ON ERGODIC PROPERTY OF A TANDEM TYPE QUEUEING PROCESS

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§ 1. Summary

In this paper ergodic property of a queueing process of tandem type is investigated. In our queue, random variables are supposed to take only integral values, and a customer is permitted to leave the service point for the next phase just when the service for the next customer starts.

For this process we have a stability criterion of a very simple form which is a straightforward extension of that for a single phase queue. From the proof of our theorem it will become clear that when the initial input is random or geometric, our stability criterion holds even without the modification of the input to the second phase. Restriction to integral variables is mainly due to their ease in treatment. But the results of this paper will be sufficient for any queue which is able to be simulated or measured by some digital method.

§ 2. Introduction

In this paper random variables are supposed to take only integral values. Let t_i be the time interval between the i-th and $(i+1)$-th arrivals at the first phase. t_i's are supposed to follow independently one and the same distribution with $E(t) < +\infty$ and to be independent of any other variables. Let s_i be the service time for the i-th customer at the first phase. s_i's are supposed to follow independently one and the same distribution with $E(s) < +\infty$ and to be independent of any other variables. In this paper only the case $E(s) < E(t)$ is considered, and the queue dicipline "first come, first served" is adopted.

Input to the second phase is defined as follows. At the very outset of the service, we consider one customer at the second phase and we suppose the service for him starts just at the same time as the service for the first customer at the first phase starts. Further, the customer getting through with his service at the first phase is supposed to join in the second phase just when the service for the customer next to him starts at the first phase. This modification of input to the second phase may seem rather artificial except for the case where the machine automatically sends its finished work to the next phase when the next new service starts, but it is essential to utilize the Markov process theory for our problem. Let τ_i denote the time interval between the above defined i-th and $(i+1)$-th arrivals at the second phase, where the $(i+1)$-th customer at the second phase is the i-th customer at the first phase, and let σ_i be the service time for the i-th customer at the second phase. σ_i's are supposed to follow independently one and the same distribution with $E(\sigma) < +\infty$ and to be independent of any other variables. From the definition of our tandem queue, the input to the second phase is of regenerative type and the imbedded Markov process theory may conveniently be applied to investigate the ergodic property of the queue ([1], [2]).

§ 3. **Ergodic property of the queue:** Case $E(t) > E(\sigma)$

First we shall concern ourselves with the case $E(\sigma) < E(t)$. We shall distinguish as non-waiting the customers whose waiting times at the first phase are zero. Let ν_i denote the number of the i-th non-waiting customer at the first phase. Then under our assumption of the process $\{\nu_i\}$ will be an infinite sequence with probability one and $E\{\nu_i - \nu_{i-1}\} \equiv r < +\infty$.

Now let ω_i be the waiting time for the i-th customer at the second phase. For ω_i we have the following recurrence formula $\omega_i = \mathrm{Max}\,(\omega_{i-1} + \sigma_{i-1} - \tau_{i-1},\ 0)$, $\omega_1 \equiv 0$. Putting $\Delta_i(\omega_{\nu_{i-1}}) \equiv \omega_{\nu_i} - \omega_{\nu_{i-1}}$ we know from our definition of the process that $\Delta_i(\omega_{\nu_{i-1}})$ depends only on $\omega_{\nu_{i-1}}$, $\{\tau_k;\ \nu_{i-1} \leq k < \nu_i\}$ and $\{\sigma_k;\ \nu_{i-1} \leq k < \nu_i\}$. This means that $\{\omega_{\nu_i}\}$ defines an imbedded Markov process with $\omega_{\nu_1} \equiv \omega_1 = 0$ by definition. We shall hereafter mainly concern ourselves with the general structure of this temporally homogeneous Markov process apart from the process $\{\omega_{\nu_i}\}$, but we shall use the same notations as for $\{\omega_{\nu_i}\}$ and the meanings should be taken along the context.

Let $\Delta_i^* \equiv \sum_{\nu_{i-1} \leq k < \nu_i} (\sigma_k - \tau_k)$, then we have the following:

Lemma 1. *We have*

$$\Delta_i(\omega_{\nu_{i-1}}) \to \Delta_i^* \ \omega.p.\ 1 \text{ as } \omega_{\nu_{i-1}} \to +\infty,$$

and

$$E(\Delta_i(\omega_{\nu_{i-1}})) \to E(\Delta_i^*) \ \text{as} \ \omega_{\nu_{i-1}} \to +\infty.$$

Note: $\omega_{\nu_{i-1}}$ takes only integral values.

PROOF. $|\Delta_i|$ and $|\Delta_i^*|$ are bounded from above by $\sum_{\nu_{i-1}\leqq k<\nu_i}(\sigma_k+\tau_k)$. Now from our definition of τ_k's we have $\sum_{\nu_{i-1}\leqq k<\nu_i}\tau_k=\sum_{\nu_{i-1}\leqq h<\nu_i}t_h$ and $E(\sum_{\nu_{i-1}\leqq h<\nu_i}t_h)=rE(t)$(*). Thus, it can be seen $E(\sum_{\nu_{i-1}\leqq k<\nu_i}\sigma_k+\tau_k)=r(E(\sigma)+E(t))<+\infty$, so $\sum_{\nu_{i-1}\leqq k<\nu_i}(\sigma_k+\tau_k)$ is finite with probability one and the convergence of $\Delta_i(\omega_{\nu_{i-1}})$ to Δ_i^* with probability one is obvious. Lebesgue's limit theorem for dominated convergence assures the convergence of $E(\Delta_i(\omega_{\nu_{i-1}}))$ to $E(\Delta_i^*)$ as $\omega_{\nu_{i-1}}\to+\infty$.

As a direct consequence of Lemma 1, we have:

LEMMA 2. *There exist some finite positive numbers L and K such that for all $\omega_{\nu_{i-1}}>L$ we have $E(\Delta_i(\omega_{\nu_{i-1}}))<-K$.*

PROOF. We have $E\Delta_i^*=r(E(\sigma)-E(t))<0$. Thus for $K\equiv-\frac{1}{2}E\Delta_i^*$ the existence of L in the lemma is assured by lemma 1.

LEMMA 3. *If for the process $\{v_i\}$ there exists a sequence $\{u_i\}$ of mutually independent non-negative random variables following one and the same distribution with $E(u_i)<+\infty$ and $|v_i|\leq u_i$ for all i then we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\{v_\nu-E(v_\nu|v_1\cdots v_{\nu-1})\}=0 \quad \text{w.p.l.}$$

PROOF. We have

$$\sum_{\nu=1}^{\infty}\frac{E\{v_\nu\wedge\nu\}^2}{\nu^2}\leq\sum_{\nu=1}^{\infty}\frac{E\{u_\nu\wedge\nu\}^2}{\nu^2}<3E(u_i)$$

and

$$\sum_{\nu=1}^{\infty}P_r\{(v_\nu-\nu)\vee0\neq0\}\leq\sum_{\nu=1}^{\infty}P_r\{(u_\nu-\nu)\vee0\}\leq E(u_i)\,.$$

Thus using the truncation procedure we get our desired result by the stability theorem and the Borel-Cantelli lemma ([3]).

Now for L obtained in Lemma 2 and $W>L$, consider the following

(*) The last equation is a direct consequence of the general martingale theory.

process: $\{X_i;\ i=1, 2, \cdots\}$ where $X_1 \equiv W(>L)$ and

$$X_i = \Delta_i(X_{i-1}) + X_{i-1} \quad \text{if } X_{i-1} > L$$
$$W \qquad\qquad \text{if } X_{i-1} \leqq L\,.$$

For this process $\{X_i\}$ we have the following:

LEMMA 4. *The event that $X_i = W$ for some i is a certain recurrent event and for its duration time $T(W)$ of non-W period we have*

$$E(T(W)) \leqq \frac{W}{K}\,,$$

where K is as given in Lemma 2.

PROOF. Let X_{μ_i} denote the i-th X of the process which is $\leqq L$. Using the result of Lemma 3 we have

$$\frac{1}{n}\left\{X_n - \sum_{k=2}^{n} E(X_k - X_{k-1} | X_1, X_2, \cdots, X_{k-1})\right\} \to 0 \quad \omega.p.\ 1\,.$$

From our definition of the process we have

$$E(X_i - X_{i-1} | X_1 \cdots X_{i-1}) = E(\Delta_i(X_{i-1})) \leqq -K \quad \text{for } X_{i-1} > L\,,$$

so that it can be seen that $X_i = W$ for some i is a certain recurrent event or $\{\mu_i\}$ will be an infinite sequence with probability one, otherwise we are led to such absurd conclusion that

$$0 \leqq \overline{\lim} \frac{1}{n} X_n \leqq -K$$

with positive probability. Now let $\delta_i \equiv X_{\mu_i} - L$. Then $\delta_i \geqq -L$ and we have

$$\lim_k \frac{\delta_1 + \delta_2 + \cdots + \delta_k}{k} = E(\delta) \geqq -L \quad \omega.p.\ 1.$$

Now

$$\frac{\delta_1 + \delta_2 + \cdots + \delta_k}{k} = \frac{\sum_{j=1}^{k}(X_{\mu_j} - W)}{\mu_k - k} \cdot \frac{\mu_k - k}{k} + (W - L)\,.$$

Therefore, we have

$$-L \leqq \lim_k \frac{\delta_1 + \delta_2 + \cdots + \delta_k}{k} = \lim_k \frac{\sum_{j=1}^{k}(X_{\mu_j} - W)}{\mu_k - k} \cdot \frac{\mu_k - k}{k} + (W - L)$$

$$\leqq \overline{\lim_k} \frac{\sum_{j=1}^{k}(X_{\mu_j}-W)}{\mu_k-k} \cdot \overline{\lim_k} \frac{\mu_k-k}{k} + (W-L) .$$

Now by the stability theorem and the definition of our process, it holds with probability one that

$$\overline{\lim_k} \frac{\sum_{j=1}^{k}(X_{\mu_j}-W)}{\mu_k-k} \leqq -K .$$

Thus

$$-L \leqq -K \cdot \overline{\lim_k} \frac{\mu_k-k}{k} + (W-L)$$

holds with probability one, and from this it can be seen that $\overline{\lim_k} \frac{\mu_k-k}{k}$ must be finite almost surely. Therefore we can assert that $\lim_k \frac{\mu_k-k}{k}$ exists and is finite and equal to the $E(T(W))$ with probability one.

This $E(T(W))$ must satisfy the inequality

$$KE(T(W)) \leqq W .$$

Using this lemma we have:

LEMMA 5. *For the Markov process $\{\omega_{\nu_i}\}$ the event that $\omega_{\nu_{i+j}} \leqq L$ for some $j > 0$ starting from $\omega_{\nu_i} = \omega \leqq L$ is a certain event and the mean value of the first passage time from some $\omega(\leqq L)$ to any one of $\omega'(\leqq L)$ is bounded by a constant T which is independent of ω.*

PROOF. We have

$$E(T(\omega, L)) \equiv E\{\text{first passage time from } \omega(\leqq L) \text{ to any one of } \omega'(\leqq L)\}$$

$$= 1 + \int_{(W>L)} E(T(W)) dP_{\Delta(\omega)+\omega}(W) ,$$

and by lemma 4

$$\leqq 1 + \int_{(W>L)} \frac{W}{K} dP_{\Delta(\omega)+\omega}(W)$$

$$\leqq 1 + \frac{1}{K}\left\{\sum_{W=L+1}^{\infty}(1-\text{Prob}(\Delta(\omega)+\omega < W)) + L\right\} .$$

Obviously

$$\mathrm{Prob}(\Delta(\omega)+\omega < W) \geqq \mathrm{Prob}(\sum_{\nu_i \leqq k < \nu_{i+1}} (\sigma_k+\tau_k)+\omega < W)$$
$$\geqq \mathrm{Prob}(\sum_{\nu_i \leq k < \nu_{i+1}} (\sigma_k+\tau_k)+L < W) \qquad \text{for } \omega \leqq L .$$

Thus we have

$$\sum_{W=L+1}^{\infty}(1-\mathrm{Prob}(\Delta(\omega)+\omega < W)) \leqq \sum_{W=L+1}^{\infty}(1-\mathrm{Prob}(\sum_{\nu_i \leq k < \nu_{i+1}} (\sigma_k+\tau_k)+L < W))$$
$$=E(\sum_{\nu_i \leq k < \nu_{i+1}} (\sigma_k+\tau_k))$$
$$=r(E(\sigma)+E(t)) ,$$

and

$$E(T(\omega, L)) \leqq 1+\frac{1}{K}\{L+r(E(\sigma)+E(t))\} \equiv T .$$

From this lemma we can consider a modified Markov process whose stochastic matrix is given by $\{p_{ij}^*\}(0 \leqq i, j \leqq L)$ with

$$p_{ij}^* = p_{ij} + \sum p_{i\omega_1} p_{\omega_1\omega_2} \cdots p_{\omega_k j}$$

where p_{ij} is the (ij)-th element of the stochastic matrix for the original process $\{\omega_{\nu_i}\}$ and the summation extends over all k and $\omega_j(> L)$'s. From the condition $E(t) > E(s)$, $E(\sigma)$, it is seen that $p_{00} > 0$ and for any i there is some n for which $p_{i0}^{(n)} > 0$ where $p_{ij}^{(n)}$ denotes the transition probability from i to j in n steps. Thus our chain is aperiodic and for any $i \leqq L$ there is some n for which $p_{i0}^{*(n)} > 0$ where $p_{ij}^{*(n)}$ denotes the transition probability from i to j in n steps for the modified Markov chain.

From this fact it follows that the state 0 is ergodic in the modified Markov process. Otherwise all states are null or transient, but obviously this is impossible for a finite chain ([4]). There is only one closed set of ergodic states containing 0 for the modified chain, and we shall denote by $\{\pi(i)\}$ the stationary absolute probabilities for the modified chain.

Now we have:

LEMMA 6. *0 is an ergodic state in the original $\{\omega_{\nu_i}\}$ process.*

PROOF. Let us denote by $T_{00}^{(i)}$ the i-th recurrence time from 0 state to itself. Then $T_{00}^{(i)}$ may be represented by some $(i_1, i_2 \cdots i_{k_i})$ as follows

$$T_{00}^{(i)}=T_{0i_1}^{(i)}+T_{i_1i_2}^{(i)}+\cdots+T_{i_{h_i}0}^{(i)}$$

where $i_j \leqq L$ and $T_{i_1i_2}^{(i)}$ represents the number of steps required to reach i_2, starting from state i_1, through states $>L$. Now in the sequence

$$T_{01_1}^{(1)}T_{1_11_2}^{(1)}T_{1_21_3}^{(1)}\cdots T_{1_{h_1}0}^{(1)}T_{02_1}^{(2)}T_{2_12_2}^{(2)}\cdots T_{2_{h_2}0}^{(2)}T_{03_1}^{(3)}\cdots T_{i_\nu i_{\nu+1}}^{(i)}\cdots$$

the relative frequency of state i tends with probability one to $\pi(i)$. If we enumerate $T_{i\cdot}^{(\cdot)}$'s for a fixed i in the order of their appearances, we get a sequence of realizations of passage time from i to some state $\omega'\leqq L$ through $\omega>L$ given by mutually independent trials. Thus it can be seen that

$$\begin{aligned}\lim_{k\to\infty}\frac{T_{00}^{(1)}+T_{00}^{(2)}+\cdots+T_{00}^{(k)}}{k}&=\lim_{k\to\infty}\frac{T_{00}^{(1)}+T_{00}^{(2)}+\cdots+T_{00}^{(k)}}{h_1+h_2+\cdots+h_k+k}\cdot\frac{h_1+h_2+\cdots+h_k+k}{k}\\&=\sum_{0\leqq i\leqq L}\pi(i)E(T(i,\ L))\cdot T^*(0)\\&\leqq T^*(0)\cdot T\end{aligned}$$

where $T^*(0)$ is the mean recurrence time from 0 to 0 in the modified Markov chain and T is given by Lemma 5. Thus in our original Markov process there is only one ergodic class containing 0 and other states are transient and the absorption into the ergodic class is certain.

Summarising these results we get the following:

THEOREM. *Under the condition $E(t)>E(s)$, $E(\sigma)$ our tandem queue is of regenerative type, or its initial state recurres certainly and the mean recurrence time is finite.*

PROOF. $\omega_{\nu_i}=0$ is the regeneration point.

Our modification at the first phase is assumed only for the assurance of the regenerative property of the input to the second phase. But if the input to the first phase is random or follows a geometric distribution $P(t=n)=qp^n$ $(p,\ q>0,\ p+q=1)$ the free time for the serviceman follows, under the condition that he is free, the same distribution as that of input interval independently of any other variables. In this case introducing this free time before the first input to the first phase, it can be seen that our modification is unnecessary and that the usual stability criterion $E(t)>E(s)$, $E(\sigma)$ suffices in our modified sense for the usual tandem queue with general s and σ.

§ 4. Case $E(t) \leqq E(\sigma)$

The case $E(t) > E(s)$, $E(t) = E(\sigma)$:

In this case we have $E(\Delta^*) = 0$ and so $E(\Delta(\omega)) \geqq E(\Delta^*) = 0$.

Thus $\{\omega_{\nu_i}\}$ constitutes a semi-martingale, and $\lim_{i\to\infty} E(\omega_{\nu_i}) = +\infty$. If $\lim_{i\to\infty} E(\omega_{\nu_i}) < +\infty$, there exists a random variable ω_0 with $E(\omega_0) < +\infty$ and $\lim_{i\to\infty} \omega_{\nu_i} = \omega_0$ with probability one ([5]). This means that $\Delta = \omega_{\nu_i} - \omega_{\nu_{i-1}}$ tends to 0 with probability one as $i \to \infty$, which is a contradiction.

The case $E(t) > E(s)$, $E(t) < E(\sigma)$:

In this case we have $E(\Delta^*) > 0$ and so $E(\Delta(\omega)) \geqq E(\Delta^*) > 0$.

Thus, $\omega_{\nu_i} = 0$ is a transient state and all states are transient and $\lim \omega_{\nu_i} = +\infty$ with probability one.

§ 5. Case with more than two phases

Let $t_\cdot^{(j)}$, $s_\cdot^{(j)}$ and $w_\cdot^{(j)}$ $(1 \leqq j \leqq k)$ denote the inter arrival time, service time and waiting time at the j-th phase with $E\{(t_\cdot^{(j)})^2\}$, $E\{(s_\cdot^{(j)})^2\}$, $< +\infty$, respectively.

Let $\nu_i^{(j)}$ be the i-th ν for which $w_\nu^{(j)} = 0$ holds and $\nu_i^{(1,2,\cdots,j)}$ be the i-th ν for which $w_\nu^{(1)} = w_\nu^{(2)} = \cdots = w_\nu^{(j)} = 0$ holds. When we adopt the modification made at the first and second phases for all latter phases we have $w_1^{(j)} = 0$ $1 \leqq j \geqq k$ and the ν-th customer at the first phase is the $(\nu + j)$-th customer at the $(j+1)$-th phase. Our definition of queue implies that

$$\sum_{\nu_{i-1}^{(j)} \leq h < \nu_i^{(j)}} t_h^{(j+1)} = \sum_{\nu_{i-1}^{(j)} \leq h < \nu_i^{(j)}} t_h^{(j)}$$

holds. So we have

$$\sum_{\nu_{i-1}^{(1,\cdots,j)} \leq h < \nu_i^{(1,\cdots,j)}} t_h^{(j+1)} = \sum_{\nu_{i-1}^{(1,\cdots j)} \leq h < \nu_i^{(1,\cdots j)}} t_h^{(1)} .$$

From this fact it can be seen inductively following the proofs of the former section that $w_{\nu_i^{(1,\cdots,j)}}^{(j+1)} = 0$ is an ergodic state for each $j (0 \leqq j < k)$ under the assumption $E(t^{(1)}) > E(s^{(j)})$ $1 \leqq j \leqq k$. Further, taking into account the possibility of such event $t_h^{(1)} > s_h^{(j)}$ $1 \leqq j \leqq k$, $1 \leqq h \leqq N$ for arbitrary large N with positive probability from the restriction imposed on expectations it can be seen that the state 0 is accessible from any state with positive probability. This assures that there is only one ergodic class and other states are transient and we have:

THEOREM. *Under the condition* $E(t^{(1)}) > E(s^{(j)})$ $1 \leqq j \leqq k$ *our tandem queue shows an ergodic property, and the arithmetic (time) means of some statistics tend to fixed constants with probability one.*

Thus we have proved that the usual stability criterion holds for our tandem queue. Cases when there is at least one $s^{(j)}$ with $E(s^{(j)}) \geqq E(t^{(1)})$ are entirely analogously treated as in the former section.

Simple proof of the main result of this paper can be given by using a slightly generalized version of Foster's result but we shall defer it for another occasion ([6]).

Some results of Monte Carlo experiments of our queue will be reported in the forthcoming paper.

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P. 13–21, throughout the paper, read "queuing" instead of "queueing".

P. 13, lines 8–10, "From the proof of our theorem.....input to the second phase" and

P. 19, lines 22–31, "Our modification.....for usual tandem queue with general s and σ" are should be deleted.