ON THE HOEFFDING'S COMBINATRIAL CENTRAL LIMIT THEOREM

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1. Introduction

Let $(X_1, X_2, \cdots, X_n)$ be a permutation random variable of $(1, 2, \cdots, N)$, each of which has probability $1/N!$, and $\{c_{ij}^N \ (1 \leqq i \leqq N,\ 1 \leqq j \leqq N)\}$ a set of n^2 real numbers. Then we shall consider the limit distribution of

$$(1) \qquad S_N = \sum c_{iX_i}^N .$$

Many authors have studied this problem (see Wald-Wolfowitz [6], Noether [5], Madow [4], Hoeffding [2], Dwass [1])[1]. Among them, Hoeffding established this form and proved that under the condition

$$(5) \qquad \lim_{N\to\infty} \frac{\frac{1}{N}\sum d_{ij}^{N^r}}{\left(\frac{1}{N}\sum d_{ij}^{N^2}\right)^{r/2}} = 0 \qquad r = 3, 4, \cdots$$

the distribution of S_N tends to the normal distribution in law, where

$$(3) \qquad d_{ij}^N = c_{ij}^N - \frac{1}{N}\sum_i c_{ij}^N - \frac{1}{N}\sum_j c_{ij}^N + \frac{1}{N^2}\sum_{ij} c_{ij}^N .$$

In this paper we shall prove the following.

THEOREM 1. *Under a Lindeberg type condition*

$$(4) \qquad \lim_{N\to\infty} \sum_{|d_{ij}^N/d^N| > \varepsilon} \left(\frac{d_{ij}^N}{d^N}\right)^2 = 0$$

the distribution of S_N tends to the normal distribution in law, where

$$d^{N^2} = \frac{1}{N}\sum d_{ij}^{N^2}.$$

2. Preliminaries

For simplicity we shall drop the index N for all coefficients,

[1] Recently, Dwass has proved in his interesting paper the same result under the non-overlapping condition with that of Hoeffding's. But his condition is of different type from ours.

i.e., we put $c_{ij}^N=c_{ij}$, $S_N=S$, etc.

Moreover, we can assume that

i) by adding a suitable constant to S

$$(5)\qquad \sum_j c_{ij}=0\,,\quad \sum_i c_{ij}=0 \qquad \text{(there for } c_{ij}=d_{ij})$$

ii) by multiplying a suitable constant

$$(6)\qquad \frac{1}{N}\sum c_{ij}^2=1,$$

iii) by reordering i

$$(7)\qquad q_1^2\geqq q_2^2\geqq\cdots\geqq q_N^2$$

where

$$q_i^2=\frac{1}{N}\sum_j c_{ij}^2\,.$$

Then the following lemma is evident.

LEMMA 1. *We have*

$$(8)\qquad \sum q_i^2=1$$

and

$$\sum_{i>n} q_i^2\leqq\frac{N-n}{N}\,.$$

Let $\{y_{ij}\}$, $\{Y_{ij}\}$ be a set of random variables ($1\leqq i,\ j\leqq N$) such that

$$(9)\qquad y_{ij}=1 \quad \text{if} \quad X_i=j\,,$$

and

$$y_{ij}=0 \quad \text{if} \quad X_i\neq j\,,$$

$$(10)\qquad Y_{n,j}=\sum_{i\leqq n} y_{ij}\,.$$

Then we have

$$(11)\qquad S=\sum c_{ij}\,y_{ij}\,.$$

and

LEMMA 2.

$$(12)\qquad y_{ij}=0 \quad \textit{or} \quad 1$$

$$\sum_i y_{ij}=1\,,\quad \sum_j y_{ij}=1$$

(13) $p_n\{y_{k,j}=1\}=\dfrac{1-Y_{nj}}{N-n}$ *for* $k>n$

$p_n\{y_{kj}=1,\ y_{kj'}=1\}=0$ *for* $k>n,\ j\neq j'$

$p_n\{y_{kj}=1,\ y_{k'j'}=1\}=\dfrac{(1-Y_{nj})(1-Y_{nj'})}{(N-n)(N-n-1)}$ *for* $k,\ k'>n,\ k\neq k',\ j\neq j'$

etc.

where $p_n\{\cdot\}$ *means the conditional probability*

$$p_n\{\cdot\}=p\{\cdot\,|y_{ij},\ i\leqq n,\ j=1,\ 2,\ \cdots,\ N\}.$$

3. Proof of theorem 1

According to the above convention, Theorem 1 is stated as follows.

THEOREM 1′. *If*

(14) $$\frac{1}{N}\sum_{|c_{ij}|>\varepsilon} c_{ij}^2 \to 0 \quad (N\to\infty)\,,$$

the distribution of $S=\sum c_{ij}y_{ij}$ *tends to the normal distribution with* 0 *mean and unit variance.*

Let

(15) $$C_{nj}=\sum_{i\leqq n} c_{ij}$$

(16) $$T_n=\sum_j\Big(\sum_{i\leqq n} c_{ij}y_{ij}+\frac{C_{nj}Y_{nj}}{N-n}\Big) \qquad n<N\,,$$

and for an arbitrarily small η with $0<\eta<1$, take N_0 such that

(17) $$\eta<\frac{N-N_0}{N}\leqq 2\eta^{2)}.$$

Our proof of the theorem is carried out in the three steps, which consist of the proofs of the following three propositions.

PROPOSITION 1.

(18) $$E(S-T_{N_0})^2<2\eta\,.$$

PROPOSITION 2.

(19) $$|1-\bar{\sigma}_{N_0}^2|<2\eta$$

(20) $$\lim_{N\to\infty}\left|\bar{\sigma}_{N_0}^2-\frac{1}{N}\sum_{n=1}^{N_0}\sigma_n^2\right|=0 \quad \textit{for fixed } \eta,$$

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2) This is possible for sufficiently large N.

where

(21)
$$\bar{\sigma}_{N_0}^2=\frac{1}{N}\sum_j\Big(\sum_{i\leqq N_0}c_{ij}^2+\frac{C_{N_0,j}^2}{N-N_0}\Big)$$
$$\sigma_n^2=\sum_j\Big(c_{nj}+\frac{C_{nj}}{N-n}\Big)^2 .$$

PROPOSITION 3. *In fixed η we have*

(22)
$$|E(e^{T_{N_0}iu})-e^{-\frac{u^2}{2}\sum_{n=1}^{N_0}\sigma_n^2}|\to 0 \qquad (N\to\infty)$$

where u is any (fixed) real number and $i^2=-1$.

When these propositions are proved, it becomes clear that Theorem 1′ holds, for η can be chosen arbitrarily small.

PROOF OF PROPOSITION 1.

$$S-T_{N_0}=\sum_j\Big(\sum_{i>N_0}c_{ij}y_{ij}-\frac{C_{N_0j}Y_{N_0j}}{N-N_0}\Big)$$

Let

$$\overline{C}_j=\sum_{i>N_0}\frac{c_{ij}}{N-N_0},$$

then

$$\overline{C}_j+\frac{C_{N_0j}}{N-N_0}=0,$$
$$Y_{N_0j}+\sum_{i>N_0}y_{ij}=1 .$$

Therefore

$$S-T_{N_0}=\sum_j\sum_{i>N_0}(c_{ij}-\overline{C}_j)y_{ij},$$

and

$$E(S-T_{N_0})^2=\frac{1}{N-1}\sum_j\sum_{i>N_0}(c_{ij}-\overline{C}_j)^2\leqq\frac{1}{N-1}\sum_j\sum_{i>N_0}c_{ij}^3$$
$$=\frac{N}{N-1}\sum_{i>N_0}q_i^2\leq\frac{N-N_0}{N-1}\leqq 2\eta .$$

PROOF OF PROPOSITION 2.

$$1-\sigma_{N_0}^2=\frac{1}{N}\sum_{i,j=1}^{N}c_{ij}^2-\sigma_{N_0}^2=\frac{1}{N}\sum_j\Big(\sum_{i>N_0}c_{ij}^2-\frac{C_{N_0j}^2}{N-N_0}\Big)$$
$$=\frac{1}{N}\sum_j\Big(\sum_{i>N_0}c_{ij}^2-\frac{\overline{C}_j^2}{N-N_0}\Big)$$

$$=\frac{1}{N}\sum_{j}\sum_{i>N_0}(c_{ij}-\overline{C})^2 .$$

Then

$$0\leqq 1-\sigma_{N_0}^2\leqq\frac{1}{N}\sum_{i,j>N_0}c_{ij}^2=\frac{1}{N}\sum_{i>N_0}q_i^2\leqq\frac{N-N_0}{N}\leqq 2\eta .$$

Thus (19) is proved.

Before proving (20), we shall state the following remark.

Let ε be any positive number, and

(23) $$\tilde{c}_{ij}=c_{ij} \quad \text{when } |c_{ij}|>\varepsilon$$

$$\tilde{c}_{ij}=0 \quad \text{when } |c_{ij}|\leqq\varepsilon$$

(24) $$c_i'=c_{ij}-\tilde{c}_{it} .$$

We can assume by (14)

(25) $$\sum_{i,j=1}^{N}\tilde{c}_{ij}^2<\varepsilon^2 \quad \text{for sufficiently large } N.$$

Then we get:

LEMMA 3.

(26) $$\frac{|C_{nj}|}{N}\leqq 2\varepsilon$$

$$n=1,\ 2,\ \cdots,\ N$$

(27) $$\sum_{j=1}^{N}\frac{C_{nj}^2}{N^2}\leqq 1 .$$

PROOF.

$$\frac{|C_{nj}|}{N}\leqq\frac{1}{N}\sum_{i=1}^{N}|c_{ij}'|+\frac{1}{N}\sum_{i=1}^{N}|\tilde{c}_{ij}|\leqq\frac{1}{N}\sum_{i=1}^{N}\varepsilon+\sqrt{\frac{1}{N}\sum_{i=1}^{N}\tilde{c}_{ij}^2}\leqq 2\varepsilon$$

$$\sum_j\frac{C_{nj}^2}{N^2}=\sum_j\left(\frac{1}{N}\sum_{i\leqq n}c_{ij}\right)^2\leqq\sum_j\frac{1}{N}\sum_{i\leqq n}c_{ij}^2\leqq\frac{1}{N}\sum_{i,j}c_{ij}^2=1$$

Now we can prove (20) in Proposition 2.

$$\bar{\sigma}_{N_0}^2=\frac{1}{N}\sum_j\left(\sum_{i=0}^{N_0}c_{ij}^2+\frac{C_{N_0,j}^2}{N-N_0}\right)$$

$$=\frac{1}{N}\sum_{n=1}^{N_0}\sum_{j=1}^{N}\left(c_{ij}^2+\frac{C_{nj}^2}{N-n}-\frac{C_{n-1,j}^2}{N-n+1}\right)$$

$$=\frac{1}{N}\sum_{n=1}^{N_0}\sum_{j=1}^{N}\left\{\left(c_{ij}+\frac{C_{ij}}{N-n}\right)^2-\left(\frac{c_{nj}^2}{N-n}+\frac{2c_{ij}C_{nj}}{(N-n)(N-n+1)}\right.\right.$$

$$\left.\left.+\frac{C_{nj}^2}{(N-n)^2(N-n+1)}\right)\right\}.$$

Then, since $N-n\geqq N-N_0\geqq N\eta$, we have

$$\left|\bar{\sigma}_{N_0}^2-\frac{1}{N}\sum_{n=1}^{N_0}\sigma_n^2\right|\leqq\sum_{n=1}^{N_0}\sum_{j=1}^{N}\left(\frac{c_{nj}^2}{N-n}+\frac{(1+c_{nj}^2)|C_{nj}|}{(N-n)(N-n+1)}+\frac{C_{nj}^2}{(N-n)^2(N-n+1)}\right)$$

$$\leqq\frac{1}{N\eta}+\frac{1}{\eta^2}\left(1+\frac{1}{N}\right)2\varepsilon+\frac{1}{N\eta^3}.$$

This proves (20) for N may be arbitrarily large and ε arbitrarily small.

PROOF OF PROPOSITION 3. Assuming (25), and therefore Lemma 3, we divide the proof in four parts.

(I) $$\frac{1}{N}\sum_{n=1}^{N}\sigma_n^2 \quad \text{is bounded.}$$

This is easily seen by Proposion 2.

(II) $$\sum_{n=0}^{N_0-1}\int_{|\Delta T_n|>3\delta}(\Delta T_n)^2dp\to 0\ (N\to\infty)\qquad\text{for any } \delta>0.$$

where

$$\Delta T_n=T_{n+1}-T_n$$

PROOF.

$$\Delta T_n=\sum_{j=1}^{N}\left(c_{n+1,j}+\frac{C_{n+1,j}}{N-n-1}\right)y_{n+1,j}+\frac{1}{N-n-1}\left(c_{n+1,j}+\frac{C_{nj}}{N-n}\right)Y_{nj}$$

$$=u+v+w.$$

where

(28)
$$u=\sum_j\tilde{c}_{n+1,j}y_{n+1,j}$$

$$v=\sum_j\left(c'_{n+1,j}+\frac{C_{n+1,j}}{N-n-1}\right)y_{n+1,j}$$

$$w=\frac{1}{N-n-1}\sum_j\left(c_{n+1,j}+\frac{C_{nj}}{N-n}\right)Y_{nj}.$$

Then we have

(29) $$\int_{|\Delta T_n|>3\delta}(\Delta T_n)^2dp$$

$$\leqq 3\left(\int u^2dp+\int w^2dp+\int_{|u|>\delta} v^2dp+\int_{|w|>\delta} v^2dp+\int_{|v|>\delta} v^2dp\right)$$

(30) $$\int u^2dp=E(\sum_j \tilde{c}_{n+1,j}y_{n+1,j})^2=\frac{1}{N}\sum_j \tilde{c}^2_{n+1,j}$$

$$\int w^2dp=\frac{1}{(N-n-1)^2}E\left\{\sum_j\left(c_{n+1,j}+\frac{C_{nj}}{N-n}\right)Y_{nj}\right\}^2$$

(31) $$=\frac{1}{(N-n-1)^2}\frac{n(N-n)}{N(N-1)}\sum_j\left(c_{n+1,j}+\frac{C_{nj}}{N-n}\right)^2$$

$$\leqq\frac{2}{N^2\eta^2}\sum_j\left(c^2_{n+1,j}+\frac{C^2_{nj}}{(N-n)^2}\right)$$

$$\leqq\frac{2}{\eta^2N}q^2_{n+1}+\frac{1}{\eta^4N^2} \qquad \text{(by Lemma 3)}$$

For

$$|v|=\left|\sum_j\left(c'_{n+1,j}+\frac{C_{n+1,j}}{N-n}\right)y_{n+1,j}\right|\leqq\sup_j\left|c_{n+1;j}+\frac{C_{n+1,j}}{N-n}\right|$$

$$\leqq\varepsilon+\frac{2\varepsilon}{\eta}=\varepsilon\left(1+\frac{2}{\eta}\right) \qquad \text{(by Lemma 3)}$$

and

$$\text{Prob. }(|u|>\delta)\leqq\frac{1}{\delta^2N}\sum_j\tilde{c}^2_{n+1,j}$$

$$\text{Prob. }(|w|>\delta)\leqq\frac{2}{\delta^2N}\left(\frac{q^2_{n+1}}{\eta^2}+\frac{1}{\eta^4N}\right)$$

(by (30), (31) and Chebyshev's inequality),

(32) $$\int_{|u|>\delta}v^2dp+\int_{|w|>\delta}v^2dp\leqq\frac{1}{\delta^2N}\varepsilon\left(1+\frac{2}{\eta}\right)\left(\sum_j\tilde{c}^2_{n+1,j}+\frac{2q^2_{n+1}}{\eta^2}+\frac{1}{\eta^4N}\right)$$

Finally, we have

$$\int_{|v|>\delta}v^2dp\leqq\frac{1}{\delta}\int|v|^3dp\leqq\frac{1}{\delta}E\left\{\sum_j\left|c'_{n+1}+\frac{C_{n+1,j}}{N-n-1}\right|y_{n+1,j}\right\}^3$$

$$=\frac{1}{\delta N}\sum_j\left|c'_{n+1}+\frac{C_{n+1,j}}{N-n-1}\right|^3$$

(33) $$\leqq\frac{1}{\delta N}\sup_j\left|c'_{n+1,j}+\frac{C^2_{n+1,j}}{N-n-1}\right|\sum_j\left(c'_{n+1}+\frac{C^2_{n+1,j}}{N-n-1}\right)^2$$

$$\leqq\frac{2\varepsilon}{\delta}\left(1+\frac{2}{\eta}\right)\left(q^2_{n+1}+\frac{1}{\eta^2N}\right) \qquad \text{(Lemma 3)}$$

By (29), (30), (31), (32) and (33) we obtain

$$\sum_{n=0}^{N_0-1}\int_{|\Delta T_n|>3\delta}\Delta T_n^2 dp \leqq 3\Big\{\varepsilon+\frac{2}{N\eta^2}+\frac{1}{N\eta^4}+\frac{\varepsilon}{\delta^2}\Big(1+\frac{2}{\eta}\Big) \cdot\Big(\varepsilon^2+\frac{2}{N\eta^2}+\frac{1}{N\eta^4}\Big)+\frac{2\varepsilon}{\delta}\Big(1+\frac{2}{\eta}\Big)\Big(1+\frac{1}{\eta^2}\Big)\Big\},$$

which proves (II).

(III) $$E_n(\Delta T_n)=0,$$

where

$$E_n(\cdot\cdot)=E(\cdot\cdot|y_{ij},\quad j=1,2,\cdots,N,\quad i\leqq n)$$

PROOF.

$$E_n(\Delta T_n)=E_n\Big\{\sum_j\Big(c_{n+,j}+\frac{C_{n+1,j}}{N-n-1}\Big)y_{n+1,j}\Big\} +\frac{1}{N-n-1}\sum_j\Big(c_{n+1,j}+\frac{C_{nj}}{N-n}\Big)Y_{nj}$$

$$=\sum_j\Big(c_{n+1,j}+\frac{C_{n+1,j}}{N-n-1}\Big)\frac{1-Y_{nj}}{N-n}+\frac{1}{N-n-1}\sum_j\Big(c_{n+1,j} +\frac{C_{nj}}{N-n}\Big)Y_{nj}=0.$$

(IV) $$\sum_{n=0}^{N_0-1}E\Big|E_n(\Delta T_n)^2-\frac{1}{N}\sigma_{n+1}^2\Big|\to 0\qquad (N\to\infty)$$

PROOF. By using the inequality

(34) $$|(a+b)^2-a^2|\leqq pa^2+\Big(1+\frac{1}{p}\Big)b^2\quad\text{where } p>0,$$

we get for $p=\dfrac{1}{\sqrt{N}}$

$$E\Big\{\Big|E_n(\Delta T_n^2)-E_n\Big\{\sum_j\Big(c_{n+1,j}+\frac{C_{n+1,j}}{N-n-1}\Big)y_{n+1,j}\Big\}^2\Big|\Big\}$$

$$\leqq\frac{1}{\sqrt{N}}E\Big\{\sum_j\Big(c_{n+1,j}+\frac{C_{n+1,j}}{N-n-1}\Big)y_{n+1,j}\Big\}^2$$

(35) $$+\frac{2\sqrt{N}}{(N-n-1)^2}E\Big\{\sum_j\Big(c_{n+1,j}+\frac{C_{nj}}{N-n}\Big)Y_{nj}\Big\}^2$$

$$\leqq \frac{1}{\sqrt{N^3}}\sigma_{n+1}^2 + \frac{4}{\sqrt{N^3}\,\eta^2}\sum_j\left\{c_{n+1,j}^2 + \frac{C_{nj}^2}{(N-n)^2}\right\}$$

$$\leqq \frac{1}{\sqrt{N^3}}\sigma_{n+1}^2 + \frac{4}{\sqrt{N}\,\eta^2}\left(q_{n+1}^2 + \frac{1}{N\eta^2}\right)$$

and

$$(36)\qquad E\left\{\left|E_n\left\{\sum_j\left(c_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)y_{n+1,j}\right\}^2 - \frac{1}{N}\sigma_{n+1}^2\right|\right\}$$

$$= E\left\{\left|\sum_j\left(c_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2\frac{1-Y_{nj}}{N-n} - \frac{1}{N}\sigma_{n+1}^2\right|\right\}$$

$$= E\left\{\left|\frac{1}{N-n}\sum_j\left(\frac{n}{N} - Y_{nj}\right)\left(c_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2\right|\right\}$$

$$\leqq \frac{1}{N\eta}\sum_{j:|c_{n+1,j}|<\varepsilon}\left(\tilde{c}_{n+1,j}^2 + \frac{C_{n+1,j}}{N-n-1}\right)^2$$

$$+ \frac{1}{N\eta}E\left\{\left|\sum_{j:|c_{n+1,j}|\leqq\varepsilon}\left(\frac{n}{N} - Y_{nj}\right)\left(c'_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2\right|\right\}$$

But, for j, with $|c_{n+1,j}| > \varepsilon$ and

$$\frac{|C_{n+1,j}|}{N-n-1} \leqq \frac{\varepsilon}{\eta} \leqq \frac{2|c_{n+1,j}|}{\eta} = \frac{2|\tilde{c}_{n+1,j}|}{\eta},$$

we have

$$(37)\qquad \frac{1}{N\eta}\sum_{j:|c_{n+1,j}|>\varepsilon}\left(\tilde{c}_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2 \leqq \frac{1}{N\eta}\sum_j\left(1+\frac{2}{\eta}\right)^2\tilde{c}_{n+1,j}^2\,.$$

On the other hand,

$$(38)\qquad \frac{1}{N\eta}E\left\{\left|\sum_{j:|c_{n+1,j}|\leqq\varepsilon}\left(\frac{n}{N} - Y_{nj}\right)\left(c'_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2\right|\right\}$$

$$\leqq \frac{1}{N\eta}\sqrt{E\left\{\sum_{j:|c_{n+1,j}|\leqq\varepsilon}\left(\frac{n}{N} - Y_{nj}\right)\left(c'_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2\right\}^2}$$

$$\leqq \frac{1}{N\eta}\sqrt{\sum_j\left(c'_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^4\frac{n(N-n)}{N(N-1)}}$$

$$\leqq \frac{1}{N\eta}\sup\left|c'_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right|\sqrt{\sum_j\left(c'_{n+1,j} + \frac{C_{n+1,j}}{N-n-1}\right)^2}$$

$$\leqq \frac{\varepsilon}{N\eta}\left(1+\frac{2}{\eta}\right)\sigma_{n+1} \leqq \frac{\varepsilon}{2N\eta}\left(1+\frac{2}{\eta}\right)(1+\sigma_{n+1}^2)\,.$$

From (35), (36), (37) and (38) it follows that

$$\sum_{n=0}^{N_0-1} E\left\{\left|E_n(\Delta T_n)^2-\frac{1}{N}\sigma_{n+1}^2\right|\right\}$$

$$\leqq \frac{1}{\sqrt{N}}\left\{\frac{1}{N}\sum_{n=0}^{N_0-1}\sigma_{n+1}^2+\frac{4}{\eta^2}\left(1+\frac{1}{\eta^2}\right)\right\}+\frac{1}{\eta}\left(1+\frac{2}{\eta}\right)^2\varepsilon^2$$

$$+\frac{\varepsilon}{2\eta}\left(1+\frac{2}{\eta}\right)\left(1+\frac{1}{N}+\sum_{n=0}^{N_0-1}\sigma_{n+1}^2\right)$$

which proves IV.

Noting that T_n is $\{y_{ij},\ i\leqq n,\ j=1, 2, \cdots, N\}$ measurable, (I), (II), (III) and (IV) altogether show that the condition of Loève's theorem is satisfied for $T_{N_0}=\Delta T_0+\Delta T_1+\cdots+\Delta T_{N_0-1}$ (Theorem C: page 337 in [3]), and Proposition 3 is proved.

4. Relation with Hoeffding's conditian

THEOREM 2. *If* $\dfrac{\frac{1}{N}\sum |d_{ij}|^r}{\left(\frac{1}{N}\sum d_{ij}^2\right)^{r/2}} \to 0 \quad (N\to\infty) \qquad$ *for* $r>2$,

then the condition of Theorem 1 *is satisfied.*

PROOF. Normalizing c_{ij}, it is sufficient to prove that, if

$$\frac{1}{N}\sum |c_{ij}|^r\to 0\ , \quad \text{then} \quad \frac{1}{N}\sum_{|c_{ij}|>\varepsilon} c_{ij}^2 \to 0\ .$$

This is easily seen from the fact:

$$\frac{1}{N}\sum_{|c_{ij}|>\varepsilon} c_{ij}^2 \leqq \frac{1}{\varepsilon^{r-1}N}\sum |c_{ij}|^r\ .$$

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