PROOF OF THE LAW OF ITERATED LOGARITHM THROUGH DIFFUSION EQUATION

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1. The iterated logarithm type theorems for the absolute value of k-dimensional Brownian motion ($k = 1, 2, \cdots$) have been given by many authors in various ways. The aim of this paper is to prove these theorems by a unified method.

Let $\{X(t)\}$ be a diffusion process in an interval, and let $\overline{C}_1$, $\overline{C}_2$, ($\underline{C}_1$, $\underline{C}_2$) be the classes of increasing (decreasing) functions in $(0, \infty)$ such that

$$\overline{C}_1 = \{\varphi : P\{\overline{\lim_{t\to\infty}}(X(t)-\varphi(t))\geqq 0\}=1\}$$
$$\overline{C}_2 = \{\varphi : P\{\overline{\lim_{t\to\infty}}(X(t)-\varphi(t))\geqq 0\}=0\}$$
$$\underline{C}_1 = \{\psi : P\{\underline{\lim_{t\to\infty}}(X(t)-\psi(t))\leqq 0\}=1\}$$
$$\underline{C}_2 = \{\psi : P\{\underline{\lim_{t\to\infty}}(X(t)-\psi(t))\leqq 0\}=0\}$$

In section 2, we shall completely determine the classes $\overline{C}_1$, $\overline{C}_2$, $\underline{C}_1$ and $\underline{C}_2$, if $X(t)$ is recurrent and the expectation of it's recurrence time is finite. For example, Uhlenbeck's process satisfies this condition.

In section 3, we treat the absolute value of k-dimensional Brownian motion. This process is a one-dimensional diffusion process which does not satisfy the above conditions. But by change of scale and time, we can apply the theorem of section 2. And we shall prove the iterated logarithm type theorems.

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2. For a diffusion process in an interval $[a, b]$, we take notations and definitions as in Ray's paper (Ray [1]). Namely,

Def. 1. Transition probabilities and Chapman-Kolmogoroff's equations. (Definition 1 in Ray [1])
(In this definition and the following, R should be considered as the interval $[a, b]$.)

Def. 2. Probability space and the continuity of sample function. (Definition 2 in Ray [1])

In this definition, $2°$ is replaced by a stronger assumption $2°'$.

$2°'$ For every $t_0 > 0$ and every $\delta > 0$, there is $\Delta > 0$ such that $|x(t, \omega)-X(t', \omega)| \leqq \delta$ whenever $0\leqq t, t'\leqq t_0$, $|t-t'| < \Delta$.

Def. 3. The first passage relation. (Definition 3 in Ray [1])
Moreover for every open interval $I=(x, y)$, $a<x<y<b$ and every $z \in I$,

Def. 4. Communicative relation.

$$P\{x^z(\tau_1^z)=x\}>0 \quad \text{and} \quad P\{x^z(\tau_1^z)=y\}>0.$$

Then, by Dinkin (Dinkin [5]) there exist canonical scale $s(x)$ and canonical measure $m(dx)$ in (a, b)[1]. The canonical scale satisfies the following relation.

$$P\{x^z(\tau_1^z) = x\} = \frac{s(y)-s(z)}{s(y)-s(x)} P\{x^z(\tau_1^z) = y\} = \frac{s(z)-s(x)}{s(y)-s(x)}.$$

For the purpose of this section we make the next additional assumption.
A. For every $x, y\ (a<x<y<b)$

$$P\{\tau^x_{[a,y)}<\infty\}=1 \quad \text{and} \quad P\{\tau^y_{(x,b]}<\infty\}=1$$

Moreover

$$E\{\tau^x_{[a,y)}\}<\infty \quad \text{and} \quad E\{\tau^y_{(x,b]}\}<\infty^{2)} .$$

Without loss of generality, we may assume $s(x)\equiv x$ and $1,\ 0 \in (a, b)$ in this section.

THEOREM. *Let $X(t)$ be a diffusion process in an interval $[a, b]$ satisfying the assumption A, and $\varphi(t)$ ($\psi(t)$) be an increasing (decreasing) function such that $0<\varphi(o)\leqq\varphi(t)\leqq b$, $(0>\psi(o)\geqq\psi(t)\geqq a)$. Then*

$$P\{\overline{\lim_{t\to\infty}}(X^x(t)-\varphi)t))\geqq 0\}=1 \quad \textit{or} \quad 0\ , \quad \textit{for all } x \in (a, b)\ ,$$

if and only if

$$\int_0^\infty \frac{dt}{\varphi(t)} = \infty \quad \text{or} \quad <\infty\ .$$

$$\left(\begin{array}{l} P\{\underline{\lim}_{t\to\infty}(X^x(t)-\psi(t))\leqq 0\}=1 \quad \text{or} \quad 0 \\ \text{if and only if } \displaystyle\int_0^\infty \frac{dt}{|\psi(t)|}=\infty \quad \text{or} \quad <\infty \end{array}\right)$$

We consider only the case $b=s(b)=\infty$, otherwise

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[1] By these m and s infinitesimal generator A of the semigroup of the process is represented as $Af(x) = d/dm\, d^+/ds\, f(x)$ for $x \in (a, b)$Where $f(x)$ is in the domain of A.

[2] For there is no mass defect at any point in R nor trap, this assumption is eqnivalent to the condition $\int_a^b m(dx) < \infty$.

$$P\{\overline{\lim_{t\to\infty}} X^x(t)=b\}=1. \qquad \text{(c. f. foot note (2))}$$

Moreover, without loss of generality, we prove the theorem for process $X(t)=X^1(t)$, $(X(0)\equiv 1)$.

Let
$$T_0(\omega)\equiv 0$$
$$S_1(\omega)=\inf\{t: X(t)=0\}$$
$$T_1(\omega)=\inf\{t: X(t)=1 \quad t>S_1\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_n(\omega)=\inf\{t: X(t)=0 \quad t>T_{n-1}\}$$
$$T_n(\omega)=\inf_{a.s.}\{t: X(t)=1 \quad t>S_n\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

(S_n and T_n are finite with probability one by assumption).

Then we first prove the following.

LEMMA 1. *Let* $Y(t, \omega)=\sup_{s\leq t} X(s)$, *and* $\{a_n\}$ *be an increasing sequence such that* $a_n\uparrow\infty$ $(n\uparrow\infty)$. *Then the following two conditions are equivalent for almost all* ω.

(i) $\qquad Y(T_n(\omega), \omega)\geqq a_n$ *for infinitely many* n.

(ii) $\qquad \sup_{T_{n-1}\leqq t<T_n} X(t)\geqq a_n$ *for infinitely many* n.

PROOF. It is easily seen that (ii) implies (i).
If (i) is true, we only consider those ω whose T's are finite and $X(t)$ is continuous. If n_0 is arbitrarily given, we can find $n_1>n_0$ such that $a_{n1}>Y(T_{n0})$ (for $Y(T_{n0})$ is bounded and $a_n\uparrow\infty$).

Then by (i), there exists n_2 such that $n_2>n_1$ and $Y(T_{n2})\geqq a_{n2}$. Therefore $\sup_{T_{n_0}\leqq t<T_{n_2}} X(t)\geqq a_{n2}$, (for $a_{n2}\geqq a_{n1}>Y(t)$ if $t\leqq T_{n0}$) and there exists n_3; $n_2\geqq n_3>n_0$ such that

$$\sup_{T_{n_3-1}\leqq t<T_{n_3}} X(t)\geqq a_{n_2}\geqq a_{n_3}\,.$$

Thus the lemma is proved.

In the similar way as in the Lemma 1, we can get

LEMMA 2. *Let* $\varphi(t)$ *be an increasing function such that* $\varphi(t)\uparrow\infty$ $(t\uparrow\infty)$. *Then the following two conditions are equivalent for almost all* ω.

(i) $\qquad \overline{\lim_{t\to\infty}}\,(Y(t, \omega)-\varphi(t))\geqq 0\,.$

(ii) $\qquad \overline{\lim_{t\to\infty}}\,(X(t, \omega)-\varphi(t))\geqq 0\,.$

By the property of canonical scale,

$$P\{\sup_{T_{n-1}\leqq t<T_n} X(t)\geqq a_n\}=P\{\sup_{T_{n-1}\leqq t<S^n} X(t)\geqq a_n\}=1/a_n$$

for $a_n > 1$. But the Borel fields generated by $\{X(t): T_{n-1}\leqq t<T_n\}$, $n=1, 2, \cdots$ are mutually independent. Then, by the Borel-Cantelli's theorem we get the following.

LEMMA 3. *For* $a_n>1$

$$P\{\sup_{T_{n-1}\leqq t<T_n} X(t)\geqq a_n \text{ for infinitely many } n\}=1 \text{ or } 0$$

if and only if $\sum 1/a_n=\infty$ *or* $<\infty$.

Now we shall prove the theorem. (T_n-T_{n-1})'s $(n=1, 2, \cdots)$ are mutually independent, and distributed according to the same distribution. Moreover

$$E\{\tau^0([a, 1))\}=\tau_1 \quad\text{and}\quad E\{\tau^1(0, \infty))\}=\tau_2 \quad (b=\infty)$$

are finite by the assumption and so

$$E\{T_n-T_{n-1}\}=E\{T_n-S_n\}+E\{S_n-T_{n-1}\}=\tau_1+\tau_2=\tau$$

is finite. Therefore, by the strong law of large numbers, we get

$$(1/n)T_n\to\tau \text{ (a.s.) } (n\to\infty)\ .$$

Hence (especially) for almost all ω, there exists $n_0(\omega)$ such that

$$(1)\qquad (n/2)\tau < T_n(\omega) < 2n\tau \quad n\geqq n_0(\omega).$$

Now, let $\varphi(t)$ be an increasing function such that $\varphi(t)>0$.

$$\varphi(t)\uparrow\infty \quad (t\to\infty)^{3)} \quad\text{and}\quad \int\frac{dt}{\varphi(t)}<\infty\ .$$

Then

$$\begin{aligned}
&\{\omega: \overline{\lim_{t\to\infty}}(X(t)-\varphi(t))\geqq 0\}\\
\underset{\text{a.s.}}{=}\ &\{\omega: \overline{\lim_{t\to\infty}}(Y(t)-\varphi(t))\geqq 0\} && \text{(by Lemma 2)}\\
\subset &\left\{\omega: Y\left(\frac{n}{2}\tau\right)\geqq\frac{1}{2}\varphi\left(\frac{n-1}{2}\tau\right) \text{ for infinitely many } n\right\}\\
\subset &\left\{\omega: Y\left(\frac{n}{2}\tau\right)\geqq\frac{1}{2}\varphi\left(\frac{n}{4}\tau\right) \text{ for infinitely many } n\right\}\\
\underset{\text{a.s.}}{\subset} &\left\{\omega: Y(T_n)\geqq\frac{1}{2}\varphi\left(\frac{n}{4}\tau\right) \text{ for infinitely many } n\right\} && \text{(by (1))}
\end{aligned}$$

3) The condition $\varphi(t)\uparrow\infty(t\to\infty)$ is not essential. For if $\varphi(t)\leqq k$, then

$$P\{\overline{\lim_{t\to\infty}} X(t)-\varphi(t)\geqq 0\}=1.$$

$$\underset{\text{a.s.}}{=}\left\{\omega : \sup_{T_{n-1}\leqq t<T_n} X(t) \geqq \frac{1}{2}\,\varphi\left(\frac{n}{4}\,\tau\right) \text{ for infinitely many } n\right\}$$

(by Lemma 1).

As $\sum_{n=1}^{\infty} \dfrac{1}{\frac{1}{2}\,\varphi\left(\frac{n}{4}\,\tau\right)} \leqq \dfrac{8}{\tau}\displaystyle\int_0^{\infty} \dfrac{dt}{\varphi(t)} < \infty$, the probability of the last set is zero by Lemma 3. Therefore

$$P\{\overline{\lim_{t\to\infty}}(X(t)-\varphi(t)) \geqq 0\} = 0 \,.$$

Now, if $\varphi(t)$ be an increasing function, $\varphi(o)>C>0$ and $\displaystyle\int_0^{\infty}\frac{dt}{\varphi(t)}=\infty$,

then

$$\begin{aligned}
&\left\{\omega : \overline{\lim_{t\to\infty}}\,(X(t)-\varphi(t)) \geqq 0\right\} \\
&\underset{\text{a.s.}}{=}\left\{\omega : \overline{\lim_{t\to\infty}}\,(Y(t)-\varphi(t)) \geqq 0\right\} && \text{(by Lemma 2)} \\
&\supset \{\omega : Y(2n\,\tau) \geqq \varphi(2n\,\tau) \text{ for infinitely many } n\} \\
&\supset \{\omega : Y(T_n) \geqq \varphi(2n\,\tau) \text{ for infinitely many } n\} && \text{(by (1))} \\
&\underset{\text{a.s.}}{=}\left\{\omega : \sup_{T_{n-1}\leqq t<T_n} X(t) \geqq \varphi(2n\,\tau) \text{ for infinitely many } n\right\} && \text{(by Lemma 1)}
\end{aligned}$$

For $\sum_{n=1}^{\infty} \dfrac{1}{\varphi(2n\,\tau)} \geqq \dfrac{1}{2\tau}\displaystyle\int_{2\tau}^{\infty} \dfrac{dt}{\varphi(t)} = \infty$, the probability of the last set is 1 by Lemma 3. Therefore

$$P\left\{\omega : \overline{\lim_{t\to\infty}}\, X(t)-\varphi(t) \geqq 0\right\} = 1.$$

Hence the theorem was proved.

Example. Uhlenbeck process.

We consider the process whose infinitesimal generator A is given by $A=-\alpha x(d/dx)+(d^2/dx^2)$ for $C^2(-\infty,\infty)$. Then its canonical scale and canonical measure are given as follows:

$$s(x)=\int_0^x \exp\left(\frac{\alpha u^2}{2}\right)du$$

$$m(dx)=\exp\left(-\frac{\alpha x^2}{2}\right)dx.$$

As $s(x)\sim(1/\alpha^2 x)\exp(\alpha^2 x^2/2)(|x|\to\infty)$ and $m(-\infty,\infty)<\infty$, we get

$$P\left\{\overline{\lim_{t\to\infty}}(X(t)-\varphi(t)) \geqq 0\right\} = 1 \quad \text{or} \quad 0$$

if and only if $\int_0^\infty \varphi(t) \exp\left(-\frac{\alpha^2\varphi^2(t)}{2}\right)dt = \infty$ or $< \infty$ for increasing φ which is positive.

3. Now we shall consider the application of the theorem to the k-dimensional Brownian motion.

Let $Z_k(t)=(X_1(t), \cdots, X_k(t))$ be k-dimensional Brownian motion (i. e. whose coordinates $X_1(t), \cdots, X_k(t)$ are mutually independent one-dimensional Brownian motion), and

$$R_k(t)=|Z_k(t)|=\sqrt{X_1(t)^2+\cdots+X_k(t)^2}.$$

Then $R_k(t)$ is a one-dimensional stationary diffusion process, but does not satisfy our assumption made in section 2.

So we set

$$Y_k(s)=e^{-s}R_n^2(e^s-1)=\frac{1}{1+t}R_k^2(t) \quad s=\log(1+t)$$
$$\text{for} \quad s \geqq 0.$$

Then $Y_k(s)$ becomes also a one-dimensional stationary diffusion process whose infinitesimal generator A is given by

$$A=(k-x)\frac{d}{dx}+2x\frac{d^2}{dx^2} \quad \text{for } C[0, \infty].$$

And if $S_k(\cdot)$ and $m_k(\cdot)$ are its canonical scale and canonical measure respectively, it is easily seen that

$$m_k(0, \infty)<\infty \quad \text{for all } k$$
$$s_k(0)=-\infty \text{ and } s_k(\infty)=\infty \quad \text{for} \quad k\geqq 2,$$

$s_k(0)>-\infty$ and $s_k(\infty)=\infty$, and 0 is a reflecting barrier of the process for $k=1$.[4)] Therefore for all k process $\{Y_k(s)\}$ satisfies our condition. So

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4)
$$s_k(x)=\int_1^x u^{-(k/2)}\exp(u/2)du$$
$$m_k(dx)=(1/2)x^{(k/2)-1}\exp(-(x/2))dx.$$
$$s_k(x)\sim x^{-(k/2)}\exp(x/2) \quad (x\to\infty)$$
$$s_k(x)\sim x^{-(k-2/2)} \quad (x\to 0) \quad \text{for} \quad k\geqq 3$$
$$\sim|\log x| \quad (x\to 0) \quad \text{for} \quad k=2.$$

$$\left\{\omega : \overline{\lim_{s\to\infty}}\,(Y_k(s)-\varphi(s)) \geqq 0\right\}=\left\{\omega : \overline{\lim}\,(R_k(t)-\sqrt{\varphi(\log(1+t))(1+t)})\geqq 0\right\}$$

$$\left(\begin{array}{l}\text{or}\\ \left\{\omega : \underline{\lim}_{s\to\infty}\,(Y_k(s)-\psi(s))\leqq 0\right\}=\left\{\omega : \underline{\lim}_{t\to\infty}\,(R_k(t)-\sqrt{\psi(\log(1+t))(1+t)}) \geqq 0\right\}\end{array}\right)$$

and the probability of the left-hand side is one or zero if and only if

$$\int_0^\infty \frac{ds}{s_k(\varphi(s))}=\infty \quad \text{or} < \quad \infty, \left(\int_0^\infty \frac{ds}{|s_k(\psi(s))|}=\infty \quad \text{or} \quad <\infty\right).$$

Therefore we get the following theorems.

THEOREM. *For an increasing $\varphi(t)$*

$$P\left\{\overline{\lim_{t\to\infty}}\,R_k(t)-\sqrt{1+t}\varphi(t)\geqq 0\right\}=1 \quad \textit{or} \quad 0$$

if and only if

$$\int_0^\infty \varphi(t)^k e^{-\varphi^2(t)/2}\frac{dt}{1+t}=\infty \quad \text{or} \quad <\infty \qquad \textit{for all } k.$$

THEOREM. *For a decreasing $\psi(t)$*

$$P\left\{\underline{\lim}_{t\to\infty}\,R_k(t)-\sqrt{1+t}\,\psi(t)\leqq 0\right\}=1 \quad \text{or} \quad 0$$

if and only if

$$\int_0^\infty \psi(t)^{k-2/2}\frac{dt}{1+t}=\infty \quad \text{or} \quad <\infty \qquad \textit{for } k\geqq 3,$$ [5]

and

$$\int_0^\infty \frac{dt}{|\log\psi(t)|(1+t)}=\infty \quad \text{or} \quad <\infty \qquad \textit{for } k=2.$$ [6]

4. The method of section 3 can be applied for a little general process with a certain regular property.

Let $X(t)$ be a diffusion process in $[a,b]$, where a is the entrance boundary and b is the natural boundary.[7] And let $s(x)\equiv x$ and $m(dx)$ be it's canonical scale and measure respectively.

Now we assume the following two conditions.

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[5] This is given by Dvoretzky and Erdös [2].

[6] This is given by Spitzer [3].

[7] This condition is not essential. By a slight change of wording we can get similar results for 2-natural boundaries etc.

1. The measure $m(dx)$ has a strictly positive density function $p(x)$ in (a, b).

2. Setling $m(x)=\int_a^x p(x)dx$ and $n(x)=\int_a^x m(x)dx$,[8)]

$$c=\frac{m(x)p(x)}{n(x)^2} \text{ is a strictly positive constant.}$$

Then, for this process we define a new process $Y(t)$ as

$$Y(s) = \frac{1}{t+1}n(X(t)) \quad \text{where} \quad s=\log(1+t).$$

Then $T(s)$ is a (stationary) diffusion process in $[0, \infty]$ whose infinitesimal generator is given by the formula:

$$A=(1-y)\frac{d}{dy}+cy\frac{d^2}{dy^2}.$$

It is easily seen that this process satisfies our condition in section 2. Thus our theorem is applicable to $Y(t)$ and therefore $X(t)$. Processes in section 3 satisfy the above conditions 1 and 2.

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8) These integrals are finite for b is the entrance boundary.