A REMARK ON PRICE ANALYSIS IN LEONTIEF'S OPEN INPUT-OUTPUT MODEL

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(Received Jan. 9, 1958)

1. Introduction.

In Leontief's open input-output model, two systems are used : one in terms of quantitative value, the other in terms of money value. Theoretically input coefficients in terms of quantity are assumed to be constant. Leontief says, though, as far as I can see, without detailed discussions on it, that money value system can be used as quantitative system, in so far as the quantities are measured by "the amount purchasable by one million dallars in base period" rather than by other usual units. In order that Leontief's whole system is valid, that presumption of his must hold. We discuss here the relation between these two systems, in order to see whether the two systems lead to the equal result.

In output estimation, outputs are determined by final demands. And when we use quantitative system, final demands are given in terms of quantity, and outputs corresponding to them are also determined in terms of quantity. When we use, on the contrary, money value system with coefficients obtained from basic value, outputs corresponding to final demands measured by prices of base period are obtained, as is well known, as measured by basic prices. Before we go into our main topic, we show in section 2 that when we measure the outputs obtained by quantitative system by basic prices, they are to be equal to the outputs obtained by the money value system.

In price analysis, prices are determined by values added. When, on the one hand, quantitative system is used, prices are determined by giving, for instance, values added per unit quantity in all industries. When we use, on the other hand, money value system, ratios of final prices to basic prices are obtained by giving values added per money unit. In section 3 we examine the relation between the prices obtained by quantitative system and the price ratios obtained by money value system.

In section 4 we further discuss the effect of price change in some industry on prices of other industries.

2. Output estimation.

Let X_i represent the total output of industry i in a year (measured by appropriate physical unit), x_{ij} the amount of product of industry i used in industry j, Y_i the amount of product of industry i consumed finally by houshold etc. Then the over-all quantitative input-output balance of the whole economy comprising n separate industries can be described as follows:

$$(1)\qquad \begin{cases} x_{11}+x_{12}+ \cdots +x_{1n}+Y_1=X_1 \\ x_{21}+x_{22}+ \cdots +x_{2n}+Y_2=X_2 \\ \cdots\cdots\cdots\cdots \\ x_{n1}+x_{n2}+ \cdots +x_{nn}+Y_n=X_n\ . \end{cases}$$

It is assumed hereby that there are fixed ratios between each input and the corresponding output:

$$(2)\qquad x_{ij}=a_{ij}X_j\ , \qquad i,j=1, 2, \cdots, n.$$

These a_{ij}'s are called technical input coefficients. The relation (1), on substitution of (2), becomes:

$$(3)\qquad \begin{cases} a_{11}X_1+a_{12}X_2+ \cdots +a_{1n}X_n+Y_1=X_1 \\ a_{21}X_1+a_{22}X_2+ \cdots +a_{2n}X_n+Y_2=X_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}X_1+a_{n2}X_2+ \cdots +a_{nn}X_n+Y_n=X_n\ . \end{cases}$$

Now the elements a_{ij}'s collectively form a square matrix A, which is called structural matrix. And using X to represent the column vector $\{X_1, X_2, \cdots, X_n\}$, and Y the column vector $\{Y_1, Y_2, \cdots, Y_n\}$, we can write the above system in the following form:

$$AX+Y=X$$

or, equivalently,

$$(I-A)X=Y.$$

Then we have

$$(4)\qquad X=(I-A)^{-1}Y.$$

When we express each matrix and vectors in terms of their elements, (4) can be rewritten as follows:

$$(5)\qquad \begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix},$$

where $(b_{ij})=(I-A)^{-1}$, i.e. the Leontief inverse.

On the other hand, let p_i denote the price of the product of industry i, for $i=1, 2, \cdots, n$, money value balance of the whole economy can be written as follows:

$$(6)\qquad \begin{cases} x_{11}P_1+x_{12}P_1+\cdots+x_{1n}P_1+Y_1P_1=X_1P_1 \\ x_{21}P_2+x_{22}P_2+\cdots+x_{2n}P_2+Y_2P_2=X_2P_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n1}P_n+x_{n2}P_n+\cdots+x_{nn}P_n+Y_nP_n=X_nP_n\ . \end{cases}$$

Here, when we use the input coefficients in money value $\alpha_{ij}=x_{ij}P_i/X_jP_j=a_{ij}P_i/P_j$, (6) can be rewritten in the following form:

$$(7)\qquad \begin{cases} \alpha_{11}X_1P_1+\alpha_{12}X_2P_2+\cdots+\alpha_{1n}X_nP_n+Y_1P_1=X_1P_1 \\ \alpha_{21}X_1P_1+\alpha_{22}X_2P_2+\cdots+\alpha_{2n}X_nP_n+Y_2P_2=X_2P_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{n1}X_1P_1+\alpha_{n2}X_2P_2+\cdots+\alpha_{nn}X_nP_n+Y_nP_n=X_nP_n\ . \end{cases}$$

These input coefficients in money value are no longer constants; they depend on the prices as well as on the fixed input coefficients in quantity a_{ij}'s. But when we proceed as follows, similar relation to the system of equations (4) can be established here too.

The matrix of input coefficients in money value gives the determinants:

$$(8)\qquad \Delta=\begin{vmatrix} 1-\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & 1-\alpha_{22} & \cdots & -\alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -\alpha_{n1} & -\alpha_{n2} & \cdots & 1-\alpha_{nn} \end{vmatrix} = \begin{vmatrix} (1-a_{11})\dfrac{P_1}{P_2} & -a_{12}\dfrac{P_1}{P_2} & \cdots & -a_{1n}\dfrac{P_1}{P_n} \\ -a_{21}\dfrac{P_2}{P_1} & (1-a_{22})\dfrac{P_2}{P_2} & \cdots & -a_{2n}\dfrac{P_2}{P_n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1}\dfrac{P_n}{P_1} & -a_{n2}\dfrac{P_n}{P_2} & \cdots & (1-a_{nn})\dfrac{P_n}{P_n} \end{vmatrix}$$

$$=\frac{P_1P_2\cdots P_n}{P_1P_2\cdots P_n}\left|\begin{array}{cccc} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{array}\right| .$$

Put

$$D=\left|\begin{array}{cccc} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{array}\right| .$$

Now if we let Δ_{ij} and D_{ij} represent the cofactor of (i, j) element in Δ and in D, respectively, we have:

$$(9)\quad \Delta_{ij}=(-1)^{i+j}\left|\begin{array}{cccccc} 1-\alpha_{11} & \cdots & -\alpha_{1,j-1} & -\alpha_{1,j-1} & \cdots & -\alpha_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\alpha_{i-1,1} & \cdots & -\alpha_{i-1,j-1} & -\alpha_{i-1,j+1} & \cdots & -\alpha_{i-1,n} \\ -\alpha_{i+1,1} & \cdots & -\alpha_{i+1,j-1} & -\alpha_{i+1,j+1} & \cdots & -\alpha_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\alpha_{n1} & \cdots & -\alpha_{n,j-1} & -\alpha_{n,j+1} & \cdots & -\alpha_{nn} \end{array}\right|$$

$$=(-1)^{i+j}\left|\begin{array}{cccccc} (1-a_{11})\frac{P_1}{P_1} & \cdots & -a_{1,j-1}\frac{P_1}{P_{j-1}} & -a_{1,j+1}\frac{P_1}{P_{j+1}} & \cdots & -a_{1n}\frac{P_1}{P_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{i-1,1}\frac{P_{i-1}}{P_1} & \cdots & -a_{i-1,j-1}\frac{P_{i-1}}{P_{j-1}} & -a_{i-1,j+1}\frac{P_{i-1}}{P_{j+1}} & \cdots & -a_{i-1,n}\frac{P_{i-1}}{P_n} \\ -a_{i+1,1}\frac{P_{i+1}}{P_1} & \cdots & -a_{i+1,j-1}\frac{P_{i+1}}{P_{j-1}} & -a_{i+1,j+1}\frac{P_{i+1}}{P_{j+1}} & \cdots & -a_{i+1,n}\frac{P_{i+1}}{P_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{n1}\frac{P_n}{P_1} & \cdots & -a_{n,j-1}\frac{P_n}{P_{j-1}} & -a_{n,j-1}\frac{P_n}{P_{j+1}} & \cdots & (1-a_{nn})\frac{P_n}{P_n} \end{array}\right|$$

$$=(-1)^{i+j}\frac{P_1\cdots P_{i-1}\ P_{i+1}\cdots P_n}{P_1\cdots P_{j-1}\ P_{j+1}\cdots P_n}\left|\begin{array}{cccccc} 1-a_{11} & \cdots & -a_{1,j-1} & -a_{1,j+1} & \cdots & -a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{i-1,1} & \cdots & -a_{i-1,j-1} & -a_{i-1,j+1} & \cdots & -a_{i-1,n} \\ -a_{i+1,1} & \cdots & -a_{i+1,j-1} & -a_{i+1,j+1} & \cdots & -a_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & \cdots & -a_{n,j-1} & -a_{n,j+1} & \cdots & (1-a_{nn}) \end{array}\right|$$

$$=\frac{P_j}{P_i}D_{ij} .$$

Then we have:

$$(10)\quad \begin{pmatrix} \beta_{11} & \beta_{12} \cdots \beta_{1n} \\ \beta_{21} & \beta_{22} \cdots \beta_{2n} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \beta_{n1} & \beta_{n2} \cdots \beta_{nn} \end{pmatrix} = \begin{pmatrix} 1-\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & 1-\alpha_{22} & \cdots & -\alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -\alpha_{n1} & -\alpha_{n2} & \cdots & 1-\alpha_{nn} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{\Delta_{11}}{\Delta} & \frac{\Delta_{21}}{\Delta} \cdots & \frac{\Delta_{n1}}{\Delta} \\ \frac{\Delta_{12}}{\Delta} & \frac{\Delta_{22}}{\Delta} \cdots & \frac{\Delta_{n2}}{\Delta} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\Delta_{1n}}{\Delta} & \frac{\Delta_{2n}}{\Delta} \cdots & \frac{\Delta_{nn}}{\Delta} \end{pmatrix} = \begin{pmatrix} \frac{D_{11}}{D}\frac{P_1}{P_1} & \frac{D_{21}}{D}\frac{P_1}{P_2} \cdots & \frac{D_{n1}}{D}\frac{P_1}{P_n} \\ \frac{D_{12}}{D}\frac{P_2}{P_1} & \frac{D_{22}}{D}\frac{P_2}{P_2} \cdots & \frac{D_{n2}}{D}\frac{P_2}{P_n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{D_{1n}}{D}\frac{P_n}{P_1} & \frac{D_{2n}}{D}\frac{P_n}{P_2} \cdots & \frac{D_{nn}}{D}\frac{P_n}{P_n} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}\frac{P_1}{P_1} & b_{12}\frac{P_1}{P_2} \cdots & b_{1n}\frac{P_1}{P_n} \\ b_{21}\frac{P_2}{P_1} & b_{22}\frac{P_2}{P_2} \cdots & b_{2n}\frac{P_2}{P_n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ b_{n1}\frac{P_n}{P_1} & b_{n2}\frac{P_n}{P_2} \cdots & b_{nn}\frac{P_n}{P_n} \end{pmatrix}.$$

Thus for values in the same period, we have the following relation:

$$(11)\quad \begin{pmatrix} X_1P_1 \\ X_2P_2 \\ \cdots \\ X_nP_n \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix} \begin{pmatrix} Y_1P_1 \\ Y_2P_2 \\ \cdots \\ Y_nP_n \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}\frac{P_1}{P_1} & b_{12}\frac{P_1}{P_2} & \cdots & b_{1n}\frac{P_1}{P_n} \\ b_{21}\frac{P_2}{P_1} & b_{22}\frac{P_2}{P_2} & \cdots & b_{2n}\frac{P_2}{P_n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1}\frac{P_n}{P_1} & b_{n2}\frac{P_n}{P_2} & \cdots & b_{nn}\frac{P_n}{P_n} \end{pmatrix} \begin{pmatrix} Y_1P_1 \\ Y_2P_2 \\ \cdots \\ Y_nP_n \end{pmatrix}.$$

The analysis deals with changes between two periods, i.e. the base and final periods. Now appropriate superscripts "0" and "1" are used to indicate the difference between the two sets of values. For example, α_{ij}^0 represents the (i, j) input coefficient in the base period;

α^1_{ij}, the corresponding value in the final period. The constant values over periods, for example, a_{ij}, b_{ij}, are used without superscripts. And from (5) we obtain the outputs X^1_i which fulfill the quantitative system for final demands Y^1_i estimated for final period, as follows:

$$(12)\qquad \begin{cases} X^1_1 = b_{11}Y^1_1 + b_{12}Y^1_2 + \cdots + b_{1n}Y^1_n \\ X^1_2 = b_{21}Y^1_1 + b_{22}Y^1_2 + \cdots + b_{2n}Y^1_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X^1_n = b_{n1}Y^1_1 + b_{n2}Y^1_2 + \cdots + b_{nn}Y^1_n . \end{cases}$$

In practice, the structural matrix in money value, and consequently also the inverse matrix are obtained for base period. And what we should note here is that the structural matrix can be obtained in money value only. Final demands in the final period are given in terms of basic prices as $Y^1_iP^0_i$. Putting these values as independent variables into (7), we have corresponding outputs $(X_iP_i)^1$ as follows:

$$(13)\qquad \begin{pmatrix} (X_1P_1)^1 \\ (X_2P_2)^1 \\ \cdots \\ (X_nP_n)^1 \end{pmatrix} = \begin{pmatrix} \beta^0_{11} & \beta^0_{12} & \cdots & \beta^0_{1n} \\ \beta^0_{21} & \beta^0_{22} & \cdots & \beta^0_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta^0_{n1} & \beta^0_{n2} & \cdots & \beta^0_{nn} \end{pmatrix} \begin{pmatrix} Y^1_1P^0_1 \\ Y^1_2P^0_2 \\ \cdots \\ Y^1_nP^0_n \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}\dfrac{P^0_1}{P^0_1} & b_{12}\dfrac{P^0_1}{P^0_2} & \cdots & b_{1n}\dfrac{P^0_1}{P^0_n} \\ b_{21}\dfrac{P^0_2}{P^0_1} & b_{22}\dfrac{P^0_2}{P^0_2} & \cdots & b_{2n}\dfrac{P^0_2}{P^0_n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}\dfrac{P^0_n}{P^0_1} & b_{n2}\dfrac{P^0_n}{P^0_2} & \cdots & b_{nn}\dfrac{P^0_n}{P^0_n} \end{pmatrix} \begin{pmatrix} Y^1_1P^0_1 \\ Y^1_2P^0_2 \\ \cdots \\ Y^1_nP^0_n \end{pmatrix}$$

$$= \begin{pmatrix} (b_{11}Y^1_1 + b_{12}Y^1_2 + \cdots + b_{1n}Y^1_n)P^0_1 \\ (b_{21}Y^1_1 + b_{22}Y^1_2 + \cdots + b_{2n}Y^1_n)P^0_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (b_{n1}Y^1_1 + b_{n2}Y^1_2 + \cdots + b_{nn}Y^1_n)P^0_n \end{pmatrix}.$$

Then we have:

$$(14)\qquad (X_iP_i)^1 = X^1_iP^0_i , \qquad i=1, 2, \cdots, n.$$

This means that if final demands are measured by basic price, the practically obtained outputs fulfilling money value system in base period are also measured by basic prices, and they are equal to the products of outputs obtained by quantitive system and basic prices.

3. Price analysis.

The relationship between the total outputs of industry i (in money value), and the money value of the products of all industries used by industry i can be described as follows:

$$
(15) \qquad \begin{cases} x_{11}P_1+x_{21}P_2+\cdots+x_{n1}P_n+R_1=X_1P_1 \\ x_{12}P_1+x_{22}P_2+\cdots+x_{n2}P_n+R_2=X_2P_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{1n}P_1+x_{2n}P_2+\cdots+x_{nn}P_n+R_n=X_nP_n\ , \end{cases}
$$

where R_i is the value added which originated in industry i. Dividing both members of (15) by X_i, we have:

$$
(16) \qquad \begin{cases} a_{11}P_1+a_{21}P_2+\cdots+a_{n1}P_n+\dfrac{R_1}{X_1}=P_1 \\ a_{12}P_1+a_{22}P_2+\cdots+a_{n2}P_n+\dfrac{R_2}{X_2}=P_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n}P_1+a_{2n}P_2+\cdots+a_{nn}P_n+\dfrac{R_n}{X_n}=P_n\ . \end{cases}
$$

Solving this system of equations for given R_i/X_i, $i=1, 2, \cdots, n$, we can determine the prices P_i's corresponding to the values added per unit products. But as available coefficients we only have α_{ij}^0's rather than a_{ij}'s. Therefore we transform (15) like this:

$$
(17) \qquad \begin{cases} \alpha_{11}+\alpha_{21}+\cdots+\alpha_{n1}+\dfrac{R_1}{X_1P_1}=1 \\ \alpha_{12}+\alpha_{22}+\cdots+\alpha_{n2}+\dfrac{R_2}{X_2P_2}=1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{1n}+\alpha_{2n}+\cdots+\alpha_{nn}+\dfrac{R_n}{X_nP_n}=1\ . \end{cases}
$$

Now what is the numerical value of R_i/X_iP_i in practical analysis? R_i can be splitted into two component parts, the wage W_i and profit π_i. If there exist the following relations between basic and final values of the same variables:

$$(18)\qquad \begin{cases} W_i^1 = s_i W_i^0 \\ \pi_i^1 = t_i \pi_i^0 \end{cases} \qquad i=1, 2, \cdots, n\,,$$

then we have

$$(19)\qquad \frac{R_i^1}{X_i^1 P_i^0} = \frac{W_i^1+\pi_i^1}{X_i^1 P_i^0} = \frac{s_i W_i^0 + t_i \pi_i^0}{W_i^0+\pi_i^0} \cdot \frac{W_i^0+\pi_i^0}{X_i^0 P_i^0} \cdot \frac{X_i^0 P_i^0}{X^1 P_i^0}$$

$$= \frac{s_i W_i^0 + t_i \pi_i^0}{W_i^0+\pi_i^0} \cdot \frac{R_i^0}{X_i^0 P_i^0} \cdot \frac{X_i^0 P_i^0}{X_i^1 P_i^0}\,.$$

In this relation, $W_i^0, \pi_i^0, R_i^0, X_i^0 P_i^0$ are known as basic values, and we have shown how $X_i^1 P_i^0$ are obtained. Then for given s_i, t_i representing the rise of wage and of profit in industry i respectively, $R_i^1/X_i^1 P_i^0$ can be computed.

Let q_i's be numerical values fulfilling the following relations:

$$(20)\qquad \begin{cases} \alpha_{11}^0 q_1 + \alpha_{21}^0 q_2 + \cdots + \alpha_{n1}^0 q_n + \dfrac{R_1^1}{X_1^1 P_1^0} = q_1 \\ \alpha_{12}^0 q_1 + \alpha_{22}^0 q_2 + \cdots + \alpha_{n2}^0 q_n + \dfrac{R_2^1}{X_2^1 P_2^0} = q_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{1n}^0 q_1 + \alpha_{2n}^0 q_2 + \cdots + \alpha_{nn}^0 q_n + \dfrac{R_n^1}{X_n^1 P_n^0} = q_n\,. \end{cases}$$

Now these q_i's show the price changes practically computable. This is, as is well known, the price change which Leontief computed. Since these q_i's fulfill (20), it can be represented as follows:

$$(21)\qquad \begin{pmatrix} q_1 \\ q_2 \\ \cdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1-\alpha_{11}^0 & -\alpha_{21}^0 & \cdots & -\alpha_{n1}^0 \\ -\alpha_{12}^0 & 1-\alpha_{22}^0 & \cdots & -\alpha_{n2}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1n}^0 & -\alpha_{2n}^0 & \cdots & 1-\alpha_{nn}^0 \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{R_1^1}{X_1^1 P_1^0} \\ \dfrac{R_2^1}{X_2^1 P_2^0} \\ \cdots \\ \dfrac{R_n^1}{X_n^1 P_n^0} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}\dfrac{P_1^0}{P_1^0} & b_{21}\dfrac{P_2^0}{P_1^0} & \cdots & b_{n1}\dfrac{P_n^0}{P_1^0} \\ b_{12}\dfrac{P_1^0}{P_2^0} & b_{22}\dfrac{P_2^0}{P_2^0} & \cdots & b_{n2}\dfrac{P_n^0}{P_2^0} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n}\dfrac{P_1^0}{P_n^0} & b_{2n}\dfrac{P_2^0}{P_n^0} & \cdots & b_{nn}\dfrac{P_n^0}{P_n^0} \end{pmatrix} \begin{pmatrix} \dfrac{R_1^1}{X_1^1 P_1^0} \\ \dfrac{R_2^1}{X_2^1 P_2^0} \\ \cdots \\ \dfrac{R_n^1}{X_n^1 P_n^0} \end{pmatrix}$$

$$= \begin{pmatrix} \left(b_{11}\dfrac{R_1^1}{X_1^1}+b_{21}\dfrac{R_2^1}{X_2^1}+\cdots+b_{n1}\dfrac{R_n^1}{X_n^1}\right)\dfrac{1}{P_1^0} \\ \left(b_{12}\dfrac{R_1^1}{X_1^1}+b_{22}\dfrac{R_2^1}{X_2^1}+\cdots+b_{n2}\dfrac{R_n^1}{X_n^1}\right)\dfrac{1}{P_2^0} \\ \cdots\cdots\cdots\cdots \\ \left(b_{1n}\dfrac{R_1^1}{X_1^1}+b_{2n}\dfrac{R_2^1}{X_2^1}+\cdots+b_{nn}\dfrac{R_n^1}{X_n^1}\right)\dfrac{1}{P_n^0} \end{pmatrix}.$$

On the other hand, for final period the following relations are established:

$$\begin{aligned} a_{11}P_1^1+a_{21}P_2^1+\cdots+a_{n1}P_n^1+\frac{R_1^1}{X_1^1}&=P_1 \\ a_{12}P_1^1+a_{22}P_2^1+\cdots+a_{n2}P_n^1+\frac{R_2^1}{X_2^1}&=P_2 \\ \cdots\cdots\cdots\cdots \\ a_{1n}P_1^1+a_{2n}P_2^1+\cdots+a_{nn}P_n^1+\frac{R_n^1}{X_n^1}&=P_n , \end{aligned}$$

or

$$(22)\qquad \begin{cases} \alpha_{11}^1+\alpha_{21}^1+\cdots+\alpha_{n1}^1+\dfrac{R_1^1}{X_1^1P_1^1}=1 \\ \alpha_{12}^1+\alpha_{22}^1+\cdots+\alpha_{n2}^1+\dfrac{R_2^1}{X_2^1P_2^1}=1 \\ \cdots\cdots\cdots\cdots \\ \alpha_{1n}^1+\alpha_{2n}^1+\cdots+\alpha_{nn}^1+\dfrac{R_n^1}{X_n^1P_n^1}=1 . \end{cases}$$

Therefore we have the following equivalent matrix representation:

$$\begin{pmatrix} \alpha_{11}^1 & \alpha_{21}^1 & \cdots & \alpha_{n1}^1 \\ \alpha_{12}^1 & \alpha_{22}^1 & \cdots & \alpha_{n2}^1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n}^1 & \alpha_{2n}^1 & \cdots & \alpha_{nn}^1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix} + \begin{pmatrix} \dfrac{R_1^1}{X_1^1P_1^1} \\ \dfrac{R_2^1}{X_2^1P_2^1} \\ \cdots \\ \dfrac{R_n^1}{X_n^1P_n^1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} 1-\alpha_{11}^1 & -\alpha_{21}^1 & \cdots & -\alpha_{n1}^1 \\ -\alpha_{12}^1 & 1-\alpha_{22}^1 & \cdots & -\alpha_{n2}^1 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1n}^1 & -\alpha_{2n}^1 & \cdots & 1-\alpha_{nn}^1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{R_1^1}{X_1^1 P_1^1} \\ \frac{R_2^1}{X_2^1 P_2^1} \\ \cdots \\ \frac{R_n^1}{X_n^1 P_n^1} \end{pmatrix},$$

or

$$\text{(23)} \qquad \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_{11}^1 & \beta_{21}^1 & \cdots & \beta_{n1}^1 \\ \beta_{12}^1 & \beta_{22}^1 & \cdots & \beta_{n2}^1 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{1n}^1 & \beta_{2n}^1 & \cdots & \beta_{nn}^1 \end{pmatrix} \begin{pmatrix} \frac{R_1^1}{X_1^1 P_1^1} \\ \frac{R_2^1}{X_2^1 P_2^1} \\ \cdots \\ \frac{R_n^1}{X_n^1 P_n^1} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}\frac{P_1^1}{P_1^1} & b_{21}\frac{P_2^1}{P_1^1} & \cdots & b_{n1}\frac{P_n^1}{P_1^1} \\ b_{12}\frac{P_1^1}{P_2^1} & b_{22}\frac{P_2^1}{P_2^1} & \cdots & b_{n2}\frac{P_n^1}{P_2^1} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n}\frac{P_1^1}{P_n^1} & b_{2n}\frac{P_2^1}{P_n^1} & \cdots & b_{nn}\frac{P_n^1}{P_n^1} \end{pmatrix} \begin{pmatrix} \frac{R_1^1}{X_1^1 P_1^1} \\ \frac{R_2^1}{X_2^1 P_2^1} \\ \cdots \\ \frac{R_n^1}{X_n^1 P_n^1} \end{pmatrix}$$

$$= \begin{pmatrix} \left(b_{11}\frac{R_1^1}{X_1^1} + b_{21}\frac{R_2^1}{X_2^1} + \cdots + b_{n1}\frac{R_n^1}{X_n^1}\right)\frac{1}{P_1^1} \\ \left(b_{12}\frac{R_1^1}{X_1^1} + b_{22}\frac{R_2^1}{X_2^1} + \cdots + b_{n2}\frac{R_n^1}{X_n^1}\right)\frac{1}{P_2^1} \\ \cdots \\ \left(b_{1n}\frac{R_1^1}{X_1^1} + b_{2n}\frac{R_2^1}{X_2^1} + \cdots + b_{nn}\frac{R_n^1}{X_n^1}\right)\frac{1}{P_n^1} \end{pmatrix}.$$

Form (21) and (23), we have:

$$\text{(24)} \qquad q_i = \frac{q_i}{1} = \frac{\left(b_{1i}\frac{R_1^1}{X_1^1} + b_{2i}\frac{R_2^1}{X_2^1} + \cdots + b_{ni}\frac{R_n^1}{X_n^1}\right)\frac{1}{P_i^0}}{\left(b_{2i}\frac{R_1^1}{X_1^1} + b_{2i}\frac{R_2^1}{X_2^1} + \cdots + b_{ni}\frac{R_n^1}{X_n^1}\right)\frac{1}{P_i^1}} = \frac{P_i^1}{P_i^0}.$$

Thus practically obtained q_i from money value system denotes the price change between base and final period. This shows that we have obtained, as far as price change is concerned, the same result that is obtained by quantitative system. And since the money value system which we have followed is the same with Leontief's system, we can now say that Leontief was right when he computed price change by his so called quantitative system with " the amount purchasable by one million dollars " as units.

Now

$$q_i=\beta^0_{1i}\frac{R^1_1}{X^1_1P^0_1}+\beta^0_{2i}\frac{R^1_2}{X^1_2P^0_2}+\ \cdots\ +\beta^0_{ni}\frac{R^1_n}{X^1_nP^0_n}\ ,$$

And here each $\frac{R^1_i}{X^1_iP^0_i}$ can be written as : $\frac{s_iW^0_i+t_i\pi^0_i}{W^0_i+\pi^0_i}\cdot\frac{R^0_i}{X^0_iP^0_i}\cdot\frac{X^0_i}{X^1_i}$. This means that prices rise as wage and profit rise, but prices fall as outputs increase.

4. The effect of the price change in an industry.

In this section, we discuss the effect of the price change in an industry on price changes in other industries. Suppose that one price, for example, P_k, changes from P^0_k to P^1_k, that is, $q_k=P^1_k/P^0_k$ is given, but the values added per unit product remain unchanged for other industries. Then (20) can be rewritten as follows :

$$(25)\qquad \left\{\begin{array}{l} \alpha^0_{11}q_1+\alpha^0_{21}q_2+\ \cdots\ +\alpha^0_{n1}q_n+\frac{R^0_1}{X^0_1P^0_1}=q_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha^0_{1k}q_1+\alpha^0_{2k}q_2+\ \cdots\ +\alpha^0_{nk}q_n+\frac{R^1_k}{X^1_kP^0_k}=q_k \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a^0_{1n}q_1+\alpha^0_{2n}q_2+\ \cdots\ +\alpha^0_{nn}q_n+\frac{R^0_n}{X^0_nP^0_k}=q_n \end{array}\right.$$

because

$$\frac{R^1_i}{X^1_iP^0_i}=\frac{R^0_i}{X^0_iP^0_i}\ ,\ i=1,2,\cdots,k-1,k+1,\cdots,n\ .$$

The independent variables we have here are q_k ; $\frac{R^0_i}{X^0_iP^0_i}$, $i=1,2,\cdots,k-1,k+1,\cdots,n$. Solving the system of $n-1$ equations which can be obtained by omitting the kth equation $\alpha^0_{1k}q_1+\alpha^0_{2k}q_2+\ \cdots\ +a^0_{nk}q_n\frac{R^1_k}{X^1_kP^0_k}$

$=q_k$, we can have q_i's, the price rises in industries other than industry k corresponding to q_k, the price rise in industry k. But, inverse (β_{ij}^0) for basic value is, in many cases, already obtained for computing outputs. Therefore we try with this inverse. For this purpose, we need not omit the kth equation in (25), but we should rather use it. When we represent (25) in matrix form, we have:

$$(26)\qquad \begin{pmatrix} 1-\alpha_{11}^0 & -\alpha_{21}^0 & \cdots & -\alpha_{n1}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1k}^0 & -\alpha_{2k}^0 & \cdots & -\alpha_{nk}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1n}^0 & -\alpha_{2n}^0 & \cdots & 1-\alpha_{nn}^0 \end{pmatrix} \begin{pmatrix} q_1 \\ \cdots \\ q_k \\ \cdots \\ q_n \end{pmatrix} \begin{pmatrix} \dfrac{R_1^0}{X_1^0 P_1^0} \\ \cdots \\ \dfrac{R^1}{X_k^1 P_k^0} \\ \cdots \\ \dfrac{R^0}{X_n^0 P_n^0} \end{pmatrix}.$$

And basic values, on the other hand, fulfill the following relation:

$$(27)\qquad \begin{pmatrix} 1-\alpha_{11}^0 & -\alpha_{21}^0 & \cdots & -\alpha_{n1}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1k}^0 & -\alpha_{2k}^0 & \cdots & -\alpha_{nk}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1n}^0 & -\alpha_{2n}^0 & \cdots & 1-\alpha_{nn}^0 \end{pmatrix} \begin{pmatrix} 1 \\ \cdots \\ 1 \\ \cdots \\ 1 \end{pmatrix} \begin{pmatrix} \dfrac{R_1^0}{X_1^0 P_1^0} \\ \cdots \\ \dfrac{R_k^0}{X_k^0 P_k^0} \\ \cdots \\ \dfrac{R^0}{X_n^0 P_n^0} \end{pmatrix}.$$

Subtracting (27) from (26) side by side, we have:

$$\begin{pmatrix} 1-\alpha_{11}^0 & -\alpha_{21}^0 & \cdots & -\alpha_{n1}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1k}^0 & -\alpha_{2k}^0 & \cdots & -\alpha_{nk}^0 \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{1n}^0 & -\alpha_{2n}^0 & \cdots & 1-\alpha_{nn}^0 \end{pmatrix} \begin{pmatrix} q_1-1 \\ \cdots \\ q_k-1 \\ \cdots \\ q_n-1 \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ S_k \\ \cdots \\ 0 \end{pmatrix},$$

where $S_k=\dfrac{R_k^1}{X_k^1 P_k^0}-\dfrac{R_k^0}{X_k^0 P_k^0}$. Then we have:

$$(28)\qquad \begin{pmatrix} q_1-1 \\ \cdots \\ q_k-1 \\ \cdots \\ q_n-1 \end{pmatrix} = \begin{pmatrix} \beta_{11}^0 & \beta_{21}^0 & \cdots & \beta_{n1}^0 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{1k}^0 & \beta_{2k}^0 & \cdots & \beta_{nk}^0 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{1n}^0 & \beta_{2n}^0 & \cdots & \beta_{nn}^0 \end{pmatrix} \begin{pmatrix} 0 \\ \cdots \\ S_k \\ \cdots \\ 0 \end{pmatrix}.$$

Consequently we have:

$$q_1-1=\beta_{k1}^0 S_k$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_k-1=\beta_{kk}^0 S_k$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_n-1=\beta_{kn}^0 S_k\ .$$

Therefore we also have:

$$q_1-1=\frac{\beta_{k1}^0}{\beta_{kk}^0} S_k$$
$$q_2-1=\frac{\beta_{k2}^0}{\beta_{kk}^0} S_k$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_n-1=\frac{\beta_{kn}^0}{\beta_{kk}^0} S_k\ .$$

In this way we can have the effect of price change in an industry far easier than by solving the system of $n-1$ equations. The case where prices in two or more industries change at the same time can also be treated in the same way.

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Leontief, Wassily W.: The Structure of American Economy 1919-1939, New York, 1951.

ERRATA

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P. 204, in the determinant of the second member in formula (9): read "$1-\alpha_{nn}$" instead of "$-\alpha_{nn}$".

P. 207, 1st line: read "quantitative" instead of "quantitive".

P. 211, the last line: read "$\cdots + a^0_{nk} q_n + \frac{R'_k}{X'_k P^0_k}$" instead of

"$\cdots + a^0_{nk} q_n \frac{R'_k}{X'_k P^0_k}$".

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P. 33, Theorems 1~4: read "X" instead of "x".