

TABLES OF THE PROBABILITY DENSITY FUNCTION OF RANGE IN NORMAL SAMPLES

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1. Summary and Introduction

In this paper tables of the p. d. f. $f_n(w)$ of the range w in normal samples of size n running from 3 to 20 (for short, we express this by =3(1)20) are given for $w=0, 0.05, 0.10, 0.15, \dots$, each value being calculated up to 4 decimal places (Table 2).

The probability integral of the same distribution was tabulated by E. S. Pearson and H. O. Hartley [1, 2], and its moments have also been calculated by some authors. But any table of its p. d. f. seems yet unpublished, although K. C. S. Pillai [3] gave the distribution of range $f_n(w)$ the form of power series in w , and G. Elfving [4], E. J. Gumbel [5], D. R. Cox [6] and J. H. Cadwell [6] considered its asymptotic distributions.

Here we used Cadwell's expansion formula, which was obtained also by Z. Yamauti independently.

2. Computation

Cadwell's method is as follows:

In general, the p. d. f. of range in samples of size n from a population with a p. d. f. of $\phi(x)$ is

$$(1) \quad f_n(w) = n(n-1) \int_{-\infty}^{\infty} \phi\left(x + \frac{w}{2}\right) \phi\left(x - \frac{w}{2}\right) \left\{ \Phi\left(x + \frac{w}{2}\right) - \Phi\left(x - \frac{w}{2}\right) \right\}^{n-2} dx \\ = n(n-1) \int_{-\infty}^{\infty} \exp[\varphi(x, w)] \exp[(n-2)\psi(x, w)] dx$$

where

$$(2) \quad \varphi(x, w) = \log \phi\left(x + \frac{w}{2}\right) + \log \phi\left(x - \frac{w}{2}\right)$$

$$(3) \quad \psi(x, w) = \log \left[\Phi\left(x + \frac{w}{2}\right) - \Phi\left(x - \frac{w}{2}\right) \right]$$

If $\phi(x)$ is a symmetrical unimodal distribution, the integrand in (1) has a single maximum at $x=0$ and descends rapidly as $|x|$ increases. Ex-

panding the functions (2) and (3) in Maclaurin's series with respect to x and performing the integration of the right hand side of (1), we get

$$f_n(w) = \frac{(2\pi)^{1/2}n(n-1)\left\{\phi\left(\frac{w}{2}\right)\right\}^2\left\{\phi\left(\frac{w}{2}\right) - \phi\left(-\frac{w}{2}\right)\right\}^{n-2}}{a} \quad (4)$$

$$\times \left\{ 1 + b_4 + b_6 + \left(b_8 + \frac{35}{6}b_4^2\right) + (b_{10} + 21b_4b_6) \right. \\ \left. + \left(b_{12} + 33b_4b_8 + \frac{231}{10}b_6^2 + \frac{385}{6}b_4^3\right) + \dots \right\},$$

where

$$b_{2m} = \frac{(n-2)\psi_{2m} + \varphi_{2m}}{m! 2^m a^{2m}}, \quad (5)$$

$$a^2 = -\{(n-2)\psi_2 + \varphi_2\}, \quad (6)$$

$$\varphi_{2m} = \frac{\partial^{2m}\varphi(x, w)}{\partial x^{2m}} \Big|_{x=0}, \quad (7)$$

$$\psi_{2m} = \frac{\partial^{2m}\psi(x, w)}{\partial x^{2m}} \Big|_{x=0}.$$

Or, if rearranged in order of the power of $1/n$ (in appearance), $f_n(w)$ is rewritten as

$$f_n(w) = \frac{(2\pi)^{1/2}n(n-1)\left\{\phi\left(\frac{w}{2}\right)\right\}^2\left\{\phi\left(\frac{w}{2}\right) - \phi\left(-\frac{w}{2}\right)\right\}^{n-2}}{a} \quad (8)$$

$$\times \left\{ 1 + b_4 + \left(b_6 + \frac{35}{6}b_4^2\right) + \left(b_8 + 21b_4b_6 + \frac{385}{6}b_4^3\right) \right. \\ \left. + \left(b_{10} + 33b_4b_8 + \frac{231}{10}b_6^2 + \frac{1001}{2}b_4^2b_6 + \frac{25025}{24}b_4^4\right) \right. \\ \left. + \left(b_{12} + \frac{143}{3}b_4b_{10} + \frac{429}{5}b_6b_8 + \frac{2145}{2}b_4^2b_6 \right. \right. \\ \left. \left. + \frac{3003}{2}b_4b_6^2 + \frac{85085}{6}b_4^3b_6 + \frac{1616615}{72}b_4^5\right) + \dots \right\}.$$

When the population is normal, namely when

$$(9) \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

we have

$$(10) \quad \varphi_2 = -2, \quad \varphi_4 = \varphi_6 = \dots = 0$$

$$(11) \quad \psi_2 = -H_1$$

$$\psi_4 = -H_3 - 3H_1^2$$

$$\psi_6 = -H_5 - 15H_3H_1 - 30H_1^3$$

$$\psi_8 = -H_7 - (28H_5H_1 + 35H_3^2) - 420H_3H_1^2 - 630H_1^4$$

$$\psi_{10} = -H_9 - (45H_7H_1 + 210H_5H_3) - (1260H_5H_1^2 + 3150H_3^2H_1) \\ - 18900H_3H_1^3 - 22680H_1^5$$

$$\psi_{12} = -H_{11} - (66H_9H_1 + 495H_7H_3 + 462H_5^2) \\ - (2970H_7H_1^2 + 27720H_5H_3H_1 + 11550H_3^3) \\ - (83160H_5H_1^3 + 311850H_3^2H_1^2) \\ - 1247400H_3H_1^4 - 1247400H_1^6$$

where H_{2m-1} are Hermite's polynomials of order $2m-1$ with the argument value

$$2\phi(w/2) / \{\Phi(w/2) - \Phi(-w/2)\}.$$

To evaluate the accuracy we calculated the right hand side of (4), taking first all the terms given in the bracket and then the terms up to $(b_{10} + 21b_4b_6)$, and also the right hand side of (8), taking first all the terms given in the bracket and then the terms up to $(b_{10} + 33b_4b_6 + \dots)$.

Table 1. Convergencies of the expansion.

$w=2.00$

Degree	$n=5$		$n=10$		$n=20$	
	Eq. (4)	Eq. (8)	Eq. (4)	Eq. (8)	Eq. (4)	Eq. (8)
1	0.459746		0.225018		0.0150618	
2	0.458837		0.224675		0.0150479	
3	0.458892	0.458903	0.224686	0.224689	0.0150481	0.0150482
4	0.458897	0.458895	0.224689	0.224688	0.0150482	0.0150482
5	0.458896	0.458896	0.224688	0.224688	0.0150482	0.0150482
6	0.458896	0.458896	0.224688	0.224688	0.0150482	0.0150482
exact	0.458896		0.224688		0.0150482	

$w=4.00$						
Degree	$n=5$		$n=10$		$n=20$	
	Eq. (4)	Eq. (8)	Eq. (4)	Eq. (8)	Eq. (4)	Eq. (8)
1	0.0776464		0.232121		0.487294	
2	0.0761051		0.226048		0.476001	
3	0.0763376	0.0765161	0.226692	0.227619	0.476752	0.478279
4	0.0764556	0.0763197	0.227501	0.226880	0.478193	0.477438
5	0.0763811	0.0764563	0.227178	0.227388	0.477841	0.477893
6	0.0763870	0.0763312	0.227043	0.226927	0.477542	0.477567
exact	0.0763947		0.227160		0.477714	

$w=6.00$						
Degree	$n=5$		$n=10$		$n=20$	
	Eq. (4)	Eq. (8)	Eq. (4)	Eq. (8)	Eq. (4)	Eq. (8)
1	0.000677231		0.00291467		0.0113159	
2	0.000667715		0.00281821		0.0106448	
3	0.000666659	0.000667439	0.00280815	0.00282677	0.0105823	0.0108145
4	0.000668133	0.000668324	0.00283299	0.00283320	0.0108489	0.0107753
5	0.000668202	0.000667720	0.00283793	0.00282330	0.0109166	0.0107417
6	0.000667834	0.000667972	0.00282546	0.00283312	0.0107070	0.0108529
exact	0.000667932		0.00282855		0.0107751	

Although these series converge rapidly for smaller or larger values of w , they do not so rapidly near the values $w=4 \sim 5$ (see Table 1). Therefore, we performed the numerical integration of the original equation (1), to find the values corresponding to the values $w=4 \sim 5$, to 4 decimal places. The integration was actually carried out for $w=3.5(0.1)7.0$. The interval of integration was $\Delta x=0.05$, and the results are accurate up to about 7 decimal places.

For a special case $n=3$, the p. d. f. is expressed as

$$(12) \quad f_3(w) = \frac{6e^{-w^2/4}}{\sqrt{2\pi}} \int_0^{w/\sqrt{6}} e^{-x^2/2} dx.$$

This equation was also computed to check errors.

The values $\phi(x)$, $\Phi(x) - \Phi(-x)$ were taken from the N.B.S. Table [8].

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Table 2. Probability density function of

$w \backslash z$	3	4	5	6	7	8	9	10	11
0.00	0.0000	0.0000							
.05	.0275	.0010	0.0000						
.10	.0550	.0038	.0002	0.0000					
.15	.0822	.0085	.0008	.0001					
.20	.1090	.0150	.0018	.0002	0.0000				
0.25	0.1355	0.0233	0.0035	0.0005	0.0001				
.30	.1613	.0333	.0059	.0010	.0001	0.0000			
.35	.1865	.0449	.0093	.0018	.0003	.0001			
.40	.2109	.0580	.0137	.0030	.0006	.0001	0.0000		
.45	.2345	.0724	.0192	.0047	.0011	.0002	.0001		
0.50	0.2572	0.0881	0.0260	0.0070	0.0018	0.0004	0.0001	0.0000	
.55	.2788	.1049	.0339	.0101	.0028	.0008	.0002	.0001	
.60	.2993	.1226	.0432	.0139	.0043	.0013	.0004	.0001	0.0000
.65	.3187	.1411	.0537	.0188	.0062	.0020	.0006	.0002	.0001
.70	.3368	.1602	.0655	.0246	.0087	.0030	.0010	.0003	.0001
0.75	0.3537	0.1798	0.0785	0.0315	0.0119	0.0044	0.0015	0.0005	0.0002
.80	.3693	.1997	.0928	.0396	.0160	.0062	.0023	.0009	.0003
.85	.3835	.2197	.1081	.0488	.0209	.0086	.0034	.0013	.0005
.90	.3963	.2397	.1245	.0594	.0268	.0116	.0049	.0020	.0008
.95	.4077	.2595	.1418	.0711	.0337	.0154	.0068	.0030	.0013
1.00	0.4177	0.2790	0.1599	0.0841	0.0418	0.0200	0.0093	0.0042	0.0019
1.05	.4263	.2980	.1787	.0983	.0511	.0256	.0124	.0059	.0028
1.10	.4335	.3163	.1979	.1136	.0617	.0322	.0163	.0081	.0039
1.15	.4393	.3338	.2175	.1300	.0734	.0399	.0211	.0109	.0055
1.20	.4438	.3505	.2373	.1473	.0865	.0488	.0268	.0143	.0076
1.25	0.4468	0.3662	0.2572	0.1656	0.1008	0.0590	0.0335	0.0186	0.0102
1.30	.4486	.3807	.2769	.1845	.1162	.0704	.0414	.0238	.0135
1.35	.4491	.3941	.2962	.2040	.1328	.0831	.0505	.0300	.0175
1.40	.4483	.4062	.3151	.2240	.1504	.0971	.0609	.0373	.0225
1.45	.4464	.4170	.3334	.2441	.1690	.1124	.0726	.0459	.0285
1.50	0.4433	0.4264	0.3509	0.2644	0.1883	0.1288	0.0856	0.0556	0.0355
1.55	.4392	.4344	.3675	.2846	.2082	.1464	.1000	.0668	.0438
1.60	.4341	.4410	.3830	.3045	.2287	.1651	.1157	.0793	.0533
1.65	.4280	.4461	.3974	.3240	.2494	.1846	.1326	.0931	.0642
1.70	.4210	.4498	.4105	.3428	.2703	.2048	.1507	.1084	.0766
1.75	0.4133	0.4521	0.4223	0.3608	0.2911	0.2257	0.1699	0.1250	0.0903
1.80	.4048	.4529	.4326	.3779	.3117	.2470	.1899	.1428	.1055
1.85	.3956	.4524	.4415	.3939	.3318	.2684	.2108	.1618	.1220
1.90	.3858	.4506	.4488	.4087	.3512	.2899	.2323	.1819	.1399
1.95	.3755	.4475	.4546	.4221	.3699	.3113	.2542	.2029	.1591
2.00	0.3647	0.4432	0.4589	0.4341	0.3875	0.3322	0.2764	0.2247	0.1794
2.05	.3536	.4377	.4616	.4446	.4040	.3525	.2985	.2470	.2007
2.10	.3421	.4312	.4627	.4535	.4192	.3720	.3204	.2696	.2228
2.15	.3304	.4237	.4624	.4607	.4330	.3906	.3419	.2924	.2456
2.20	.3184	.4152	.4606	.4663	.4452	.4079	.3627	.3151	.2687
2.25	0.3063	0.4059	0.4574	0.4702	0.4558	0.4240	0.3827	0.3374	0.2921
2.30	.2942	.3959	.4528	.4724	.4647	.4386	.4015	.3592	.3154
2.35	.2820	.3852	.4470	.4730	.4718	.4515	.4191	.3801	.3384
2.40	.2698	.3739	.4400	.4720	.4771	.4628	.4353	.4000	.3608
2.45	.2577	.3621	.4318	.4694	.4807	.4722	.4499	.4187	.3825
2.50	0.2457	0.3499	0.4227	0.4653	0.4825	0.4798	0.4628	0.4360	0.4031
2.55	.2339	.3374	.4126	.4598	.4825	.4856	.4739	.4516	.4225
2.60	.2222	.3246	.4017	.4529	.4808	.4894	.4830	.4655	.4403
2.65	.2108	.3116	.3901	.4448	.4774	.4913	.4902	.4775	.4566
2.70	.1996	.2985	.3779	.4355	.4725	.4913	.4953	.4876	.4710
2.75	0.1887	0.2853	0.3651	0.4252	0.4660	0.4896	0.4985	0.4956	0.4835
2.80	.1781	.2722	.3519	.4140	.4582	.4860	.4997	.5015	.4939
2.85	.1678	.2592	.3384	.4019	.4491	.4808	.4989	.5053	.5022
2.90	.1579	.2462	.3246	.3892	.4388	.4740	.4962	.5070	.5083
2.95	.1483	.2335	.3107	.3758	.4274	.4657	.4917	.5067	.5122
3.00	0.1390	0.2210	0.2967	0.3619	0.4151	0.4561	0.4855	0.5043	0.5139

the range w in normal samples of size n .

$w \backslash n$	12	13	14	15	16	17	18	19	20
0.00									
.05									
.10									
.15									
.20									
0.25									
.30									
.35									
.40									
.45									
0.50									
.55									
.60									
.65									
.70	0.0000								
0.75	0.0001								
.80	.0001	0.0000							
.85	.0002	.0001	0.0000						
.90	.0003	.0001	.0001						
.95	.0005	.0002	.0001	0.0000					
1.00	0.0008	0.0004	0.0002	0.0001	0.0000				
1.05	.0013	.0006	.0003	.0001	.0001				
1.10	.0019	.0009	.0004	.0002	.0001	0.0000			
1.15	.0028	.0014	.0007	.0003	.0002	.0001			
1.20	.0039	.0020	.0010	.0005	.0003	.0001	0.0001	0.0000	
1.25	0.0055	0.0029	0.0015	0.0008	0.0004	0.0002	0.0001	0.0001	0.0000
1.30	.0075	.0041	.0022	.0012	.0007	.0003	.0002	.0001	.0001
1.35	.0101	.0057	.0032	.0018	.0010	.0005	.0003	.0002	.0001
1.40	.0134	.0078	.0046	.0026	.0015	.0009	.0005	.0003	.0002
1.45	.0174	.0105	.0063	.0037	.0022	.0013	.0007	.0004	.0002
1.50	0.0223	0.0139	0.0085	0.0052	0.0032	0.0019	0.0011	0.0007	0.0004
1.55	.0283	.0181	.0114	.0072	.0045	.0028	.0017	.0010	.0006
1.60	.0354	.0232	.0150	.0097	.0062	.0039	.0025	.0015	.0010
1.65	.0437	.0293	.0195	.0129	.0084	.0055	.0035	.0023	.0015
1.70	.0533	.0367	.0250	.0168	.0113	.0075	.0050	.0033	.0021
1.75	0.0643	0.0452	0.0315	0.0217	0.0149	0.0101	0.0069	0.0046	0.0031
1.80	.0767	.0552	.0393	.0277	.0194	.0135	.0093	.0064	.0044
1.85	.0907	.0665	.0483	.0348	.0249	.0177	.0125	.0088	.0061
1.90	.1060	.0794	.0588	.0432	.0315	.0228	.0164	.0118	.0084
1.95	.1229	.0938	.0708	.0530	.0394	.0291	.0213	.0156	.0113
2.00	0.1411	0.1097	0.0844	0.0643	0.0487	0.0366	0.0273	0.0203	0.0150
2.05	.1607	.1271	.0995	.0772	.0594	.0455	.0346	.0262	.0197
2.10	.1814	.1459	.1161	.0916	.0717	.0558	.0432	.0332	.0254
2.15	.2032	.1661	.1343	.1077	.0857	.0677	.0532	.0416	.0324
2.20	.2258	.1874	.1540	.1254	.1013	.0813	.0649	.0515	.0407
2.25	0.2491	0.2098	0.1749	0.1446	0.1185	0.0966	0.0782	0.0630	0.0505
2.30	.2728	.2331	.1971	.1652	.1374	.1135	.0932	.0762	.0620
2.35	.2968	.2570	.2203	.1872	.1578	.1322	.1100	.0911	.0751
2.40	.3206	.2814	.2443	.2103	.1796	.1524	.1285	.1079	.0901
2.45	.3442	.3058	.2689	.2344	.2027	.1742	.1487	.1264	.1068
2.50	0.3671	0.3302	0.2939	0.2592	0.2269	0.1973	0.1705	0.1466	0.1254
2.55	.3892	.3541	.3188	.2845	.2519	.2215	.1937	.1684	.1458
2.60	.4102	.3774	.3436	.3100	.2775	.2467	.2181	.1917	.1678
2.65	.4299	.3998	.3678	.3353	.3034	.2726	.2435	.2164	.1913
2.70	.4481	.4209	.3912	.3603	.3293	.2989	.2697	.2421	.2162
2.75	0.4645	0.4406	0.4135	0.3846	0.3549	0.3253	0.2964	0.2686	0.2423
2.80	.4790	.4586	.4345	.4079	.3799	.3515	.3232	.2956	.2691
2.85	.4914	.4748	.4539	.4299	.4040	.3771	.3499	.3229	.2965
2.90	.5018	.4890	.4715	.4504	.4269	.4019	.3760	.3500	.3242
2.95	.5098	.5010	.4870	.4691	.4483	.4255	.4014	.3766	.3517
3.00	0.5156	0.5107	0.5004	0.4858	0.4679	0.4476	0.4256	0.4024	0.3788

Table 2. Probability density function of

n	w									
w	3	4	5	6	7	8	9	10	11	
3.00	0.1390	0.2210	0.2967	0.3619	0.4151	0.4561	0.4855	0.5043	0.5139	
3.05	.1302	.2088	.2827	.3477	.4020	.4452	.4776	.5000	.5135	
3.10	.1217	.1968	.2687	.3332	.3882	.4332	.4683	.4939	.5110	
3.15	.1135	.1852	.2549	.3185	.3739	.4203	.4576	.4861	.5064	
3.20	.1058	.1740	.2413	.3037	.3591	.4065	.4456	.4766	.5000	
3.25	0.0984	0.1631	0.2279	0.2889	0.3439	0.3920	0.4326	0.4657	0.4917	
3.30	.0914	.1527	.2148	.2741	.3285	.3769	.4186	.4535	.4819	
3.35	.0848	.1426	.2020	.2595	.3131	.3614	.4038	.4402	.4705	
3.40	.0785	.1330	.1896	.2452	.2975	.3455	.3884	.4258	.4578	
3.45	.0726	.1238	.1777	.2311	.2821	.3295	.3725	.4107	.4439	
3.50	0.0670	0.1151	0.1661	0.2173	0.2668	0.3134	0.3562	0.3948	0.4290	
3.55	.0618	.1067	.1550	.2039	.2517	.2972	.3396	.3783	.4132	
3.60	.0569	.0989	.1444	.1909	.2370	.2812	.3229	.3615	.3967	
3.65	.0523	.0914	.1342	.1784	.2225	.2654	.3062	.3444	.3797	
3.70	.0480	.0843	.1245	.1664	.2085	.2498	.2896	.3272	.3624	
3.75	0.0440	0.0777	0.1153	0.1548	0.1949	0.2346	0.2731	0.3100	0.3447	
3.80	.0403	.0715	.1066	.1438	.1818	.2198	.2570	.2928	.3270	
3.85	.0368	.0657	.0983	.1332	.1692	.2054	.2411	.2759	.3093	
3.90	.0336	.0602	.0906	.1232	.1572	.1915	.2257	.2593	.2917	
3.95	.0306	.0551	.0832	.1137	.1456	.1782	.2108	.2430	.2744	
4.00	0.0278	0.0504	0.0764	0.1048	0.1347	0.1654	0.1964	0.2272	0.2574	
4.05	.0253	.0459	.0700	.0964	.1243	.1532	.1825	.2118	.2409	
4.10	.0229	.0419	.0640	.0885	.1145	.1416	.1692	.1971	.2248	
4.15	.0208	.0381	.0584	.0810	.1053	.1306	.1566	.1829	.2092	
4.20	.0188	.0346	.0532	.0741	.0966	.1202	.1445	.1693	.1943	
4.25	0.0170	0.0313	0.0484	0.0676	0.0884	0.1104	0.1331	0.1564	0.1799	
4.30	.0153	.0284	.0440	.0616	.0808	.1011	.1223	.1441	.1663	
4.35	.0138	.0256	.0399	.0560	.0737	.0925	.1122	.1325	.1533	
4.40	.0124	.0232	.0361	.0509	.0671	.0845	.1027	.1216	.1410	
4.45	.0112	.0209	.0326	.0461	.0610	.0769	.0938	.1113	.1294	
4.50	0.0100	0.0188	0.0295	0.0417	0.0553	0.0700	0.0855	0.1017	0.1184	
4.55	.0090	.0169	.0265	.0377	.0501	.0635	.0777	.0927	.1082	
4.60	.0080	.0151	.0239	.0340	.0453	.0575	.0706	.0843	.0986	
4.65	.0072	.0136	.0214	.0306	.0408	.0520	.0639	.0765	.0897	
4.70	.0064	.0121	.0192	.0275	.0368	.0469	.0578	.0694	.0814	
4.75	0.0057	0.0108	0.0172	0.0247	0.0331	0.0423	0.0522	0.0627	0.0738	
4.80	.0051	.0097	.0154	.0221	.0297	.0380	.0470	.0566	.0667	
4.85	.0045	.0086	.0137	.0198	.0266	.0341	.0423	.0510	.0602	
4.90	.0040	.0076	.0122	.0176	.0238	.0306	.0379	.0458	.0542	
4.95	.0035	.0068	.0109	.0157	.0212	.0273	.0340	.0411	.0487	
5.00	0.0031	0.0060	0.0097	0.0140	0.0189	0.0244	0.0304	0.0368	0.0437	
5.05	.0028	.0053	.0086	.0124	.0168	.0218	.0271	.0329	.0391	
5.10	.0024	.0047	.0076	.0110	.0150	.0194	.0242	.0294	.0349	
5.15	.0022	.0042	.0067	.0098	.0133	.0172	.0215	.0262	.0312	
5.20	.0019	.0037	.0059	.0086	.0118	.0153	.0191	.0233	.0277	
5.25	0.0017	0.0032	0.0052	0.0076	0.0104	0.0135	0.0169	0.0207	0.0246	
5.30	.0015	.0028	.0046	.0067	.0092	.0119	.0150	.0183	.0219	
5.35	.0013	.0025	.0040	.0059	.0081	.0105	.0132	.0162	.0194	
5.40	.0011	.0022	.0036	.0052	.0071	.0093	.0117	.0143	.0171	
5.45	.0010	.0019	.0031	.0046	.0063	.0082	.0103	.0126	.0151	
5.50	0.0009	0.0017	0.0027	0.0040	0.0055	0.0072	0.0090	0.0111	0.0133	
5.55	.0007	.0015	.0024	.0035	.0048	.0063	.0079	.0097	.0117	
5.60	.0007	.0013	.0021	.0031	.0042	.0055	.0070	.0085	.0103	
5.65	.0006	.0011	.0018	.0027	.0037	.0048	.0061	.0075	.0090	
5.70	.0005	.0010	.0016	.0023	.0032	.0042	.0053	.0065	.0079	
5.75	0.0004	0.0008	0.0014	0.0020	0.0028	0.0037	0.0046	0.0057	0.0069	
5.80	.0004	.0007	.0012	.0018	.0024	.0032	.0040	.0050	.0060	
5.85	.0003	.0006	.0010	.0015	.0021	.0028	.0035	.0043	.0052	
5.90	.0003	.0005	.0009	.0013	.0018	.0024	.0031	.0038	.0045	
5.95	.0002	.0005	.0008	.0011	.0016	.0021	.0026	.0033	.0039	
6.00	0.0002	0.0004	0.0007	0.0010	0.0014	0.0018	0.0023	0.0028	0.0034	

the range w in normal samples of size n (continued).

$w \backslash n$	12	13	14	15	16	17	18	19	20
3.00	0.5156	0.5107	0.5004	0.4858	0.4679	0.4476	0.4256	0.4024	0.3788
3.05	.5192	.5181	.5115	.5004	.4856	.4680	.4483	.4271	.4050
3.10	.5204	.5232	.5203	.5126	.5011	.4864	.4692	.4503	.4300
3.15	.5194	.5258	.5266	.5224	.5142	.5026	.4882	.4717	.4535
3.20	.5163	.5262	.5304	.5298	.5248	.5164	.5049	.4910	.4752
3.25	0.5110	0.5242	0.5318	0.5345	0.5329	0.5277	0.5192	0.5081	0.4948
3.30	.5039	.5201	.5308	.5368	.5385	.5363	.5309	.5227	.5121
3.35	.4949	.5138	.5276	.5366	.5414	.5424	.5400	.5347	.5268
3.40	.4843	.5056	.5221	.5340	.5417	.5457	.5463	.5439	.5389
3.45	.4722	.4956	.5145	.5290	.5396	.5464	.5499	.5504	.5482
3.50	0.4587	0.4840	0.5050	0.5219	0.5350	0.5446	0.5508	0.5540	0.5546
3.55	.4440	.4708	.4937	.5128	.5282	.5402	.5490	.5549	.5581
3.60	.4284	.4564	.4808	.5017	.5192	.5335	.5447	.5531	.5588
3.65	.4119	.4409	.4665	.4890	.5083	.5246	.5380	.5486	.5567
3.70	.3948	.4244	.4510	.4748	.4956	.5137	.5290	.5417	.5520
3.75	0.3772	0.4071	0.4345	0.4592	0.4814	0.5009	0.5179	0.5325	0.5448
3.80	.3592	.3893	.4171	.4426	.4657	.4865	.5050	.5212	.5352
3.85	.3411	.3711	.3991	.4251	.4489	.4707	.4904	.5080	.5235
3.90	.3229	.3526	.3806	.4068	.4312	.4537	.4743	.4930	.5099
3.95	.3048	.3340	.3618	.3880	.4127	.4356	.4570	.4766	.4945
4.00	0.2869	0.3154	0.3428	0.3689	0.3936	0.4168	0.4386	0.4589	0.4777
4.05	.2693	.2970	.3238	.3495	.3741	.3975	.4195	.4403	.4597
4.10	.2521	.2789	.3050	.3302	.3545	.3777	.3998	.4208	.4406
4.15	.2354	.2612	.2864	.3110	.3348	.3577	.3797	.4007	.4207
4.20	.2192	.2439	.2682	.2920	.3152	.3377	.3594	.3803	.4003
4.25	0.2036	0.2271	0.2504	0.2733	0.2958	0.3177	0.3390	0.3597	0.3796
4.30	.1886	.2110	.2332	.2552	.2768	.2981	.3188	.3390	.3586
4.35	.1743	.1954	.2165	.2375	.2583	.2788	.2989	.3186	.3378
4.40	.1607	.1806	.2006	.2205	.2403	.2600	.2793	.2984	.3170
4.45	.1478	.1665	.1853	.2041	.2230	.2417	.2603	.2786	.2966
4.50	0.1356	0.1531	0.1708	0.1885	0.2064	0.2241	0.2418	0.2594	0.2767
4.55	.1241	.1404	.1570	.1737	.1904	.2073	.2240	.2407	.2573
4.60	.1134	.1285	.1439	.1595	.1753	.1911	.2070	.2228	.2386
4.65	.1033	.1174	.1317	.1462	.1610	.1758	.1907	.2057	.2206
4.70	.0940	.1069	.1202	.1337	.1474	.1613	.1752	.1893	.2034
4.75	0.0853	0.0972	0.1094	0.1219	0.1347	0.1476	0.1606	0.1738	0.1870
4.80	.0772	.0882	.0994	.1109	.1227	.1347	.1468	.1591	.1714
4.85	.0698	.0798	.0901	.1007	.1116	.1226	.1339	.1453	.1568
4.90	.0629	.0721	.0815	.0912	.1012	.1114	.1218	.1323	.1430
4.95	.0566	.0650	.0736	.0825	.0916	.1010	.1105	.1202	.1301
5.00	0.0509	0.0584	0.0663	0.0744	0.0827	0.0913	0.1001	0.1090	0.1181
5.05	.0456	.0524	.0596	.0669	.0746	.0824	.0904	.0986	.1069
5.10	.0408	.0470	.0534	.0601	.0670	.0741	.0815	.0889	.0966
5.15	.0364	.0420	.0478	.0539	.0601	.0666	.0733	.0801	.0870
5.20	.0325	.0375	.0427	.0482	.0538	.0597	.0657	.0719	.0783
5.25	0.0289	0.0334	0.0381	0.0430	0.0481	0.0534	0.0588	0.0645	0.0702
5.30	.0257	.0297	.0339	.0383	.0429	.0477	.0526	.0576	.0628
5.35	.0227	.0263	.0301	.0341	.0382	.0425	.0469	.0514	.0561
5.40	.0201	.0233	.0267	.0302	.0339	.0377	.0417	.0458	.0500
5.45	.0178	.0206	.0236	.0268	.0301	.0335	.0370	.0407	.0445
5.50	0.0157	0.0182	0.0209	0.0237	0.0266	0.0296	0.0328	0.0361	0.0395
5.55	.0138	.0160	.0184	.0209	.0235	.0262	.0290	.0320	.0350
5.60	.0121	.0141	.0162	.0184	.0207	.0231	.0256	.0282	.0309
5.65	.0106	.0124	.0142	.0162	.0182	.0204	.0226	.0249	.0273
5.70	.0093	.0108	.0125	.0142	.0160	.0179	.0199	.0219	.0240
5.75	0.0081	0.0095	0.0109	0.0124	0.0140	0.0157	0.0175	0.0193	0.0211
5.80	.0071	.0083	.0095	.0109	.0123	.0137	.0153	.0169	.0185
5.85	.0062	.0072	.0083	.0095	.0107	.0120	.0134	.0148	.0162
5.90	.0054	.0063	.0073	.0083	.0094	.0105	.0117	.0129	.0142
5.95	.0047	.0055	.0063	.0072	.0081	.0091	.0102	.0112	.0124
6.00	0.0041	0.0047	0.0055	0.0063	0.0071	0.0079	0.0088	0.0098	0.0108

the range w in normal samples of size n (continued).

$w \backslash n$	12	13	14	15	16	17	18	19	20
6.00	0.0041	0.0047	0.0055	0.0063	0.0071	0.0079	0.0088	0.0098	0.0108
6.05	.0035	.0041	.0047	.0054	.0061	.0069	.0077	.0085	.0094
6.10	.0030	.0035	.0041	.0047	.0053	.0060	.0066	.0074	.0081
6.15	.0026	.0031	.0035	.0040	.0046	.0051	.0058	.0064	.0070
6.20	.0023	.0027	.0031	.0035	.0040	.0044	.0050	.0055	.0061
6.25	0.0019	0.0023	0.0026	0.0030	0.0034	0.0038	0.0043	0.0047	0.0053
6.30	.0017	.0020	.0023	.0026	.0029	.0033	.0037	.0041	.0045
6.35	.0014	.0017	.0019	.0022	.0025	.0028	.0032	.0035	.0039
6.40	.0012	.0014	.0017	.0019	.0022	.0024	.0027	.0030	.0034
6.45	.0010	.0012	.0014	.0016	.0019	.0021	.0023	.0026	.0029
6.50	0.0009	0.0011	0.0012	0.0014	0.0016	0.0018	0.0020	0.0022	0.0025
6.55	.0008	.0009	.0010	.0012	.0014	.0015	.0017	.0019	.0021
6.60	.0007	.0008	.0009	.0010	.0012	.0013	.0015	.0016	.0018
6.65	.0006	.0007	.0008	.0009	.0010	.0011	.0012	.0014	.0015
6.70	.0005	.0006	.0006	.0007	.0008	.0009	.0011	.0012	.0013
6.75	0.0004	0.0005	0.0005	0.0006	0.0007	0.0008	0.0009	0.0010	0.0011
6.80	.0003	.0004	.0005	.0005	.0006	.0007	.0008	.0008	.0009
6.85	.0003	.0003	.0004	.0004	.0005	.0006	.0006	.0007	.0008
6.90	.0002	.0003	.0003	.0004	.0004	.0005	.0005	.0006	.0007
6.95	.0002	.0002	.0003	.0003	.0004	.0004	.0005	.0005	.0006
7.00	0.0002	0.0002	0.0002	0.0003	0.0003	0.0003	0.0004	0.0004	0.0005
7.05	.0001	.0002	.0002	.0002	.0003	.0003	.0003	.0004	.0004
7.10	.0001	.0001	.0002	.0002	.0002	.0002	.0003	.0003	.0003
7.15	.0001	.0001	.0001	.0002	.0002	.0002	.0002	.0003	.0003
7.20	.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0002	.0002
7.25	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002	0.0002
7.30	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0002	.0002
7.35	.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
7.40		.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0001
7.45			.0000	.0001	.0001	.0001	.0001	.0001	.0001
7.50				0.0000	0.0001	0.0001	0.0001	0.0001	0.0001
7.55					.0000	.0000	.0001	.0001	.0001
7.60							.0001	.0001	.0001
7.65							.0000	.0000	.0000