N \) individual points which may be correlated each other, in the \( p \)-variate normal sample is defined and discussed. It contains, as special cases, (i) the extreme deviate from the population mean or the sample mean, (ii) the extreme deviate from the control variate and (iii) the range defined by (2.10) or (2.11) below. The exact sampling distributional theory of this statistic is extremely difficult to find, even its moments. However, the method of obtaining the approximate upper 100 \( \alpha \) percentage points for the ordinary significance level \( \alpha \) is given. The lower percentage points can be obtained in the similar way if necessary. In connection with the evaluation of the approximate percentage points, the two-dimensional chi-square distribution is discussed and the asymptotic formulas for the joint distribution function of the two generalized distances are given in the special forms for the present aim. The extreme deviate from the sample mean will be explained in some detail and the tables of the approximate upper 5, 2.5 and 1\% points are given. For the cases (ii) and (iii) mentioned above the details are omitted and will be discussed in the case of need.

2. Introduction and definition.

Let \( \mathbf{y}(\alpha)=(y_{1\alpha}, y_{2\alpha}, \ldots, y_{p\alpha}), (\alpha=1, 2, \ldots, N) \), be \( N \) \( p \)-variate vectors with mean vector \( \mathbf{0}'=(0, 0, \ldots, 0) \) and with covariance matrix \( \gamma \Lambda (\gamma>0) \) and let the covariance matrix of \( \mathbf{y}_\alpha \) and \( \mathbf{y}_\beta (\alpha \neq \beta) \) be \( \delta_\Lambda \) where \( \Lambda \) is a symmetric, positive definite matrix and \( \gamma>\mid \delta \). Karl Pearson [9] defined the generalized distance of the \( \alpha \)th variate \( \mathbf{y}_\alpha \) from the origin by

\[
\chi_\alpha = \frac{1}{\gamma} \sum_{i=1}^{p} \sum_{j=1}^{p} \lambda_{ij} y_{i\alpha} y_{j\alpha} = \frac{1}{\gamma} \mathbf{y}_\alpha \Lambda^{-1} \mathbf{y}_\alpha,
\]

where \( \Lambda^{-1}=(\lambda^{ij}) \) is the inverse of \( \Lambda \). However, when \( \Lambda \) is not known a
priori, we must use the estimate of $A$. Let $L=(l_{ij})$ be the unbiasedly estimated matrix of $A$ based on the $v$ degrees of freedom, which is statistically independent of $y_n (\alpha=1, 2, \ldots, N)$ and $L^{-1}=(l^{ij})$ be the inverse of $L$. Putting $L$ instead of $A$ in (2.1), we have the so called studentized form, i.e.,

$$T^2_s = \frac{1}{\gamma} \sum_{i=1}^{p} \sum_{j=1}^{p} l^{ij} y_i y_j^{-1} = \frac{1}{\gamma} y_s' L^{-1} y_s .$$

In this paper we are concerned with the maximum values of $\gamma \chi^2_s$ and $\gamma T^2_s$,

$$\hat{\chi}^2_{\text{MAX}} = \max_{a} \{ \gamma \chi^2_s \} = \max_{a} \{ y_s' A^{-1} y_s \}$$

and

$$\hat{T}^2_{\text{MAX}} = \max_{a} \{ \gamma T^2_s \} = \max_{a} \{ y_s' L^{-1} y_s \} .$$

For the special cases of these quantities, we shall consider the random sample of size $n$ drawn from a $p$-variate population. Let $x_n=(x_{1\alpha}, x_{2\alpha}, \ldots, x_{p\alpha}), (\alpha=1, 2, \ldots, n)$, be $n$ observed values of a $p$-variate vector which has the mean vector $m'= (m_1, m_2, \ldots, m_p)$ and the covariance matrix $A=(\lambda_{ij})$. Then the maximum deviate from the population mean is defined as the positive square root of

$$\chi^2_{\text{MAX-D}} = \max_{a} \{ (x_a - m)' A^{-1} (x_a - m) \}$$

or

$$T^2_{\text{MAX-D}} = \max_{a} \{ (x_a - m)' L^{-1} (x_a - m) \}$$

according to the case when $A$ is known and the case when $A$ is not known, and the maximum deviate from the sample mean, $\bar{x}=(1/n) \sum_{a=1}^{n} x_a$, by the positive square root of

$$\hat{\chi}^2_{\text{MAX-D}} = \max_{a} \{ (x_a - \bar{x})' A^{-1} (x_a - \bar{x}) \}$$

or

$$\hat{T}^2_{\text{MAX-D}} = \max_{a} \{ (x_a - \bar{x})' L^{-1} (x_a - \bar{x}) \} .$$

In the univariate case, $p=1$, $\hat{\chi}^2_{\text{MAX-D}}$ and $\hat{T}^2_{\text{MAX-D}}$ are reduced, respectively, to the forms
\[ \hat{\chi}_{\text{MAX-D}}^2 = \max \{ (x_a - \bar{x})/\sigma^2 \} = (x_{\text{MAX}} - \bar{x})/\sigma^2 \quad \text{or} \quad (\bar{x} - x_{\text{MIN}})^2/\sigma^2 \]

and

\[ \hat{T}_{\text{MAX-D}}^2 = \max \{ (x_a - \bar{x})/S_a^2 \} = (x_{\text{MAX}} - \bar{x})/S_a^2 \quad \text{or} \quad (\bar{x} - x_{\text{MIN}})^2/S_a^2 \]

where \( S_a^2 \) is the unbiased estimate of the population variance \( \sigma^2 \) based on the \( \nu \) degrees of freedom. Positive square roots of these statistics were studied by K. R. Nair [6, 7] and H. A. David [1, 2].

If \( x_0 \) is the control variate which has the same distribution with that of \( x_a \), then the maximum deviate from \( x_0 \) is defined by the positive square root of

(2.9) \[ \hat{\chi}_{\text{MAX-C}}^2 = \max \{ (x_a - x_0)^\prime \Lambda^{-1}(x_a - x_0) \}, \quad \text{or} \]

\[ \hat{T}_{\text{MAX-C}}^2 = \max \{ (x_a - x_0)^\prime L^{-1}(x_a - x_0) \} . \]

In the same way we define the range of the sample in the multivariate case by the positive square root of the maximum of the squares of the generalized distances between two any observed points, that is, the positive square root of

(2.10) \[ R_{\text{MAX}}^2 = \max \{ R_{a\beta}^2 \} = \max \{ (x_a - x_\beta)^\prime \Lambda^{-1}(x_a - x_\beta) \} \]

or

(2.11) \[ \mathcal{R}_{\text{MAX}}^2 = \max \{ \mathcal{R}_{a\beta}^2 \} = \max \{ (x_a - x_\beta)^\prime L^{-1}(x_a - x_\beta) \} . \]

When \( p = 1 \), we have the square of the usual range in the univariate case.

It is easily seen that the various statistics defined in (2.5)–(2.11) are the special cases of (2.3) and (2.4) and that the values of \( \gamma \) and \( \delta \) of the respective cases are as in the following table:

<table>
<thead>
<tr>
<th>Statistics</th>
<th>( U_a ) (( U_{a\beta} ))</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\chi}<em>{\text{MAX-D}}^2 ) and ( \hat{T}</em>{\text{MAX-D}}^2 )</td>
<td>( x_a - m )</td>
<td>1</td>
<td>0</td>
<td>( n )</td>
</tr>
<tr>
<td>( \hat{\chi}<em>{\text{MAX-D}}^2 ) and ( \hat{T}</em>{\text{MAX-D}}^2 )</td>
<td>( x_a - \bar{x} )</td>
<td>( n-1 )/( n )</td>
<td>( -1/n )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \hat{\chi}<em>{\text{MAX-C}}^2 ) and ( \hat{T}</em>{\text{MAX-C}}^2 )</td>
<td>( x_a - x_0 )</td>
<td>2</td>
<td>1</td>
<td>( n )</td>
</tr>
<tr>
<td>( R_{\text{MAX}}^2 ) and ( \mathcal{R}_{\text{MAX}}^2 )</td>
<td>( x_a - x_\beta )</td>
<td>2</td>
<td>0 or 1</td>
<td>( \frac{1}{2} n(n-1) )</td>
</tr>
</tbody>
</table>
First we shall deal with the general statistics defined in (2.3) and (2.4) and then with their special cases. To find the sampling distributions of \( \hat{\chi}^{\text{MAX}} \) and \( \hat{T}^{\text{MAX}} \) exactly is extremely difficult, but the values of the upper (or the lower) percentage points of these statistics can be approximately evaluated for the ordinary significance levels with the accuracy sufficient for the most practical applications. Though we treat, in this paper, the case of the upper percentage point, we can evaluate the lower percentage point by the analogous way if necessary.

In order to evaluate the upper percentage points of \( \hat{\chi}^{\text{MAX}} \) and \( \hat{T}^{\text{MAX}} \), we shall use the following well-known formula:

The probability \( P \) of the realization of at least one among the \( N \) events \( E_1, E_2, \ldots, E_N \) is given by

\[
P = S_1 - S_2 + S_3 - \cdots \pm S_N,
\]

where \( S_r \) is the sum of the \( \binom{N}{r} \) joint probabilities of \( r(\leq N) \) events with \( r \) different subscripts.

In our cases \( E_a \) corresponds to the event \( \hat{\chi}_s = \gamma\chi_s > a^2 \) or \( \hat{T}_s = \gamma T_s > b^2 \) and \( P \) corresponds to \( P_r(\hat{\chi}^{\text{MAX}}_r > a^2) \) or \( P_r(\hat{T}^{\text{MAX}}_r > b^2) \) where \( a^2 \) and \( b^2 \) are some constants. If the joint distribution of the \( E \)'s is a symmetric function of the \( E \)'s, then (2.12) becomes

\[
P = NP_r(E_1) - \frac{1}{2} N(N-1)P_r(E_1, E_2) + \cdots \pm P_r(E_1, E_2, \cdots, E_N).
\]

The maximum deviates, (2.5)–(2.9) are of this case.

3. First approximate upper percentage points of \( \hat{\chi}^{\text{MAX}} \) and \( \hat{T}^{\text{MAX}} \).

For our cases of \( \hat{\chi}^{\text{MAX}} \) and \( \hat{T}^{\text{MAX}} \), (2.12) can be written as

\[
P_r(\hat{\chi}^{\text{MAX}} > a^2) = NP_r(\hat{\chi} > a^2) - S_2 + \cdots \pm S_N
\]

and

\[
P_r(\hat{T}^{\text{MAX}} > b^2) = NP_r(\hat{T} > b^2) - S_3 + \cdots \pm S_N,
\]

respectively. For reasonably large values of \( a^2 \) and \( b^2 \), the good first approximations to the left-hand sides of (3.1) and (3.2) are provided by \( NP_r(\hat{\chi} > a^2) \) and \( NP_r(\hat{T} > b^2) \).

Now let us assume that the distribution of \( \nu \) is normal and that the unbiased estimate \( L \) of \( \Lambda \) has the Wishart distribution with \( \nu \) degrees
of freedom. Then \( \chi^2 = (1/\gamma)\mathbf{y}'\mathbf{A}^{-1}\mathbf{y} \) is distributed according to the chi-square distribution with \( p \) degrees of freedom and \( T^2 = (1/\gamma)\mathbf{y}'\mathbf{L}^{-1}\mathbf{y} \) has the Hotelling’s \( T^2 \)-distribution with \( \nu \) degrees of freedom, i.e.,

\[
(3.3) \quad dF\left( \frac{T^2}{\nu} \right) = \frac{\Gamma[(\nu+1)/2]}{\Gamma(p/2)\Gamma[(\nu+1-p)/2]} \left( \frac{T^2}{\nu} \right)^{(\nu/2)-1} \left( 1 + \frac{T^2}{\nu} \right)^{-(\nu+1)/2} d\left( \frac{T^2}{\nu} \right).
\]

Consequently we have as the first approximations

\[
(3.4) \quad P_r\{\chi_{\text{MAX}}^2 > a^2\} = NP_r\{\chi^2 > a^2/\gamma\} = N \int_{(1/\gamma)a^2}^{\infty} \frac{t^{p/2-1}e^{-t/2}}{2^{p/2}\Gamma(p/2)} dt
\]

and

\[
P_r\{\hat{T}_{\text{MAX}}^2 > b^2\} = NP_r\{\hat{T}^2 > b^2\} = NP_r\left\{ \frac{T^2}{\nu} > \frac{b^2}{\nu/\gamma} \right\} = \frac{N\Gamma[(\nu+1)/2]}{\Gamma(p/2)\Gamma[(\nu+1-p)/2]} \int_{\nu/\gamma}^{\infty} u^{p/2-1}(1+u)^{-(\nu+1)/2}du
\]

\[
= \frac{N}{B[(\nu+1-p)/2, p/2]} \int_{\nu/\gamma+b^2}^{\infty} \nu^{(1+1-p)/2}v^{2-1}(1-v)^{p/2-1}dv
\]

\[
= NI_{\nu, \nu/\gamma+b^2} \left( \frac{1}{2}(\nu+1-p), \frac{1}{2}p \right),
\]

where \( I_c(e, d) \) is the K. Pearson’s incomplete beta function [10]. If we denote the upper 100\( \alpha \)% point of the chi-square distribution with \( p \) degrees of freedom by \( \chi^2(\alpha^*; p) \) and the lower 100\( \alpha \)% point of the beta-distribution with parameters \((\nu+1-p)/2 \) and \( p/2 \) by \( C(\alpha^*; (\nu+1-p)/2, p/2) \), then \( A_\gamma(\alpha; p, N) \), the first approximation to the upper 100\( \alpha \)% point of \( \chi_{\text{MAX}}^2 \), is calculated from

\[
(3.6) \quad A_\gamma(\alpha; p, N) = \gamma \chi^2(\alpha/N; p)
\]

and \( B_\gamma(\alpha; p, N, \nu) \), the first approximation to upper 100\( \alpha \)% point of \( \hat{T}^2_{\text{MAX}} \) is given by

\[
(3.7) \quad B_\gamma(\alpha; p, N, \nu) = \nu\gamma \left\{ \frac{1}{C(\alpha/N; (\nu+1-p)/2, p/2)} - 1 \right\}.
\]

Next it is necessary to examine whether \( A_\gamma(\alpha; p, N) \) and \( B_\gamma(\alpha; p, N, \nu) \) have the accuracy sufficient for most practical applications,
By Bonferroni’s inequalities, we have, for the significance level $\alpha$,
\begin{equation}
\alpha - \beta(\alpha; p, N) < P_r\{\hat{\chi}_\text{MAX} > A_3(\alpha; p, N)\} < \alpha
\end{equation}
and
\begin{equation}
\alpha - \beta^*(\alpha; p, N, \nu) < P_r\{\hat{T}_\text{MAX} > \hat{B}(\nu; \alpha; p, N, \nu)\} < \alpha,
\end{equation}
where $\beta(\alpha; p, N)$ is $S_2$ for $\alpha^2 = A_3(\alpha; p, N)$ in (3.1) and $\beta^*(\alpha; p, N, \nu)$ is $S_3'$ for $B_3(\alpha; p, N, \nu)$ in (3.2). For the symmetric cases, $\beta(\alpha; p, N)$ and $\beta^*(\alpha; p, N, \nu)$ can be written respectively
\begin{equation}
\beta(\alpha; p, N) = \frac{1}{2} N(N-1) P_r\{\hat{\chi}_1 > A_3(\alpha; p, N), \hat{\chi}_3 > A_3(\alpha; p, N)\}
\end{equation}
and
\begin{equation}
\beta^*(\alpha; p, N, \nu) = \frac{1}{2} N(N-1) P_r\{\hat{T}_1 > \hat{B}(\nu; \alpha; p, N), \hat{T}_3 > \hat{B}(\nu; \alpha; p, N, \nu)\}.
\end{equation}

In order to evaluate $\beta$ and $\beta^*$ for fixed $\alpha$, $p$ and $N$, we shall find, in the following two sections, individual terms $P_r\{\hat{\chi}_1 > A_3(\alpha; p, N), \hat{\chi}_3 > A_3(\alpha; p, N)\}$ and $P_r\{\hat{T}_1 > \hat{B}(\nu; \alpha; p, N), \hat{T}_3 > \hat{B}(\nu; \alpha; p, N, \nu)\}$ ($\alpha \neq \beta$) of $S_2$ and $S_3'$, respectively.

4. Two-dimensional chi-square distribution.

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ be $m$ independent random variables, each of which is distributed according to a bivariate standard normal distribution with correlation coefficient $\rho$, that is,
\begin{equation}
f(x, y) dxdy = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left[ -\frac{1}{2} \frac{1}{1 - \rho^2} (x^2 - 2\rho xy + y^2) \right] dxdy.
\end{equation}

Then it is easy to find the joint distribution of two variables
\begin{equation}
U = \sum_{x=1}^{m} x_i \quad \text{and} \quad V = \sum_{x=1}^{m} y_i.
\end{equation}

Since the joint characteristic function of $x^2$ and $y^2$ is
\[ \varphi(t_1, t_2) = \mathbb{E}\{\exp(it_1 x^2 + it_2 y^2)\} = \{(1 - 2it_1)(1 - 2it_2) + 4\rho^2 t_1 t_2\}^{-1/2}, \]
and $(x_i^2, y_i^2), (i = 1, 2, \ldots, m)$, are independent, the characteristic function of $U$ and $V$ is, by the multiplication theorem,
\begin{equation}
\Phi_m(t_1, t_2) = \{\varphi(t_1, t_2)\}_m = \{(1 - 2it_1)(1 - 2it_2) + 4\rho^2 t_1 t_2\}^{-m/2},
\end{equation}
which is also expressible as
\( (4.3b) \quad \Phi_m(t_1, t_2) \)
\[
= \frac{(1 - \rho^m)^{m/2}}{\Gamma(m/2)} \sum_{j=0}^{m/2} \frac{\Gamma(m/2+j)}{j!} \left[ (1 - 2(1 - \rho^2)it_1) \left( 1 - 2(1 - \rho^2)it_2 \right) \right]^{m/2 + j} \cdot
\]

By the inversion theorem we have the joint probability element of \( U \) and \( V \) in the form
\[
(4.4) \quad f(U, V) \, dU \, dV
\]
\[
= \frac{1}{2^m \Gamma(m/2)(1 - \rho^m)^{m/2}} \sum_{j=0}^{m/2} \frac{1}{j! \Gamma(m/2+j)} \left( \frac{\rho}{2(1 - \rho^2)} \right)^{2j} (UV)^{m/2+j-1} \times e^{-\left( \frac{1}{2(1-\rho^2)} \right)(U^2 + V^2)} \, dU \, dV
\]
and hence the joint density function of
\[
(4.5) \quad \chi^2_1 = U/(1 - \rho^2) \quad \text{and} \quad \chi^2_2 = V/(1 - \rho^2)
\]
is written in the form
\[
(4.6) \quad f_m(\chi^2_1, \chi^2_2) = \frac{(1 - \rho^m)^{m/2}}{\Gamma(m/2)} \sum_{j=0}^{m/2} \frac{\Gamma(m/2+j)}{j!} \rho^{2j} g_{m+1}^*(\chi^2_1)g_{m+1}^*(\chi^2_2),
\]
where \( g^*_m(z) = 2^{-m/2} \Gamma(m/2)^{-1} z^{m/2} e^{-z/2} \), the density function of the chi-square distribution with \( m \) degrees of freedom.

**Definition.** The distribution defined by \( (4.6) \) is called the two-dimensional (or bivariate) chi-square distribution with \( m \) degrees of freedom and with the parameter \( \rho \) and the combined variate \((\chi^2_1, \chi^2_2)\) which has the density \( (4.6) \) is called the two-dimensional (or bivariate) chi-square variate with \( m \) degrees of freedom and with the parameter \( \rho \).

We obtain easily the following theorem from \( (4.3a) \) and \( (4.5) \) or directly from \( (4.6) \):

**Theorem 4.1.** The characteristic function of the two-dimensional chi-square distribution with \( m \) degrees of freedom and with the parameter \( \rho \) is
\[
(4.7) \quad \Psi_m(t_1, t_2) = \left( 1 - \frac{2it_1}{1 - \rho^2} \right) \left( 1 - \frac{2it_2}{1 - \rho^2} \right) + 4 \left( \frac{\rho}{1 - \rho^2} \right) \left( t_1 t_2 \right)^{-m/2}
\]

It is obvious that, when \( \rho = 0 \), \( \chi^2_1 \) and \( \chi^2_2 \) are independent (one-dimensional) chi-square variates and that the distribution of the sum of two independent bivariate chi-square variates with the common parameter \( \rho \) and with \( m_1 \) and \( m_2 \) degrees of freedom respectively is also a bivariate chi-square distribution with \( m_1 + m_2 \) degrees of freedom and with the parameter \( \rho \).

Now we shall prove the following
THEOREM 4.2. Let \((y', y')\) be a \((p+p)\)-dimensional normal variate with mean vector \((0, 0)\) and with covariance matrix \(\Sigma = \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} \times \Lambda\) (direct product) where \(\Lambda\) \((p \times p)\) is a symmetric, positive definite matrix and \(\gamma > |\delta|\). Then the combined variate of \(\tilde{\chi}_1 = y'_1 \Lambda^{-1} y_1\) and \(\tilde{\chi}_2 = y'_2 \Lambda^{-1} y_2\) is distributed as \([\gamma^2 - \delta^2]/\gamma] \cdot \chi_1^2 + \chi_2^2\), where \(\tilde{\chi}_1^2, \tilde{\chi}_2^2\), is the two-dimensional chi-square variate with \(p\) degrees of freedom and with the parameter \(\rho = \delta/\gamma\).

PROOF.

\[
f(y_1, y_2) = 2\pi^{-p} |\Sigma|^{-1/2} \exp \left\{ - \frac{1}{2} (y'_1 y'_2 \Sigma^{-1} y_1 y_2) \right\} \\
= 2\pi^{-p} |\Lambda|^{-1} \gamma^{-p/2} \exp \left\{ - \frac{1}{2} \gamma \left[ (\gamma^2 - \delta^2) y'_1 y'_2 - 2\delta \gamma y'_1 y'_2 + \delta^2 \gamma^2 y'_1 y'_2 \right] \right\}.
\]

Let us consider the non-singular linear transformation

\[
y'_1 = \sqrt{\gamma} z'_1 C' \quad \text{and} \quad y'_2 = \sqrt{\gamma} z'_2 C'
\]

such that \(C' \Lambda^{-1} C = I\), where \(I\) is the unit matrix of order \(p\). Under these transformations \((z'_1, z'_2)\) has the density

\[
f(z_1, z_2) = \left( \frac{1}{2\pi \gamma (1 - \delta^2/\gamma^2)} \right)^p \exp \left\{ - \frac{1}{2} \sum_{j=1}^{p} \frac{1}{(1 - \delta^2/\gamma^2)} z_{1j}^2 - 2 \frac{\delta}{\gamma} z_{1j} z_{2j} + z_{2j}^2 \right\},
\]

from which it is seen that \((z_{11}, z_{12}), (z_{12}, z_{22}), \ldots, (z_{1p}, z_{2p})\) are considered as \(p\) independent combined variables, each of which has the bivariate normal distribution with mean \((0, 0)\), with unit variances and the correlation coefficient \(\rho = \delta/\gamma\). Then \(\chi'_1 = \sum_{i=1}^{p} z_{1i}^2/(1 - \rho^2)\) and \(\chi'_2 = \sum_{i=1}^{p} z_{2i}^2/(1 - \rho^2)\) have a two-dimensional chi-square distribution with \(p\) degrees of freedom and with \(\rho = \delta/\gamma\). Since \(\chi'_1 = \gamma z'_1/((\gamma^2 - \delta^2)) = \gamma [\gamma/(\gamma^2 - \delta^2)] y'_1 \Lambda^{-1} y_1 = [\gamma/(\gamma^2 - \delta^2)] \tilde{\chi}_1^2\) and \(\chi'_2 = \gamma [\gamma/(\gamma^2 - \delta^2)] \tilde{\chi}_2^2\), our theorem is proved.

From this theorem and (4.6), we have

\[
P_r \{ \tilde{\chi}_1 > a^2, \tilde{\chi}_2 > a^2 \} = P_r \{ y'_1 \Lambda^{-1} y_1 > a^2, y'_2 \Lambda^{-1} y_2 > a^2 \} \]

\[
= \left[ (\gamma^2 - \delta^2)/\gamma \right]^{p/2} \left\{ \Gamma(p/2) \right\}^{-1} \sum_{j=0}^{p} \frac{\gamma (p/2 + j)}{j!} \left( \delta/\gamma \right)^j G_{\gamma^2/\delta}^j(\gamma)
\]

where \(\eta = (1/2)[\gamma/(\gamma^2 - \delta^2)] a^2\) and \(G_m(\eta) = \left[ \Gamma(m) \right]^{-1} \int_\eta^\infty t^{m-1} e^{-t} dt = \int_\eta^\infty g_m(\chi') d\chi'.\)

5. Evaluation of \(P_r \{ y'_1 L^{-1} y_1 > B_1(\alpha; p, N, \nu), y'_2 L^{-1} y_2 > B_2(\alpha; p, N, \nu) \}\). Let \((y'_1, y'_2)\) be the \((p+p)\)-dimensional normal variate defined in Theorem 4.2 and let \(L\) be the unbiased estimate of \(\Lambda\), which has the Wishart distribution with \(\nu\) degrees of freedom,
\[(5.1) \quad W_x(L; \Lambda, \nu) = \frac{1}{\pi^{p(p-1)/4} \prod_{i=1}^{p} \left( \frac{\nu - i + 1}{2} \right)^{\nu/2}} L^{(\nu - p - 1)/2} \exp \left\{ -\frac{\nu}{2} tr \Lambda^{-1} L \right\}. \]

and is independent of \( u_i \) and \( u_i \). In order to evaluate \( \beta^*(\alpha; p, N, \nu) \) in (3.9), we shall consider, in this section, \( P_r\{y_i^t L^{-1} y_i > B_i(\alpha; p, N, \nu), y_i^t L^{-1} y_i > B_i(\alpha; p, N, \nu)\} \), where \( B_i(\alpha; p, N, \nu) \) is the first approximation to the upper 100\(\alpha\)% point of \( \hat{T}_i^{\alpha}_{\text{MAX}} \), that is, the solution of the equation
\[(5.2) \quad \alpha = NP_r\{y_i^t L^{-1} y_i > B_i(\alpha; p, N, \nu)\} \cdot \]

However the exact evaluation of
\[P_r\{y_i^t L^{-1} y_i > B_i(\alpha; p, N, \nu), y_i^t L^{-1} y_i > B_i(\alpha; p, N, \nu)\}\]
is very difficult and hence we shall try to obtain the asymptotic formula for this probability with respect to \( \nu^{-1} \).

To do this we shall preliminarily prove several lemmas.

**Lemma 5.1.** \( B_i(\alpha; p, N, \nu) \) has the following asymptotic expression up to the order \( \nu^{-3} \),
\[(5.3) \quad B_i(\alpha; p, N, \nu) = \chi_i^2 + \frac{1}{2\nu} \chi_i^2(\chi_i^2 + p) + \frac{1}{24\nu^2} \chi_i^2(4\chi_i^2 + (13p - 2)\chi_i^2 + 7p^2 - 4) + O(\nu^{-3}), \]
where \( \chi_i^2(\alpha/N; p) \) the upper 100\(\alpha\)/\(N\)% point of the chi-square distribution with \( p \) degrees of freedom.

**Proof.** This is the direct consequence from the fact that \( (1/\gamma)B_i(\alpha; p, N, \nu) \) is the upper 100\(\alpha\)% point of the Hotelling's \( T^2 \)-distribution with \( \nu \) degrees of freedom and from the Hotelling and Frankel's formula [3]
\[T^2(\alpha/N; p, \nu) = \chi_i^2 + \frac{1}{2\nu} \chi_i^2(\chi_i^2 + p) + \frac{1}{24\nu^2} \chi_i^2(4\chi_i^2 + (13p - 2)\chi_i^2 + 7p^2 - 4) + O(\nu^{-3}) , \]
where \( T^2(\alpha/N; p, \nu) \) is the 100\(\alpha\)/\(N\)% point of \( T^2 \)-distribution.

**Lemma 5.2.** (The integral of Dirichlet's type)
\[(5.4) \quad \int_{z_{j_1}^{j_1} \cdots \int_{z_{j_1}^{j_1}} \exp \left\{ -\sum_{j=1}^{p} z_j \right\}}^{\infty} \int_{l_{j_1}^{l_1}}^{\infty} \frac{\Gamma(l_1)}{\Gamma\left( \sum_{j=1}^{p} l_j \right)} \cdot \left( \prod_{j=1}^{p} z_j^{l_j} \right)^{-1} \cdot u^{2\nu_{i-1}} e^{-u} du \quad \zeta_{j_1}^{\gamma} \]

\(*\) If necessary, we can obtain the asymptotic formulas for \( P_r\{y_i^t L^{-1} y_i < \beta_i^1, y_i^t L^{-1} y_i < \beta_i^2\} \)
and \( P_r\{y_i^t L^{-1} y_i < B_i^1(\alpha; p, N, \nu), y_i^t L^{-1} y_i < B_i^2(\alpha; p, N, \nu)\} \) by the obvious modification of the method in this section.
where \( z_j > 0, j = 1, 2, \ldots, p \).

The next lemma concerns the simple expression of the multiple integral

\[
J = \left| \frac{I - \zeta}{\pi^p} \right|^{(\gamma^2 - \delta^2)^{p/2}} \prod_{i \neq j} \exp \left\{ -w_i'(I - \zeta)w_j \right\} \prod_{i = 1}^{p} dw_i dw_j
\]

where \( w_i, i = 1, 2 \) are \( p \)-dimensional column vector variates, \( \zeta \) is a diagonal matrix, \( \text{diag} \{ \zeta_1, \zeta_2, \ldots, \zeta_p \} \), with \( |\zeta_i| < 1, i = 1, 2, \ldots, p, \gamma > |\delta|, \eta > 0 \) and \( \mathcal{D} \) is the domain defined by \( w_i w_j > \gamma \) and \( w_i w_j > \gamma \).

**Lemma 5.3.** \( J \) can be expressed in the form

\[
J = [1 - (\delta/\gamma)^2]^{p/2}(I - \zeta)^{-1}(I - \zeta E_i)(I - \zeta E_j) - (\delta/\gamma)^2 E_i E_j \left\{ \frac{G_{p}^{(1)}(\zeta) G_{p}^{(2)}(\zeta)}{G_{p}^{(1)}(\gamma) G_{p}^{(2)}(\gamma)} \right\}.
\]

where \( E_i, i = 1, 2 \) are operators such that \( E_i^m G_{p}^{(i)}(\gamma) = G_{p}^{(i)}(\gamma) \) for any integer \( m \geq 0 \) and \( G_{p}^{(1)}(\gamma) = G(\gamma) = [\Gamma(\gamma)]^{-1} \int_{\gamma}^{\infty} t^{-1} e^{-t} dt \), \( (i) \) being attached for convenience of the operation of \( E_i. \)

**Proof.** This lemma is proved by expanding the exponential function in the integrand of (5.5), by integrating term by term (this is allowable) and by using the integral formula of Dirichlet's type. From (5.5)

\[
J = \left| I - \zeta \right| \pi^{-p} [1 - (\delta/\gamma)^2]^{p/2} \sum_{\nu_1, \nu_2, \mu a = 0}^{\infty} \frac{\prod_{a = 1}^{p} \zeta_{1a}^{*} \gamma_{2a}^{*} \nu_{1a} \nu_{2a} \mu_{a}}{\prod_{a = 1}^{p} \mu_{a}!} \times
\]

\[
\times \left\{ \prod_{a = 1}^{p} w_{1a}^{\nu_{1a}} w_{2a}^{\nu_{2a}} e^{\nu_{1a}} \right\} \prod_{a = 1}^{p} dw_{1a} \times
\]

\[
\times \left\{ \prod_{a = 1}^{p} w_{1a}^{\nu_{1a}} w_{2a}^{\nu_{2a}} e^{\nu_{2a}} \right\} \prod_{a = 1}^{p} dw_{2a},
\]

\[
= \left| I - \zeta \right| \pi^{-p} [1 - (\delta/\gamma)^2]^{p/2} \sum_{\nu_1, \nu_2, \mu a = 0}^{\infty} \frac{\prod_{a = 1}^{p} \zeta_{1a}^{*} \gamma_{2a}^{*} \nu_{1a} \nu_{2a} \mu_{a}}{\prod_{a = 1}^{p} \mu_{a}! (2 \mu_{a})!} \times
\]

\[
\times \left\{ \prod_{a = 1}^{p} z_{1a}^{\nu_{1a}} z_{2a}^{\nu_{2a}} e^{-\nu_{1a}} \right\} \prod_{a = 1}^{p} dz_{1a} \times
\]

\[
\times \left\{ \prod_{a = 1}^{p} z_{1a}^{\nu_{1a}} z_{2a}^{\nu_{2a}} e^{-\nu_{2a}} \right\} \prod_{a = 1}^{p} dz_{2a}
\]

The integrals are of Dirichlet's type and from Lemma (5.2)
\[
J = |I - \zeta|^{1 - \gamma} \left[ \frac{1 - (\delta/\gamma)^2}{[1 - (\delta/\gamma)^2]} \right]^{p/2} \sum_{\nu_{1a}, \ldots, \nu_{p}} (2\delta/\gamma)^{\nu_{1a} + \ldots + \nu_{p}} \frac{\prod_{a=1}^{p} \Gamma(\nu_{1a} + \ldots + \nu_{p})}{\prod_{a=1}^{p} \Gamma(\nu_{1a})} (2\mu_{a})!
\times \prod_{a=1}^{p} \Gamma(\nu_{1a} + \mu_{a} + 1/2) \Gamma\left( \nu_{2a} + \mu_{a} + 1/2 \right) G_{\nu_{1a} + \mu_{a}}^{(i)} \left[ G_{\nu_{2a} + \mu_{a}}^{(i)}(\gamma) \right] .
\]

Using the operators \( E_{i} (i=1, 2) \) and the superscripts \((i)\) for convenience defined in the lemma, we can have the expression

\[
\prod_{a=1}^{p} \sum_{\nu_{1a}} \frac{\Gamma(\nu_{1a} + \mu_{a} + 1/2)}{\Gamma(\mu_{a} + 1/2)} G_{\nu_{1a} + \mu_{a}}^{(i)} \left[ G_{\nu_{2a} + \mu_{a}}^{(i)}(\gamma) \right] .
\]

Hence we can write \( J \) as

\[
J = \frac{|I - \zeta E_{i}|^{1 - \gamma}}{|I - \zeta|^{1 - \gamma}} \left[ \frac{1 - (\delta/\gamma)^2}{[1 - (\delta/\gamma)^2]} \right]^{p/2}
\times \prod_{a=1}^{p} \frac{\Gamma(\mu_{a} + 1/2)}{\Gamma(\mu_{a})} \left( \frac{1 - \zeta_{a}^{2}}{1 - \zeta_{a}^{2} E_{i}} \right) \left( \frac{\delta}{\gamma} \right)^{\nu_{1a} + \mu_{a}} G_{\nu_{1a} + \mu_{a}}^{(i)}(\gamma) G_{\nu_{2a} + \mu_{a}}^{(i)}(\gamma)
\]

\[
= [1 - (\delta/\gamma)^2]^{p/2} |I - \zeta E_{i} - (I - \zeta E_{i}) (I - \zeta)^{-1} (I - \zeta E_{i})|^{-1/2}
\times |I - (I - \zeta E_{i})^{-1} (I - \zeta)(I - \zeta E_{i})^{-1} (I - \zeta)(\delta)^{2} E_{i} E_{2}|^{-1/2}
\times G_{\nu_{1a}}^{(i)}(\gamma) G_{\nu_{2a}}^{(i)}(\gamma) ,
\]

which proves the lemma.

**Lemma 5.4.**

(5.7) \( D_{\gamma} \{ u^{\Lambda - 1} u, u^{\Lambda - 1} u > a^{2} \} = -[\gamma/(\gamma^{2} - \delta^{2})][1 - (\delta/\gamma)^{2}]^{p/2} \left[ \Gamma\left( \frac{1}{2} p \right) \right]^{-1}
\]

\[
\times \sum_{j=0}^{\infty} \frac{\Gamma(p/2 + j)}{j!} \left( \frac{\delta}{\gamma} \right)^{j} g_{p/2 + j}(\gamma) G_{p/2 + j}(\gamma) .
\]

(5.8) \( D_{p} \{ u^{\Lambda - 1} u, u^{\Lambda - 1} u > a^{2} \} = \frac{1}{2} \left[ \gamma/(\gamma^{2} - \delta^{2}) \right][1 - (\delta/\gamma)^{2}]^{p/2} \left[ \Gamma\left( \frac{1}{2} p \right) \right]^{-1}
\]

\[
\times \sum_{j=0}^{\infty} \frac{\Gamma(p/2 + j)}{j!} \left( \frac{\delta}{\gamma} \right)^{j} \left[ g_{p/2 + j}(\gamma) - g_{p/2 + j - 1}(\gamma) \right] G_{p/2 + j}(\gamma) + g_{p/2 + j}(\gamma) ,
\]

where \( g_{m}(\gamma) = [\Gamma(m)]^{-1} \gamma^{m-1} e^{-\gamma} (m > 0) \) with the following exceptions,

\[
g_{p/2 + j}(\gamma) = \begin{cases} 0 & \text{for } p=2, j=0 , \\ \frac{-1}{2} \gamma^{-3/2} e^{-\gamma} & \text{for } p=1, j=0 , \\ \end{cases}
\]
and \( \gamma = (1/2) (\gamma(\gamma^2 - \delta^2)) a^2 \) and \( D = \partial / \partial a^2 \).

**Proof** of this lemma is easily obtained by differentiating (4.8) directly with respect to \( a^2 \).

Now we shall consider the evaluation of the derivatives of \( P_r \{ u'_1, \Lambda^{-1} u_1, > a^2, u'_2, \Lambda^{-1} u_2, > a^2 \} \) with respect to \( \lambda_{rs} \) which is an element of \( \Lambda \). To do this, we consider

\[
(5.9) \quad J = P_r \{ u'_1 (\Lambda + \epsilon)^{-1} u_1, > a^2, u'_2 (\Lambda + \epsilon)^{-1} u_2, > a^2 \} = \left\{ 1 + \sum_{r,s} \epsilon_{rs} \partial_{rs} + \frac{1}{2} \sum_{r,s,t} \epsilon_{rs} \epsilon_{tu} \partial_{rs} \partial_{tu} + \cdots \right\} P_r \{ u'_1, \Lambda^{-1} u_1, > a^2, u'_2, \Lambda^{-1} u_2, > a^2 \}
\]

where \( \partial_{rs} = (1/2)(1 + \delta_{rs}) \partial / \partial \lambda_{rs} \) (\( \delta_{rs} \) is Kronecker’s delta), and \( \epsilon \) is a \( p \times p \) symmetric matrix consisting of small increments \( \epsilon_{ij} \), to \( \lambda_{ij} \) \((i, j = 1, 2, \cdots, p)\).

On the other hand we can also express \( J \) in the form

\[
J = \frac{1}{(2\pi)^p |\Lambda|^{(\gamma^2 - \delta^2)/2}} \int_{D^*} \ldots \int_{D^*} \exp \left\{ - \frac{1}{2} [\gamma(\gamma^2 - \delta^2)] (u'_1, \Lambda^{-1} u_1 - 2(\delta/\gamma) u'_1, \Lambda^{-1} u_2 + u'_2, \Lambda^{-1} u_1) \right\} d u_1 d u_2,
\]

where \( D^* : u'_1 (\Lambda + \epsilon)^{-1} u_1, > a^2, u'_2 (\Lambda + \epsilon)^{-1} u_2, > a^2 \).

Making the non-singular linear transformations

\[
u_i = \sqrt{(\gamma^2 - \delta^2)/\gamma} C w_1 \quad \text{and} \quad \nu_2 = \sqrt{(\gamma^2 - \delta^2)/\gamma} C w_2
\]

such that

\[
\frac{1}{2} C' (\Lambda + \epsilon)^{-1} C = I \quad \text{and} \quad \frac{1}{2} C' \Lambda^{-1} C = I - \zeta
\]

where \( I \) is the unit matrix and \( \zeta \) is a diagonal matrix, \( \text{diag} \{ \xi_1, \xi_2, \cdots, \xi_p \} \), with \( |\xi_i| < 1 \) for all \( i \), \( J \) becomes

\[
J = |I - \zeta| [1 - (\delta/\gamma)]^{p/2} \int_{D^*} \ldots \int_{D^*} \exp \left\{ - w'_1 (I - \zeta) w_1 + 2(\delta/\gamma) w'_1 (I - \zeta) w_2 - w'_2 (I - \zeta) w_1 \right\} d w_1 d w_2,
\]

where \( 2\gamma = [\gamma(\gamma^2 - \delta^2)] a^2 \). This is the form of (5.5) and hence, from Lemma 5.3, \( J \) can be expressed in the form (5.6). Putting \( \Delta_i = E_i - 1 \), \((i = 1, 2)\), \( H = [1 - (\delta/\gamma)] E_1 E_2 \) and \( X = (\Lambda + \epsilon)^{-1} \Lambda - I \), we have

\[
(5.10) \quad J = [1 - (\delta/\gamma)]^{p/2} |I - \{ X (\Delta_1 + \Delta_2) - X^2 \Delta_1 \Delta_2 \} H |^{1/2} H^{p/2} G_{p/2}^{(1)}(\gamma) G_{p/2}^{(2)}(\gamma)
\]
In order to obtain the derivatives of \( P_r(u; \Delta^{-1}u > a^2, u; \Delta^{-1}u > a^2) \) with respect to \( \lambda_{rs} \), we expand (5.10) in power series of \( \varepsilon \)'s and compare the resultant with (5.9). Since, for fixed \( p \) and for small \( \varepsilon \)'s, we have

\[
\begin{align*}
| I - \{ (\Delta_1 + \Delta_2) - X^2 \Delta_1 \Delta_2 \} H |^{-3/2} &= \exp \left[ -\frac{1}{2} \log | I - \{ (\Delta_1 + \Delta_2) - X^2 \Delta_1 \Delta_2 \} H | \right] \\
= \exp \left[ \frac{1}{2} \text{tr} \{ (\Delta_1 + \Delta_2) - X^2 \Delta_1 \Delta_2 \} H + \frac{1}{4} \text{tr} \{ (\Delta_1 + \Delta_2) - X^2 \Delta_1 \Delta_2 \}^2 H^2 + \cdots \right] \\
= 1 + \frac{1}{2} \text{tr} X (\Delta_1 + \Delta_2) H \\
&+ \frac{1}{4} \left( \text{tr} X^2 + \frac{1}{2} \left( \text{tr} X \right)^2 \right) (\Delta_1 + \Delta_2) H^2 - \frac{1}{2} \left( \text{tr} X \right) \Delta_1 \Delta_2 H \\
&+ \frac{1}{6} \left( \text{tr} X^2 + \frac{3}{4} \left( \text{tr} X \right)^2 \left( \text{tr} X \right) + \frac{1}{8} \left( \text{tr} X \right)^3 \right) (\Delta_1 + \Delta_2) H^3 \\
&- \frac{1}{2} \left( \text{tr} X^2 + \frac{1}{2} \left( \text{tr} X \right) \left( \text{tr} X \right) \right) \Delta_1 \Delta_2 H^2 \\
&+ \frac{1}{8} \left( \text{tr} X^2 + \frac{2}{3} \left( \text{tr} X \right)^2 \left( \text{tr} X \right) + \frac{1}{4} \left( \text{tr} X \right)^3 \right) (\Delta_1 + \Delta_2) H^4 \\
&+ \frac{1}{4} \left( \text{tr} X^3 \right) \left( \text{tr} X \right)^3 + \frac{1}{48} \left( \text{tr} X \right)^4 \left( \Delta_1 + \Delta_2 \right) H^4 \\
&- \frac{1}{2} \left( \text{tr} X^3 + \frac{1}{2} \left( \text{tr} X \right) \left( \text{tr} X \right) \right) \Delta_1 \Delta_2 H^3 \\
&+ \frac{1}{4} \left( \text{tr} X^3 + \frac{1}{2} \left( \text{tr} X \right) \left( \text{tr} X \right) \right) \Delta_1 \Delta_2 H^2 \\
&+ \cdots, 
\end{align*}
\]

and \( X \) has the expansion, with the notation \( \Delta_{rs} = \theta_{rs} \Lambda \),

\[
X = (\Lambda + \varepsilon)^{-1} \Lambda - I = (I + \sum \varepsilon_{rs} \Lambda^{-1} \Lambda_{rs}^{-1} - I
= - \sum \varepsilon_{rs} \Lambda^{-1} \Lambda_{rs} - \sum \varepsilon_{rs} \varepsilon_{tu} \varepsilon_{sw} \Lambda^{-1} \Lambda_{tu} \Lambda^{-1} \Lambda_{sw} + \cdots,
\]

\( J \) can be expanded in the following form

\[
(5.11) \quad J = (1 - (\delta/\gamma)^2) \sum [1 - \sum \varepsilon_{rs} (M_1) + \sum \varepsilon_{rs} \varepsilon_{tu} (M_2) - \sum \varepsilon_{rs} \varepsilon_{tu} \varepsilon_{uw}(M_3) \\
+ \sum \varepsilon_{rs} \varepsilon_{tu} \varepsilon_{uw} (M_4) - \cdots] H^{p/3} G^{(1)}_{p/3} G^{(2)}_{p/3},
\]

where

\[
(M_1) = \frac{1}{2} [rs] (\Delta_1 + \Delta_2) H,
\]

\[
(M_2) = \frac{1}{4} \left( [rs tu] + \frac{1}{2} [rs][tu] \right) (\Delta_1 + \Delta_2) H^2 + \frac{1}{2} [rs tu] (\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H,
\]

\[
(M_3) = \frac{1}{6} \left( [rs tu vw] + \frac{3}{4} [rs][tu vw] + \frac{1}{8} [rs][tu][vw] \right) (\Delta_1 + \Delta_2) H^3.
\]
\[
+ \frac{1}{2} \left( [rs|tu|vw] + \frac{1}{2} [rs][tu][vw] \right) (\Delta_1 + \Delta_2) (\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H^2
+ \frac{1}{2} [rs|tu|vw] (\Delta_1 + \Delta_2 - 2 \Delta_1 \Delta_2) H^2
\]

\[
(M_8) = \frac{1}{8} \left( [rs|tu|vw|xy] + \frac{2}{3} [rs][tu|vw|xy] + \frac{1}{4} [rs|tu][vw|xy]
+ \frac{1}{4} [rs][tu][vw|xy] + \frac{1}{48} [rs][tu][vw][xy] \right) (\Delta_1 + \Delta_2) H^4
+ \frac{1}{2} \left( [rs|tu|vw|xy] + \frac{1}{2} [rs][tu|vw|xy] + \frac{1}{4} [rs|tu][vw|xy] \right) (\Delta_1 + \Delta_2) (\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H^3
+ \frac{1}{4} \left( [rs|tu|vw|xy] + [rs][tu|vw|xy] \right) (\Delta_1 + \Delta_2) (\Delta_1 + \Delta_2 - 2 \Delta_1 \Delta_2) H^2
+ \frac{1}{4} \left( [rs|tu|vw|xy] + \frac{1}{2} [rs|tu][vw|xy] \right) \Delta_1 \Delta_2 H^2
+ \frac{1}{2} [rs|tu|vw|xy] (\Delta_1 + \Delta_2 - 3 \Delta_1 \Delta_2) H^2
\]

and here we have used the abbreviated notations (see [4] and [11])

\[
[rs] = \text{tr} \Delta^{-1} \Lambda_{rs} = \lambda^r s, [rs|tu] = \text{tr} \Delta^{-1} \Lambda_{rt} \Delta^{-1} \Lambda_{sw} = (1/2)(\lambda^r s \lambda^t u + \lambda^s r \lambda^t u),
\]

and so on. Comparing (5.11) with (5.9), we have

\[
\partial_r P_r \{ y_i' \Lambda^{-1} y_i > a^2, y_i' \Lambda^{-1} y_i > a^2 \}
= - \frac{1}{2} [rs] (\Delta_1 + \Delta_2) H^{p+2} \left[ 1 - (\delta/\gamma)^p G_{\mid p}(\gamma) G_{\mid p}(\gamma) \right],
\]

\[
\partial_r \partial_u P_r \{ \cdots \} = \frac{1}{2} \left( [rs|tu] + \frac{1}{2} [rs][tu] \right) (\Delta_1 + \Delta_2) H^{p/2+2}
+ 2 [rs|tu] (\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H^{p/2+1} \left[ 1 - (\delta/\gamma)^p G_{\mid p}(\gamma) G_{\mid p}(\gamma) \right].
\]

\[
\partial_r \partial_u \partial_{vw} P_r \{ \cdots \} = - \left( [rs|tu|vw] + \frac{3}{4} [rs][tu|vw] \right)
+ \frac{1}{8} [rs][tu][vw] (\Delta_1 + \Delta_2) H^{p/2+3}
+ 3 \left( [rs|tu|vw] + \frac{1}{2} [rs][tu|vw] \right) (\Delta_1 + \Delta_2) (\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H^{p/2+2}
+ 3 [rs|tu|vw] (\Delta_1 + \Delta_2 - 2 \Delta_1 \Delta_2) H^{p/2+1} \right]
\times \left[ 1 - (\delta/\gamma)^p G_{\mid p}(\gamma) G_{\mid p}(\gamma) \right],
\]
\[
(5.15) \quad \partial_r \partial_{tu} \partial_{vw} \partial_{xy} P_r(\cdots) =\begin{cases}
2[rs|tu|vw|xy]| + |rs|vw|tu|xy| \\
\frac{1}{4} \frac{|rs|tu|vw|xy|}{|tu|xy|} + \frac{1}{2} \frac{|rs|vw|tu|xy|}{|tu|xy|} \\
\frac{1}{4} \frac{|rs|tu|vw|xy|}{|tu|xy|} + \frac{1}{2} \frac{|rs|vw|tu|xy|}{|tu|xy|} \\
+ \frac{1}{16} \frac{|rs|tu|vw|xy|}{(\Delta_1 + \Delta_2)H^{\gamma/2+2}} \\
+ \{8[rs|tu|vw|xy| + |4[rs|vw|tu|xy| + 6[rs|tu|vw|xy| \\
+ |rs|tu|vw|xy| + 2[rs|vw|tu|xy| + \frac{1}{2} |rs|tu|vw|xy| \\
+ |rs|vw|tu|xy|} (\Delta_1 + \Delta_2)(\Delta_1 + \Delta_2 - 2\Delta_3)H^{2/2+3} \\
+ \{12[rs|tu|vw|xy| + 6[rs|vw|tu|xy| + 6[rs|tu|vw|xy| \\
+ |rs|tu|vw|xy| \\
+ 2[rs|vw|tu|xy|} (\Delta_1 + \Delta_2)(\Delta_1 + \Delta_2 - 2\Delta_3)H^{\gamma/2+2} \\
+ \{4[rs|tu|vw|xy| + 2[rs|vw|tu|xy| + |rs|tu|vw|xy| \\
+ 2[rs|vw|tu|xy|} |\Delta_1 |H^{\gamma/2+2} \\
+ \{8[rs|tu|vw|xy| + 4[rs|vw|tu|xy|} (\Delta_1 + \Delta_2 - 3\Delta_3)H^{\gamma/2+1} \\
\times [1 - (\delta/\gamma)^{\gamma/2} G_{\gamma/2}(\gamma) G_{\gamma/2}(\gamma) .
\end{cases}
\]

**Lemma 5.5.** Let \( (y', y') \) be \((p+p)\)-dimensional normal vector variate defined in Theorem 4.2. Then the first, second, third and fourth order derivatives of \( P_r(y', \lambda^{-1}y, \lambda^{-1}y, > a^2) \) with respect to the elements of \( \Lambda \) are given by (5.12), (5.13), (5.14) and (5.15) respectively.

On the basis of the preliminary works discussed above, we can now obtain the asymptotic formula for

\[
P_r(\chi' L^{-1} y, < B[(\alpha; p, N, \nu), y' L^{-1} y, > B[(\alpha; p, N, \nu))
\]

up to the term of order \( \nu^{-2} \). According to James' formula [(5.16) of 4], we have

\[
(5.16) \quad P_r(\chi' L^{-1} y, > B[(\alpha; p, N, \nu), y' L^{-1} y, > B[(\alpha; p, N, \nu))
\]

\[
= \left[ 1 + \frac{1}{\nu} \sum_{\nu \tau \tau} \lambda_{\nu \tau} \lambda_{\tau \nu} \partial_r \partial_{tu} \right]
\]
\[ + \frac{1}{\nu^2} \left( \frac{4}{3} \sum \lambda_{vr} \lambda_{zt} \lambda_{rw} \lambda_{ov} \partial_{ru} \partial_{tu} \partial_{ov} + \frac{1}{2} \sum \lambda_{ur} \lambda_{zt} \lambda_{yw} \lambda_{ow} \partial_{ru} \partial_{tu} \partial_{ov} \partial_{xy} \right) \]

\[ + O \left( \frac{1}{\nu^3} \right) \] \cdot P_r \{ \nu' A_{r}^{-1} \nu' > B_i'(\alpha; p, N, \nu), \nu' A_{r}^{-1} \nu' > B_i'(\alpha; p, N, \nu) \}.

Since for fixed \( \alpha, p \) and \( N, B_i'(\alpha; p, N, \nu) \) is a function of \( \nu \) and when \( \nu \) is large, \( B_i'(\alpha; p, N, \nu) \) will approach \( A_i'(\alpha; p, N) \), the first approximation to the upper 100\% point of \( \chi^2_{\text{max}} \), we need to expand

\[ P_r \{ \nu' A_{r}^{-1} \nu' > B_i'(\alpha; p, N, \nu), \nu' A_{r}^{-1} \nu' > B_i'(\alpha; p, N, \nu) \} \]

in Taylor's series. From Lemma 5.1, we can represent \( B_i'(\alpha; p, N, \nu) \) as

\( B_i'(\alpha; p, N, \nu) = A_i'(\alpha; p, N) + h_i'(\alpha) + h_i'(\alpha) + \ldots \)

\( A_i'(\alpha; p, N) = A_i = \gamma \chi^2(\alpha/N; p) \),

\( h_i'(\alpha) = \frac{1}{2\nu} \gamma \chi^2(\alpha/N; p) \{ \chi(\alpha/N; p) + p \} \).

\[ h_2(\alpha) = \frac{1}{24\nu^2} \gamma \chi^2(\alpha/N; p) \{ 4 \chi(\alpha/N; p) + (13p - 2) \chi'(\alpha/N; p) + 7p^2 - 4 \} \].

Then by Taylor's expansion

\[ P_r \{ \nu' A_{r}^{-1} \nu' > B_i'(\alpha; p, N, \nu), \nu' A_{r}^{-1} \nu' > B_i'(\alpha; p, N, \nu) \} = \left[ 1 + \{ h_i(\alpha) + h_i(\alpha) + \ldots \} D + \frac{1}{2} \{ h_i(\alpha) + h_i(\alpha) + \ldots \}^2 D^2 + \ldots \right] \times P_r \{ \nu' A_{r}^{-1} \nu' > A_i^2 \}, \nu' A_{r}^{-1} \nu' > A_i^2 \}

where \( D = \partial / \partial A_i^2 \). Substituting this into (5.16), we obtain

\[ P_r \{ \nu' L^{-1} \nu' > B_i'(\alpha; p, N, \nu), \nu' L^{-1} \nu' > B_i'(\alpha; p, N, \nu) \} \]

\[ = \left[ 1 + \left\{ h_i(\alpha) D + \frac{1}{\nu} \sum \lambda_{ur} \lambda_{zt} \partial_{ru} \partial_{tu} \right\} \right. \]

\[ + \left\{ h_i(\alpha) D + \frac{1}{2} h_i(\alpha) D^2 + \frac{1}{\nu} \sum \lambda_{ur} \lambda_{zt} \partial_{ru} \partial_{tu} D \right. \]

\[ + \frac{4}{3} \frac{1}{\nu^2} \sum \lambda_{ur} \lambda_{zt} \lambda_{yw} \lambda_{ow} \partial_{ru} \partial_{tu} \partial_{ov} \]

\[ + \frac{1}{2} \frac{1}{\nu^2} \sum \lambda_{ur} \lambda_{zt} \lambda_{yw} \lambda_{ow} \partial_{ru} \partial_{tu} \partial_{ov} \partial_{xy} \right] \]

\[ + O \left( \frac{1}{\nu^3} \right) \] \cdot P_r \{ \nu' A_{r}^{-1} \nu' > A_i^2, \nu' A_{r}^{-1} \nu' > A_i^2 \}.

From this formula we can evaluate the desired probability with the aid of Lemmas 5.4, 5.5 and (5.3).

(a) **Term of order zero**, \( O_i(\alpha, p, N) \).
(5.18) \[ O_0(\alpha, p, N) = P_r\{\mathbf{v}'\Lambda^{-1}\mathbf{y}_1 > \Lambda^{+}_1; \mathbf{y}_1'\Lambda^{-1}\mathbf{y}_1 > \Lambda^{+}_1\} \]
\[ = [1 - (\delta/\gamma)^2]^{p/2} \sum_{j=0}^{\infty} \frac{\Gamma(p/2+j)}{j!} \left( \frac{\delta}{\gamma} \right)^j G_{p/2+j}^{(1)}(\gamma) \]
where \(2\gamma = [\gamma/(\gamma^2 - \delta^2)]\cdot A_0(\alpha; p, N) = [1 - (\delta/\gamma)^2]^{-1} \chi(\alpha/N; p).

(b) Term of order \(\nu^{-1}, O_1(\alpha, p, N) = O_{11}(\alpha, p, N) + O_{12}(\alpha, p, N).

(5.19) \[ O_{11}(\alpha, p, N) = h_1(\alpha) \cdot DP_r\{\mathbf{v}'\Lambda^{-1}\mathbf{y}_1 > \Lambda^{+}_1; \mathbf{y}_1'\Lambda^{-1}\mathbf{y}_1 > \Lambda^{+}_1\} \]
\[ = - \frac{\nu}{\nu} \chi(\alpha^2 + p) \frac{[1 - (\delta/\gamma)^2]^{p/2} \Gamma(p/2)}{\gamma} \]
\[ \times \sum_{j=0}^{\infty} \frac{\Gamma(p/2+j)}{j!} \left( \frac{\delta}{\gamma} \right)^j g_{p/2+j}^{(1)}(\gamma) G_{p/2+j}^{(2)}(\gamma) \]
where \(\chi = \chi(\alpha/N; p).

O_{12}(\alpha, p, N) \equiv \frac{1}{\nu} \sum_{r,s,t} \lambda_r \lambda_s \lambda_t \partial_{r} \partial_{s} \partial_{t} P_r\{\mathbf{y}'\Lambda^{-1}\mathbf{y}_1 > \Lambda^{+}_1; \mathbf{y}_1'\Lambda^{-1}\mathbf{y}_1 > \Lambda^{+}_1\} \]
\[ = \frac{1}{\nu} \left[1 - (\delta/\gamma)^2\right]^{p/2} \sum_{r,s,t} \lambda_r \lambda_s \lambda_t \left[ \frac{1}{2} [rs|tu] + \frac{1}{4} [rs][tu] \right] (\Delta_1 + \Delta_2)^{p/2+2} \]
\[ + [rs|tu](\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H_{p/2+1}^{(1)}(\gamma) \]
\[ G_{p/2}^{(1)}(\gamma) G_{p/2}^{(2)}(\gamma) \]
\[ = \frac{1}{\nu} \left[1 - (\delta/\gamma)^2\right]^{p/2} \frac{1}{4} p(p+2)(\Delta_1 + \Delta_2)^{p/2+2} \]
\[ + \frac{1}{2} p(p+1)(\Delta_1 + \Delta_2 - \Delta_1 \Delta_2) H_{p/2+1}^{(2)}(\gamma) \]
\[ G_{p/2}^{(1)}(\gamma) G_{p/2}^{(2)}(\gamma) . \]

Noting that
\[ H_{p/2+m}^{(1)}(\gamma) G_{p/2}^{(1)}(\gamma) = \left\{1 - (\delta/\gamma)^2 E_1 E_2\right\}^{-p/2+m} G_{p/2}^{(1)}(\gamma) G_{p/2}^{(2)}(\gamma) \]
\[ = \frac{1}{\Gamma(p/2+m)} \sum_{j=0}^{\infty} \frac{\Gamma(p/2+m+j)}{j!} \left( \frac{\delta}{\gamma} \right)^j G_{p/2+j}^{(1)}(\gamma) G_{p/2+j}^{(2)}(\gamma) , \]
\( (m = 0, 1, 2, \ldots) \)
and
\[ \Delta_i G_{p/2+j}^{(1)}(\gamma) = (E_i - 1) G_{p/2+j}^{(1)}(\gamma) = \left[ \Gamma\left( \frac{p}{2} + 1 + j \right) \right]^{-1} \gamma^{p/2+j} e^{-\gamma} \equiv g_{p/2+j}^{(1)}(\gamma) , \]
\[ \Delta_i G_{p/2+j}^{(2)}(\gamma) = (E_i - 2E_1 + 1) G_{p/2+j}^{(2)}(\gamma) = g_{p/2+j}^{(2)}(\gamma) - g_{p/2+j+1}^{(1)}(\gamma) , \]
\[ \ldots \]
we obtain after some arrangement
(5.20) \[ O_{12}(\alpha, p, N) = \frac{1}{\nu} \left[1 - (\delta/\gamma)^2\right]^{p/2} \sum_{j=0}^{\infty} \frac{\Gamma(p/2+1+j)}{j!} \left( \frac{\delta}{\gamma} \right)^j \]
\[ \times \left\{ (2\gamma + p - 2j) g_{p/2+j}^{(1)}(\gamma) G_{p/2+j}^{(2)}(\gamma) + (2j + 1) g_{p/2+j+1}^{(1)}(\gamma) \right\} , \]
where we put \( G_{p/2}^{(1)}(\nu) = G_{p/2}(\nu) \) and \( g_{p/2}^{(1)}(\nu) = g_{p/2+1}(\nu) \) after operation by \( E_i \) is carried out.

In the analogous way, though a great deal of algebra is needed, the term of order \( \nu^{-2} \) can be obtained in the following form.

(c) Term of order \( \nu^{-2} \), \( O_4(\alpha, p, N) = O_{21}(\alpha, p, N) + O_{22}(\alpha, p, N) + O_{23}(\alpha, p, N) + O_{24}(\alpha, p, N) \).

\[
(5.21) \quad O_{21}(\alpha; p, N) = h_4(\alpha) \cdot DP_r\{ y_1^2 y_2^2 > 4; y_2^2 y_3^2 > 4 \}
\]
\[
= -\frac{1}{24\nu^2} [1 - (\delta/\gamma)]^{(p-2)/2} \Gamma\left(\frac{p}{2}\right)^{-1} \times \chi^2(4\chi + (13p - 2)\chi + 7p^2 - 4)
\times \sum_{j=0}^{\infty} \frac{\Gamma(p/2 + j)}{j!} \left(\frac{\delta}{\gamma}\right)^{2j} g_{p/2+j}(\gamma) G_{p/2+j}(\gamma),
\]

\[
(5.22) \quad O_{22}(\alpha, p, N) = \frac{1}{2} h_5(\alpha) DP_r\{ \cdots \}
\]
\[
= \frac{1}{16\nu^2} [1 - (\delta/\gamma)]^{p/2 - 2} \Gamma\left(\frac{p}{2}\right)^{-1} \chi^2(p + p) \times \sum_{j=0}^{\infty} \frac{\Gamma(p/2 + j)}{j!} \left(\frac{\delta}{\gamma}\right)^{2j} \times \left[ (g_{p/2+j}(\gamma) - g_{p/2+j-1}(\gamma)) G_{p/2+j}(\gamma) + g_{p/2+j}^2(\gamma) \right]
\]

\[
(5.23) \quad O_{23}(\alpha, p, N) = h_1(\alpha) \sum_{\nu \neq 0} \lambda \lambda_{\nu} \theta_{\nu} \theta_{\nu} DP_r\{ \cdots \}
\]
\[
= -\frac{1}{2\nu^2} [1 - (\delta/\gamma)]^{(p-2)/2} \Gamma\left(\frac{p}{2}\right)^{-1} \times \chi^2(p + p) \sum_{j=0}^{\infty} \frac{\Gamma(p/2 + 1 + j)}{j!} \left(\frac{\delta}{\gamma}\right)^{2j} \left[ (\gamma + p - j) (g_{p/2+j+1}(\gamma) - g_{p/2+j}(\gamma)) G_{p/2+j}(\gamma) + g_{p/2+j+1}(\gamma) G_{p/2+j}(\gamma) \right]
\]
\[
+ (2j + 1) (g_{p/2+j+1}(\gamma) - g_{p/2+j}(\gamma)) G_{p/2+j+1}(\gamma) - g_{p/2+j+1}(\gamma) G_{p/2+j}(\gamma) \right],
\]

\[
(5.24) \quad O_{24}(\alpha, p, N) = \frac{4}{3} \sum_{\nu \neq 0} \lambda_{\nu} \lambda_{\nu} \theta_{\nu} \theta_{\nu} DP_r\{ \cdots \}
\]
\[
= \frac{1}{2\nu^2} [1 - (\delta/\gamma)]^{p/2} \Gamma\left(\frac{p}{2}\right)^{-1} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{\gamma}\right)^{2j} \left[ \frac{4}{3} H_0(\gamma) \right],
\]

where

\[
(5.25) \quad -\frac{4}{3} H_0(\gamma) = \frac{8}{3} \Gamma\left(\frac{p}{2} + 3 + j\right) \left[ (g_{p/2+j+1}(\gamma) - 2g_{p/2+j}(\gamma) \right.
\]
\[+ g_{p/2+j+1}(\gamma)) G_{p/2+j}(\gamma) + g_{p/2+j+1}(\gamma) - g_{p/2+j+1}(\gamma) G_{p/2+j}(\gamma) \]
\[+ 4(p + 3) \Gamma\left(\frac{p}{2} + 2 + j\right) \left[ g_{p/2+j+1}(\gamma) - g_{p/2+j+1}(\gamma) \right]
\]
\[ \frac{1}{\nu} \{ g_{p/2} \} \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\delta}{\gamma} \right)^i \{ \frac{1}{2} K_{i}(\gamma) \}, \]

where

\[ \frac{1}{2} K_{i}(\gamma) = \Gamma \left( \frac{p}{2} + 4 + j \right) \left\{ \{ g_{p/2+4+j}(\gamma) - \frac{3}{2} g_{p/2+2+j}(\gamma) \right. \\
+ \frac{3}{2} g_{p/2+2+j}(\gamma) - g_{p/2+1+j}(\gamma) \} G_{p/2+j}(\gamma) + 3 \{ g_{p/2+2+j}(\gamma) - g_{p/2+1+j}(\gamma) \} \right. \\
\left. + 4 \{ g_{p/2+2+j}(\gamma) - 2g_{p/2+1+j}(\gamma) + g_{p/2+1+j}(\gamma) \} g_{p/2+1+j}(\gamma) \right. \\
\left. + 2(p+5) \Gamma \left( \frac{p}{2} + 3 + j \right) \left\{ \{ g_{p/2+3+j}(\gamma) - 2g_{p/2+2+j}(\gamma) \right. \\
+ g_{p/2+j+1}(\gamma) \} G_{p/2+j}(\gamma) - \{ g_{p/2+2+j}(\gamma) - g_{p/2+1+j}(\gamma) \} \right. \\
\left. - \{ g_{p/2+3+j}(\gamma) - 5g_{p/2+2+j}(\gamma) + 4g_{p/2+1+j}(\gamma) \} g_{p/2+1+j}(\gamma) \right. \\
\left. + (p^2 + 12p + 23) \Gamma \left( \frac{p}{2} + 2 + j \right) \left\{ g_{p/2+2+j}(\gamma) - g_{p/2+1+j}(\gamma) \right. \\
\left. \times \{ G_{p/2+j}(\gamma) - 2g_{p/2+1+j}(\gamma) \} + g_{p/2+1+j}(\gamma) \right. \\
\left. + \frac{1}{2} (p^2 + 4p + 7) \Gamma \left( \frac{p}{2} + 2 + j \right) \left\{ g_{p/2+2+j}(\gamma) - g_{p/2+1+j}(\gamma) \right. \\
\left. + 2p^2 + 5p + 5) \Gamma \left( \frac{p}{2} + 1 + j \right) \left\{ g_{p/2+1+j}(\gamma) \right. \\
\left. - g_{p/2+1+j}(\gamma) \} g_{p/2+1+j}(\gamma) - g_{p/2+1+j}(\gamma) \right. \}. \]

The result are summarized in

**Theorem 5.1.** Let \((y'_i, y'_j)\) be the \((p+p)\)-dimensional normal variate defined in Theorem 4.2 and let \(L\) be the unbiased estimate of \(\Lambda\), distributed independently of \(y_i\) and \(y_j\) according to the Wishart distribution (5.1). Then, for the solution \(B^*(\alpha; p, N, \nu)\) of (5.2), the asymptotic formula for \(P_r\{y'_i L^{-1} y_i > B^*(\alpha; p, N, \nu)\} \) up to the term of order \(\nu^{-1}\) is given by (5.18), (5.19), (5.20), (5.21), (5.22), (5.23), (5.24) and (5.26).

6. The maximum deviates from the sample mean, \(\hat{\chi}^2_{\text{MAX.D}}\) and \(\hat{T}^2_{\text{MAX.D}}\). As an example of evaluating the \(\beta(\alpha; p, N)\) and the \(\beta^*(\alpha; p, N, \nu)\) by application of Theorem 5.1, we shall discuss the maximum deviates from the sample mean defined in (2.7) and (2.8), i.e.,
\[(6.1) \quad \hat{\chi}_{\text{MAX-D}}^2 = \max_{x} \{(x - \bar{x})' \Lambda^{-1} (x - \bar{x})\} \quad \text{and} \quad \hat{T}_{\text{MAX-D}} = \max_{x} \{(x - \bar{x})' L^{-1} (x - \bar{x})\} . \]

The statistical procedure based on \( \hat{\chi}_{\text{MAX-D}}^2 \) has an optimum property as a multiple decision procedure. This is the straight-forward generalization of the slippage problem treated by E. Paulson [8] in the univariate case and in the case of unknown variance.

Let \( z_\alpha, (\alpha = 1, \cdots, n) \) be normal vector variates with mean vectors \( m_\alpha, (\alpha = 1, 2, \cdots, n) \), respectively, and with a common covariance matrix \( \Lambda \) which is assumed to be known. Then we consider the null hypothesis \( H_0 \) that \( m_1 = m_2 = \cdots = m_n \) and \( n \) alternative hypotheses \( H_\alpha(\Delta m), (\alpha = 1, 2, \cdots, n) \), that \( m_1 = \cdots = m_{\alpha-1} = m_\alpha - \Delta m = m_{\alpha+1} = \cdots = m_n \) where \( \Delta m \) is a non-zero vector and \( (n+1) \) decisions \( D_0, D_1, \cdots, D_n \) which, under some restrictions [(i), (ii) and (iii) of 5], maximizes the probability of making the correct decision when one of the hypotheses \( H_0, H_1, \cdots, H_n \) is true, is given by the following:

if \( (z_m - \bar{z})' \Lambda^{-1} (z_m - \bar{z}) > L_t \), select \( D_M \) and

if \( (z_m - \bar{z})' \Lambda^{-1} (z_m - \bar{z}) \leq L_t \), select \( D_0 \)

where \( \bar{z} = \sum_{\alpha=1}^{n} z_\alpha / n \), \( M \) is defined by

\[(z_m - \bar{z})' \Lambda^{-1} (z_m - \bar{z}) = \max_{z} \{(x - \bar{x})' \Lambda^{-1} (x - \bar{x})\} . \]

and \( L_t \) is the upper 100\( \% \) point of \( \max_{z} \{(x - \bar{x})' \Lambda^{-1} (x - \bar{x})\} \) under \( H_0 \).

When \( \Lambda \) is not known to us, it is nearly certain that the procedure based on \( \hat{T}_{\text{MAX-D}}^1 \) has the analogous optimum property though the rigorous proof has not been given.

Now we return to the problem of the evaluation of the upper percentage point of \( \hat{\chi}_{\text{MAX-D}}^2 \) and \( \hat{T}_{\text{MAX-D}}^1 \).

### 6.1 When \( \Lambda \) is known --- \( \hat{\chi}_{\text{MAX-D}}^2 \)

Noting that \( \gamma = (n-1)/n, \delta = -1/n \) and \( N = n \), we have from (3.6), (3.10) and (4.8)

\[(6.3) \quad A^2(\alpha; p, n) = \frac{(n-1)/n}{\chi^2(\alpha/n; p)} \]

and
(6.4) \[ \beta(\alpha; p, n) = \frac{1}{2} n(n-1) \left[\frac{n(n-2)/(n-1)}{\Gamma(p/2)} \right]^{1/2} \times \sum_{j=0}^{\infty} \frac{\Gamma(p/2+j)}{j!} \frac{1}{(n-1)^{1/2}} \left( \int_{n^{-1/2} F_{a; p, n}}^{\infty} g_{p+1}(\chi^2) d\chi^2 \right)^2. \]

Table 6.1 shows the values of \( A^2_{\alpha}(\alpha; p, n) \) and the lower bounds, \( \alpha - \beta(\alpha; p, n) \), of \( P_r\{\hat{\chi}^2_{\text{MAX-D}} > A^2_{\alpha}(\alpha; p, n)\} \) for \( \alpha = 0.05, 0.01; p = 2, 3, 4; n = 3, 5, 10, 20. \)

<table>
<thead>
<tr>
<th>Table 6.1.</th>
</tr>
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<tbody>
<tr>
<td>Values of ( A^2_{\alpha}(\alpha; p, n) ) and the lower bounds, ( \alpha - \beta(\alpha; p, n) ), for ( \alpha = 0.05 ) and ( 0.01. )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( p )</th>
<th>( n )</th>
<th>( \alpha = 0.05 )</th>
<th>( \alpha = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>( A^2_{1} )</td>
<td>5.459</td>
<td>7.368</td>
</tr>
<tr>
<td>( \alpha - \beta )</td>
<td>0.0447</td>
<td>0.0475</td>
<td>0.0485</td>
</tr>
<tr>
<td>3</td>
<td>( A^2_{1} )</td>
<td>6.825</td>
<td>9.076</td>
</tr>
<tr>
<td>( \alpha - \beta )</td>
<td>0.0450</td>
<td>0.0477</td>
<td>0.0485</td>
</tr>
<tr>
<td>4</td>
<td>( A^2_{1} )</td>
<td>8.063</td>
<td>10.62</td>
</tr>
<tr>
<td>( \alpha - \beta )</td>
<td>0.0453</td>
<td>0.0478</td>
<td>0.0486</td>
</tr>
</tbody>
</table>

From Table 6.1, it can be seen that for the usual values of \( \alpha, A^2_{\alpha}(\alpha; p, n) \) has the accuracy sufficient for most practical application except for \( n < 5 \), though it slightly overestimates the true value. However, the more accurate approximation to the upper 100\( \alpha \)% point of \( \hat{\chi}^2_{\text{MAX-D}} \) can be obtained by calculating the modified second approximation, \( A^2_{\alpha}(\alpha; p, n) \) such that

(6.5) \[ nP_r\{\hat{\chi}^2_{\text{MAX-D}} = (x_i - \bar{x})' A^{-1}(x_i - \bar{x}) > A^2_{\alpha}(\alpha; p, n)\} = \alpha + \beta(\alpha; p, n) \]

for \( \beta(\alpha; p, n) \) calculated for \( A^2_{\alpha}(\alpha; p, n) \), which are given in Table 1 at the end of this paper.

6.2. When \( \Lambda \) is not known—Studentized form \( \hat{T}^2_{\text{MAX-D}} \)— The first approximation \( B^2(\alpha; p, n, \nu) \) to the upper 100\( \alpha \)% point of \( \hat{T}^2_{\text{MAX-D}} \) is from (3.7) for \( \gamma = (n-1)/n, \delta = -1/n \) and \( N = n, \)

(6.6) \[ B^2(\alpha; p, n, \nu) = [(n-1)\nu/n] \left\{ \frac{1}{C(\alpha/n; (\nu + 1 - p)/2, p/2)} - 1 \right\}, \]

which is calculated with the aid of Tables of the Beta-Function [10]: \( \beta^*(\alpha; p, n, \nu) \) can be obtained from Theorem 5.1 by putting \( \gamma = (n-1)/n, \)

\( \beta(\alpha; p, n, \nu) = [1 - \left( \frac{(n-1)\nu/n}{C(\alpha/n; (\nu + 1 - p)/2, p/2)} \right)]^{-1}. \)
\( \delta = -1/n \) and \( N = n \). Though the approximation by the terms up to order \( \nu^{-2} \) are not so accurate to evaluate the probability \( P_r\{\bar{T}_{1,0} = (x_i - \bar{x})'L^{-1}(x_i - \bar{x}) > B'(\alpha; p, n, \nu) , \bar{T}_{2,0} = (x_i - \bar{x})'L^{-1}(x_i - \bar{x}) > B'(\alpha; p, n, \nu)\} \) itself, we can conclude from the numerical computation based on the formulas in Theorem 5.1 that \( \beta^*(\alpha; p, n, \nu) \) is lower order than \( \alpha \) for moderately large \( \nu \). Figures 6.1 and 6.2 show the curves of \( \beta^*(\alpha; p, n, \nu) \) for \( \alpha = 0.05 \) and \( \alpha = 0.01 \), respectively and for \( p = 2 \), which are calculated by FACOM 128, the automatic relay computer, in our institute.

From these figures it is seen that the effect of \( \beta^*(\alpha; p, n, \nu) \) is significant for the moderate magnitude of \( n \) and so we need to consider the second approximation. But in the same way with the evaluation of \( A_3(\alpha; p, n) \), we use the modified second approximation procedure, that is, we calculate \( B_3(\alpha; p, n, \nu) \) such that
(6.7) \( \alpha + \beta^*(\alpha; p, n, \nu) = nP_r\{(x_i - x)\Lambda^{-1}(x_i - x) > B^*(\alpha; p, n, \nu)\} \)

for \( \beta^*(\alpha; p, n, \nu) \) calculated for \( B^*(\alpha; p, n, \nu) \), which are contained in Table II.

7. The squares of ranges, \( R_{\text{MAX}}^2 \) and \( \mathfrak{R}_{\text{MAX}}^2 \). We shall give some comments on the evaluation of the upper percentage points of the squares of the ranges in the multivariate case, \( R_{\text{MAX}}^2 \) and \( \mathfrak{R}_{\text{MAX}}^2 \):

\[
(7.1) \quad R_{\text{MAX}}^2 = \max_{i < j} R_{ij}^2 = \max_{i < j} \{(x_i - x_j)'\Lambda^{-1}(x_i - x_j)\},
\]

\[
(7.2) \quad \mathfrak{R}_{\text{MAX}}^2 = \max_{i < j} \mathfrak{R}_{ij}^2 = \max_{i < j} \{(x_i - x_j)'L^{-1}(x_i - x_j)\}.
\]

We arrange \((1/2)n(n-1)\) \( R_{ij} \)'s in a row in a certain way and rename them \( R_1^2, R_2^2, \ldots, R_N^2 \), where \( N = (1/2)n(n-1) \). Using the formula (2.12) for \( N \) events \( R_i^2 > r^2, R_j^2 > r^2, \ldots, R_N^2 > r^2 \), we have

\[
(7.3) \quad P_r\{R_{\text{MAX}}^2 > r^2\} = NP_r\{R_i^2 > r^2\} - \sum_{i < j} P_r\{R_i^2 > r^2, R_j^2 > r^2\} + \cdots.
\]

It is easily seen that, though \( P_r\{R_i^2 > r^2, R_j^2 > r^2\} \), \((i < j = 1, 2, \ldots, N)\), are not symmetric functions of \( R_i^2 \)'s as a whole, there are following two groups \( G_1 \) and \( G_2 \) in each of which \( P_r\{R_i^2 > r^2, R_j^2 > r^2\} \) has the same value:

- \( G_1 \): group of \( P_r\{R_i^2 > r^2, R_j^2 > r^2\} \)'s such that \( R_i^2 \) and \( R_j^2 \) are independent; for example, \( R_i^2 = (x_i - x_i)'\Lambda^{-1}(x_i - x_i) \) and \( R_j^2 = (x_j - x_j)'\Lambda^{-1}(x_j - x_j) \).

- \( G_2 \): group of \( P_r\{R_i^2 > r^2, R_j^2 > r^2\} \)'s such that \( R_i^2 \) and \( R_j^2 \) have a common \( x \); for example, \( R_i^2 = (x_i - x_j)'\Lambda^{-1}(x_i - x_j) \) and \( R_j^2 = (x_j - x_j)'\Lambda^{-1}(x_j - x_j) \).

Thus we have, for the first approximation \( r^2(\alpha; p, N) \) such that

\[
(7.4) \quad \alpha = NP_r\{(x_i - x_j)'\Lambda^{-1}(x_i - x_j) > r^2(\alpha; p, N)\},
\]

\[
(7.5) \quad \beta(\alpha; p, N) = \sum_{i < j} P_r\{R_i^2 > r^2(\alpha; p, N), R_j^2 > r^2(\alpha; p, N)\}
\]

\[= M_1P_r\{(x_i - x_j)'\Lambda^{-1}(x_i - x_j) > r^2(\alpha; p, N), (x_j - x_j)'\Lambda^{-1}(x_j - x_j) > r^2(\alpha; p, N)\}
\]

\[+ M_2P_r\{(x_i - x_j)'\Lambda^{-1}(x_i - x_j) > r^2(\alpha; p, N), (x_j - x_j)'\Lambda^{-1}(x_i - x_j) > r^2(\alpha; p, N)\},
\]

where \( M_1, M_2 \) are the numbers of the elements of \( G_1, G_2 \), respectively and

\[
M_i = \left(\frac{(n-1)(n-2)}{2}\right) \quad \text{and} \quad M_s = \frac{1}{2} n(n-1)(n-2),
\]

which are shown in [12].

Analogously, in the case of unknown \( \Lambda \), that is, for \( \mathfrak{R}_{\text{MAX}}^2 \), we have

\[
(7.7) \quad \alpha = NP_r\{(x_i - x_j)'L^{-1}(x_i - x_j) > r^2(\alpha; p, N, \nu)\},
\]
and

\[
(7.8) \quad \beta^*(\alpha; p, N, \nu) \\
= M_1 P_r \{(x_1 - x_2)^T L^{-1}(x_1 - x_2) > r_1(\alpha; p, N, \nu), (x_1 - x_2)^T L^{-1}(x_3 - x_4) > r_1(\alpha; p, N, \nu)\} \\
+ M_2 P_r \{(x_1 - x_2)^T L^{-1}(x_1 - x_2) > r_1(\alpha; p, N, \nu), (x_1 - x_2)^T L^{-1}(x_3 - x_4) > r_1(\alpha; p, N, \nu)\}.
\]

In the author’s separate paper [12], he has discussed the range in some details and examined the order of magnitude of \(\beta(\alpha; p, N)\) and \(\beta^*(\alpha; p, N, \nu)\) numerically by using the formulas obtained in this paper. According to this examination, \(\beta\) and \(\beta^*\) are, in the multivariate range case, significantly large and so the first approximations \(r_1(\alpha; p, N)\) and \(r_1(\alpha; p, N, \nu)\) are not accurate but the modified second approximations give values with the sufficient accuracy for practical application.

8. Acknowledgement. I wish to express my gratitude to Mr. M. Ozawa and Miss K. Yoshida for their help in numerical work.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES


[9] K. Pearson, “On the criterion that a given system of deviations from the probable in the case of a correlated system of variable is such that it can be reasonably supposed to have arisen from random sampling”, Phil. Mag., Vol. 50 (1900), pp. 157-175.


## Table I

Upper percentage points of the extreme deviate from the sample mean

\[ \hat{\chi}^2_{\text{MAX-D}} = \max_i \left\{ (x_i - \bar{x})^2 / \Lambda^{-1}(x_i - \bar{x}) \right\} \]

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## Table II

Upper percentage points of the studentized extreme deviate from the sample mean

\[ \hat{r}^2_{\text{MAX-D}} = \max_i \left\{ (x_i - \bar{x})^2 / L^{-1}(x_i - \bar{x}) \right\} \]

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\[ \cdots \sum_{j=0}^{\infty} \frac{\Gamma(p/2+1+j)}{j!} \left( \frac{\delta}{\gamma} \right)^{3j} \cdots \sum_{j=0}^{\infty} \frac{\Gamma(p/2+1+j)}{j!} \]

\[ \times \left[ (\eta+\frac{p}{2}-j) \cdots \right] \cdots \]

Vol. XI, No. 2.

108 16  

\[ [X]_n \]

112 11  

while $X_{ij}$ are defined by\cdots  

\[ X_{ij} \text{ defined by\cdots} \]