SAMPLING FLUCTUATIONS OF THE TEST RELIABILITY

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1. Introduction

Recently F. M. Lord treated the sampling fluctuations resulting from the sampling of test items [1]. In his paper he discussed only about the Kuder-Richardson reliability of type-2 sampling without the presentation of standard errors of test statistics by the parameters of the population. In this paper we shall treat the Kuder-Richardson reliability of type-1 sampling and of type-2 sampling giving the standard errors of test statistics by the parameter of the population. Moreover, we shall show the standard errors of Spearman-Brown reliability of the two sampling types.

In what follows we use the term of type-1 and type-2 sampling as was used by F. M. Lord. The former is concerned with the sampling of examinees and the latter with the sampling of test items.

2. Kuder-Richardson reliability of type-1 sampling

In the present study, we use the following notation:

$$r_{xx}=r_{20}=\frac{K}{K-1}\left(1-\sum_{i=1}^{K}p_iq_i\Big/s_x^2\right) \tag{1}$$

where K= the number of items

p_i= the observed " difficulty " of item i for n examinees sampled from N examinees population

$q_i=1-p_i$

s_x^2= the observed variance of the score x for n examinees

n= the number of examinees in a single sample

N= the number of examinees in a finite population of examinees. In what follows we neglect, however, the finite population correction.

Then the expectation of r_{xx} of type-1 sampling is represented by

$$E(r_{xx})=\frac{K}{K-1}\left(1-E\left(\frac{\sum_i p_iq_i}{s_x^2}\right)\right)$$

However, we have generally for the function $f(x, y)$ of random variables x and y

$$Ef(x, y)=f(Ex, Ey)+\frac{1}{2}(f_{xx}\sigma_x^2+2\rho_{xy}f_{xy}\sigma_x\sigma_y+f_{yy}\sigma_x^2)+\cdots$$

hence we get

$$E\left(\frac{\sum p_iq_i}{s_x^2}\right)=\frac{E(\sum_i p_iq_i)}{E(s_x^2)}+R=\frac{\sum P_iQ_i}{\sigma_x^2}+O\left(\frac{1}{n}\right)$$

where P_i, Q_i and σ_x^2 are the population values of p_i, q_i and s_x^2 respectively. Therefore, we have

$$(2)\qquad E(r_{xx})=\frac{K}{K-1}\left(1-\frac{\sum P_iQ_i}{\sigma_x^2}\right)+O\left(\frac{1}{n}\right)$$

The error term $R=O(1/n)$ is evaluated as follows.

$$(3)\qquad R=-\sum_i\frac{\operatorname{cov}(p_iq_i,\, s_x^2)}{(n-1)^2/n^2\sigma_x^4}+\frac{\sum_i P_iQ_i}{(n-1)^2/n^2\sigma_x^6}D^2(s_x^2)$$

On the other hand, we have

$$(4)\qquad D^2(s_x^2)=\frac{\mu_4-\sigma_x^4}{n}-\frac{2(\mu_4-2\sigma_x^4)}{n^2}+\frac{\mu_4-3\sigma_x^4}{n^3}$$

$$=\frac{\sigma_x^4}{n}\left(\beta_2-1-\frac{2(\beta_2-2)}{n}\right)+O(1/n^3)$$

where μ_4 is the 4th central moment of total score x of the examinees population and β_2 is the kurtosis of the distribution of x, and

$$(5)\qquad D^2(p_iq_i)=D^2(s_i^2)=\frac{\sigma_i^4}{n}\left(\beta_{2i}-1-\frac{2(\beta_{2i}-2)}{n}\right)$$

$$=\frac{P_i^2Q_i^2}{n}\left(\frac{1}{P_iQ_i}-4-\frac{2}{n}\left(\frac{1}{P_iQ_i}-5\right)\right)$$

Therefore, from (3), (4) and (5), we have

$$(6)\qquad R=\frac{\sum_i P_iQ_i}{n\sigma_x^2}\left(\beta_2-1-\sum_i\rho(p_iq_i,\, s_x^2)P_iQ_i\sqrt{(\beta_2-1)(1/P_iQ_i-4)}\right.$$

$$\left.\div\sum_i P_iQ_i\right)+O(1/n^2)$$

In the following we shall derive $D^2(r_{xx})$. At first by the relation

$$E[f(x, y)-Ef(x, y)]^2 \doteqdot E[f(x, y)-f(Ex, Ey)]^2=\frac{1}{(Ey)^2}E\left(x-y\frac{Ex}{Ey}\right)^2$$

for the function $f(x, y)=x/y$ of random variables x and y, we have

$$D^2\left(\frac{\sum p_i q_i}{s_x^2}\right)=\frac{1}{(n-1)^2/n^2\sigma_x^4}E(\sum_i p_i q_i - s_x^2 C)^2 \tag{7}$$

where $C=\sum P_i Q_i/\sigma_x^2$. On the other hand, we have

$$\begin{aligned} E(\sum_i p_i q_i - s_x^2 C)^2 &= \sum_i E(p_i^2 q_i^2) + \sum\sum_{i\neq j}(p_i q_i p_j q_j) + C^2 E(s_x^4) - 2C\sum_i E(p_i q_i s_x^2) \\ &= \left(\frac{n-1}{n}\right)^2 \sum_i P_i^2 Q_i^2 + \frac{1}{n}\sum_i P_i^2 Q_i^2\left(\frac{1}{P_i Q_i}-4\right) + \sum\sum_{i\neq j}\left\{\left(\frac{n-1}{n}\right)^2 \sigma_i^2\sigma_j^2\right. \\ &\quad \left.+\frac{1}{n}(\mu_{22}^{ij}-\sigma_i^2\sigma_j^2)\right\} + C^2\sigma_x^4\left(1+\frac{\beta_2-3}{n}\right) - 2C\sum_i\left\{\left(\frac{n-1}{n}\right)^2\sigma_i^2\sigma_x^2\right. \\ &\quad \left.+\frac{1}{n}(\mu_{22}^{ix}-\sigma_i^2\sigma_x^2)\right\} + O(1/n^2) \end{aligned} \tag{8}$$

because we have in general

$$E(s_i^2 s_j^2)=E(s_i^2)E(s_j^2)+(\mu_{22}^{ij}-\sigma_i^2\sigma_j^2)/n+O(1/n^2)^*$$

where μ_{22}^{ij} denotes the bivariate central moment of the simultaneous distribution of items i and j. Since $\sigma_i^2=P_i Q_i$, we have from (8)

$$\begin{aligned} E(\sum p_i q_i - s_x^2 C)^2 &= \frac{1}{n}\{(\sum_i P_i Q_i)^2\beta_2 + \sum_i P_i Q_i - 3\sum_i P_i^2 Q_i^2 \\ &\quad + \sum\sum_{i\neq j}\mu_{22}^{ij} - 2C\sum_i \mu_{22}^{ix}\} + O(1/n^2) \end{aligned} \tag{9}$$

Thus we get

$$\begin{aligned} D^2(r_{xx}) &= \left(\frac{K}{K-1}\right)^2 D^2\left(\frac{\sum_i p_i q_i}{s_x^2}\right) = \left(\frac{K}{K-1}\right)^2\left(\frac{n}{n-1}\right)^2\frac{1}{n\sigma_x^4}\{\beta_2(\sum_i P_i Q_i)^2 \\ &\quad + \sum_i P_i Q_i - 3\sum_i P_i^2 Q_i^2 + \sum\sum_{i\neq j}\mu_{22}^{ij} - 2(\sum_i \mu_{22}^{ix})(\sum_i P_i Q_i)/\sigma_x^2\} + O(1/n^2) \end{aligned} \tag{10}$$

In the simultaneous distribution of items i and j denote by α_{ij} the percentage of examinees who pass both items. Then we easily get

$$\mu_{22}^{ij}=\alpha_{ij}(1+4P_i P_j)-P_i P_j(P_i+P_j)-3P_i^2 P_j^2 \tag{11}$$

Without loss of generality we can assume

$$P_1 \leqq P_2 \leqq \cdots \leqq P_{K-1} \leqq P_K$$

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* See, for example, M. G. Kendall: The advanced theory of statistics, vol. I, London, Charles Griffin and Co., 1948.

Then we have

$$\text{(12)}\quad \sum_{i\neq j}\sum \mu_{22}^{ij} \leqq K(K-1)P_{K-1}+4P_{K-1}\{(\sum_i P_i)^2-\sum_i P_i^2\}-2(\sum_i P_i^2)(\sum_i P_i)$$
$$+2\sum_i P_i^3+3(\sum_i P_i^2)^2-3\sum_i P_i^4$$

Let the simultaneous distribution of x and i be the one in Table 1, and

Table 1.

Score x \ Item i	1	0
x_1	α_{ix}	β_{ix}
$\vdots$	$\vdots$	$\vdots$
x_l	γ_{ix}	δ_{ix}
Total	P_i	Q_i

$$x_1\alpha_{ix}+\cdots+x_l\gamma_{ix}\equiv L_1$$

$$x_1^2\alpha_{ix}+\cdots+x_l^2\gamma_{ix}\equiv L_2$$

We get then

$$\text{(13)}\quad \mu_{22}^{ix}=L_2(1-P_i^2)+3L_1^2P_iQ_i$$

Put $x_m=\min\,(x_1,\cdots,x_l)$, and we have

$$\text{(14)}\quad \sum_i \mu_{22}^{ix}\geqq x_m^2(\sum_i P_i+2\sum_i P_i^3-3\sum_i P_i^4)$$

Therefore, we have from (10), (12) and (14)

$$D^2(r_{xx})\leqq\left(\frac{K}{K-1}\right)^2\left(\frac{n}{n-1}\right)^2\frac{1}{n\sigma_x^4}\Big\{\beta_2K^2(P_{01}-P_{02})^2+KP_{01}-4KP_{02}+8KP_{03}$$
$$\text{(15)}\quad -6KP_{04}+3K^2P_{02}^2-2K^2P_{01}P_{02}+K(K-1)P_{K-1}+4KP_{K-1}(KP_{01}^2$$
$$-P_{02})-2x_m^2K^2(P_{01}-P_{02})(P_{01}+2P_{03}-3P_{04})/\sigma_x^2\Big\}$$

where

$$\sum_i P_i=KP_{01}\,,\quad \sum_i P_i^2=KP_{02}\,,\quad \sum_i P_i^3=KP_{03}$$

and

$$\sum_i P_i^4=KP_{04}$$

If we put particularly $P_i=0.5$ $(i=1, 2, \cdots, K)$, $\alpha_{ij}=0.5$ $(i, j=1, 2, \cdots, K)$, then

$$\sigma_x^2=\sum_i P_iQ_i+\sum_{i\neq j}\sum \mu_{11}^{ij}=\sum_i P_iQ_i+\sum_{i\neq j}\sum(\alpha_{ij}-P_iP_j)=\frac{K^2}{4}$$

consequently

$$\text{(16)}\quad D^2(r_{xx})\leqq\frac{1}{n(K-1)^2}\left(\frac{n}{n-1}\right)^2\left(\beta_2+15-\frac{14}{K}-\frac{18}{K^2}x_m^2\right)$$

3. Kuder-Richardson reliability of type-2 sampling

In this section we use the following notation:

k=the number of items in a single sample.

K=the number of items in a finite population of items. In what follows, we, however, neglect the finite population correction.

N=the number of examinees.

x_{ia}=the score of examinee a on item i

$$x_{ia}\begin{cases}=1 & \text{if item is answered correctly}\\ =0 & \text{otherwise}\end{cases}$$

z_a=the "proportion-correct score" of examinee a; the proportion of the items in a single test answered correctly by examinee a.

ζ_a=the "true" proportion-correct score of examinee a.

ζ_{ab}=the proportion of all items answered correctly by both examinee a and examinee b.

ζ_{abc}=the proportion of all items answered correctly by examinees a, b and c.

$$s_z^2=\sum_a z_a^2/N-\bar{z}^2\,,\quad \bar{z}=\sum z_a/N$$

At first we modify the formula r_{xx} and $\sum_{i=1}^{k} s_i^2/k$ as follows:

$$r_{xx}=\frac{k}{k-1}\left(1-\frac{\sum s_i^2/k}{ks_z^2}\right) \tag{17}$$

$$\frac{1}{k}\sum_i^k s_i^2=\frac{1}{k}\sum_i^k p_i q_i=\frac{1}{k}\sum_i\left\{\frac{1}{N}\sum_i x_{ia}-\frac{1}{N^2}(\sum_a x_{ia})^2\right\}$$

$$=\frac{1}{N}\sum_a z_a-\frac{1}{N^2}\sum_a\left(\frac{1}{k}\sum_i x_{ia}\right)-\frac{1}{N^2}\sum\sum_{a\neq b}\left(\frac{1}{k}\sum_i x_{ia}x_{ib}\right) \tag{18}$$

$$=\frac{1}{N}\sum_a z_a-\frac{1}{N^2}\sum_a z_a-\frac{1}{N^2}\sum\sum_{a\neq b}(s_{ab}+z_a z_b)$$

Therefore, by the well-known formulae**

$$E(m_{11})=\frac{k-1}{k}\mu_{11}$$

$$E(m_2)=\frac{k-1}{k}\mu_2$$

$$E(m_{10},\, m_{01})=\mu_{10}\mu_{01}+\frac{\mu_{11}}{k}$$

** See the text book of M.G. Kendall, loc. cit.

we have

$$E\left(\frac{1}{k}\sum_i s_i^2\right)=\frac{1}{N}\sum_a \zeta_a-\frac{1}{N^2}\left(\sum_a \zeta_a+\sum_{a\neq b}\sum \zeta_{ab}\right)+O\left(\frac{1}{k}\right)$$
$$=\frac{1}{K}\sum_{i=1}^{K} P_i Q_i+O\left(\frac{1}{k}\right) \tag{19}$$

On the other hand, we get

$$E(s_z^2)=\frac{1}{N}\sum_a E(z_a^2)-\frac{1}{N^2}\left(\sum_a E(z_a^2)+\sum_{a\neq b}\sum E(z_a z_b)\right)$$
$$=\frac{1}{N}\sum_a \zeta_a^2-\bar{\zeta}^2+\frac{1}{kN}\left(1-\frac{1}{N}\right)\sum_a \zeta_a(1-\zeta_a) \tag{20}$$

Hence from (17), (19) and (20) we have

$$E(r_{xx})=\frac{k}{k-1}\left(1-\frac{\sum_{i=1}^{K} P_i Q_i}{Kk\sigma_z^2}\right)+O\left(\frac{1}{k}\right)$$
$$=\frac{K}{K-1}\left(1-\frac{\sum_i P_i Q_i}{\sigma_x^2}\right)+O\left(\frac{1}{k}\right)=\rho_{xx}+O\left(\frac{1}{k}\right) \tag{21}$$

Now, we shall derive the variance of r_{xx}. As before we modify

$$\left(\frac{1}{k}\sum_i s_i^2\right)^2$$

as follows:

$$\left(\frac{1}{k}\sum_i s_i^2\right)^2=\frac{1}{k^2}\sum_i\left\{\frac{1}{N^2}\left(\sum_a x_{ia}+\sum_{a\neq b}\sum x_{ia}x_{ib}\right)-\frac{2}{N^3}\left(\sum_a x_{ia}+3\sum_{a\neq b}\sum x_{ia}x_{ib}\right.\right.$$
$$\left.+\sum_{a\neq b\neq c}\sum\sum x_{ia}x_{ib}x_{ic}\right)+\frac{1}{N^4}\left(\sum_a x_{ia}+7\sum_{a\neq b}\sum x_{ia}x_{ib}\right.$$
$$\left.\left.+6\sum_{a\neq b\neq c}\sum\sum x_{ia}x_{ib}x_{ic}+\sum_{a\neq b\neq c\neq d}\sum\sum\sum x_{ia}x_{ib}x_{ic}x_{id}\right)\right\}$$
$$+\frac{1}{k^2}\sum_{i\neq j}\sum\left\{\frac{1}{N^2}\sum_a x_{ia}x_{ja}+\frac{1}{N^2}\sum_{a\neq b}\sum x_{ia}x_{jb}-\frac{1}{N^3}\left(2\sum_a x_{ia}x_{ja}\right.\right. \tag{22}$$
$$\left.+2\sum_{a\neq b}\sum x_{ia}x_{jb}+4\sum_{a\neq b}\sum x_{ia}x_{ja}x_{ib}+2\sum_{a\neq b\neq c}\sum\sum x_{ia}x_{ib}x_{jc}\right)$$
$$+\frac{1}{N^4}\left(\sum_a x_{ia}x_{ja}+\sum_{a\neq b}\sum x_{ia}x_{jb}+4\sum_{a\neq b}\sum x_{ia}x_{ib}x_{ja}\right.$$
$$+2\sum_{a\neq b\neq c}\sum\sum x_{ia}x_{ib}x_{jc}+2\sum_{a\neq b}\sum x_{ia}x_{ib}x_{ja}x_{jb}$$

$$+4\sum\sum_{a\neq b\neq c}\sum x_{ia}x_{ib}x_{ja}x_{jc}+\sum\sum_{a\neq b\neq c\neq d}\sum\sum x_{ia}x_{ib}x_{jc}x_{jd})\Big\}$$

From this formula we get

$$\begin{aligned}
E\Big(\frac{1}{k}\sum_i s_i^2\Big)^2 \doteq \frac{1}{N^2}(\sum_a \zeta_a^2+\sum_{a\neq b}\sum\zeta_a\zeta_b)-\frac{2}{N^3}(\sum_a\zeta_a^2+\sum_{a\neq b}\sum\zeta_a\zeta_b+2\sum_{a\neq b}\sum\zeta_a\zeta_{ab}\\
+\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c)+\frac{1}{N^4}(\sum_a\zeta_a^2+\sum_{a\neq b}\sum\zeta_a\zeta_b+4\sum_{a\neq b}\sum\zeta_{ab}\zeta_a\\
+2\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c+2\sum_{a\neq b}\sum\zeta_{ab}^2+4\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_{ac}\\
\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{ab}\zeta_{cd})+\frac{1}{k}\Big\{\frac{1}{N^2}(\sum_a\zeta_a+\sum_{a\neq b}\sum\zeta_{ab}-\sum_a\zeta_a^2\\
-\sum_{a\neq b}\sum\zeta\zeta_{ab})+\frac{1}{N^3}(-2\sum_a\zeta_a-6\sum_{a\neq b}\sum\zeta_{ab}-2\sum\sum_{a\neq b\neq c}\sum\zeta_{abc}\\
+2\sum_a\zeta_a^2+2\sum_{a\neq b}\sum\zeta_a\zeta_b+4\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2+2\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c)\\
+\frac{1}{N^4}(\sum_a\zeta_a+7\sum_{a\neq b}\sum\zeta_{ab}+6\sum\sum_{a\neq b\neq c}\sum\zeta_{abc}+\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{abcd}\\
-\sum_a\zeta_a^2-\sum_{a\neq b}\sum\zeta_a\zeta_b-4\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2-2\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c-2\sum_{a\neq b}\sum\zeta_{ab}^2\\
-4\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_{ac}-\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{ab}\zeta_{cd})\Big\}
\end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned}
D^2\Big(\frac{1}{k}\sum_i s_i^2\Big)=\frac{1}{k}\Big\{\frac{1}{N^2}(N\overline{\zeta}+\sum_{a\neq b}\sum\zeta_{ab}-N^2\overline{\zeta}^2)-\frac{2}{N^3}(N\overline{\zeta}+3\sum_{a\neq b}\sum\zeta_{ab}\\
+\sum\sum_{a\neq b\neq c}\sum\zeta_{abc}+N^2\overline{\zeta}^2+2\sum_{a\neq b}\sum\zeta_{ab}\zeta_a+\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c)\\
+\frac{1}{N^4}(N\overline{\zeta}+7\sum_{a\neq b}\sum\zeta_{ab}+6\sum\sum_{a\neq b\neq c}\sum\zeta_{abc}+\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{abcd}\\
-N^2\overline{\zeta}^2-4\sum_{a\neq b}\sum\zeta_{ab}\zeta_a-2\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c-2\sum_{a\neq b}\sum\zeta_{ab}^2\\
-4\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_{ac}-\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{ab}\zeta_{cd})\Big\}\equiv A/k\quad\text{(say)}
\end{aligned} \tag{24}$$

In the next place we shall calculate $\mathrm{cov}(\sum s_i^2/k,\ s_z^2)$. At first we also modify the product $\sum s_i^2 s_z^2/k$ as follows:

$$\Big(\frac{1}{k}\sum_i s_i^2\Big)\Big(\frac{1}{N}\sum_a z_a^2-\overline{z}^2\Big)=\frac{N-1}{N^3}(\sum_a z_a^3+\sum_{a\neq b}\sum z_a^2 z_b)-\frac{N-1}{k^3N^4}\{\sum_a(\sum_i x_{ia}^2$$

$$+\sum_{i\neq j}\sum x_{i\iota}x_{j\iota})+\sum_{a\neq b}\sum(\sum_i x_{ia}x_{ib}+\sum_{i\neq j}\sum x_{ia}x_{jb})+2\sum_{i\neq j}\sum\sum_a x_{i\iota}x_{ja}$$

$$+2\sum_{i\neq j}\sum\sum_{a\neq b}\sum x_{ia}x_{j\iota}x_{ib}+\sum_{i\neq j\neq l}\sum\sum\sum_a x_{ia}x_{ja}x_{la}+\sum_{i\neq j\neq l}\sum\sum\sum_{a\neq b}\sum x_{i\iota}x_{j\iota}x_{lb}$$

$$+2\sum_i\sum_{a\neq b}\sum x_{ia}x_{ib}+\sum_i\sum_{a\neq b\neq c}\sum\sum x_{ia}x_{ib}x_{ic}+2\sum_{i\neq j}\sum\sum_{a\neq b}\sum x_{i\iota}x_{ib}x_{j\iota}$$

$$+\sum_{i\neq j}\sum\sum_{a\neq b\neq c}\sum\sum x_{ia}x_{ib}x_{jc}+4\sum_{i\neq j}\sum\sum_{a\neq b}\sum x_{ia}x_{ja}x_{ib}+2\sum_{i\neq j\neq l}\sum\sum\sum_{a\neq b}\sum x_{i\iota}x_{ja}x_{la}x_{lb}$$

$$+2\sum_{i\neq j}\sum\sum_{a\neq b\neq c}\sum\sum x_{ic}x_{jc}x_{ia}x_{ib}+\sum_{i\neq j\neq l}\sum\sum\sum_{a\neq b\neq c}\sum\sum x_{ic}x_{jc}x_{la}x_{lb}\}$$

$$-\frac{1}{N^3}(2\sum_{a\neq b}\sum z_a^2z_b+\sum_{a\neq b\neq c}\sum\sum z_az_bz_c)+\frac{1}{k^3N^4}\{2\sum_{a\neq b}\sum\sum_i x_{ia}x_{ib}$$

$$(25)\quad +2\sum_{a\neq b\neq i\neq j}\sum\sum\sum x_{ia}x_{ib}x_{j\iota}+\sum_{a\neq b\neq c}\sum\sum\sum_i x_{ia}x_{ib}x_{ic}+\sum_{a\neq b\neq c}\sum\sum\sum_{i\neq j}\sum x_{ia}x_{ib}x_{jc}$$

$$+4\sum_{a\neq b}\sum\sum_{i\neq j}\sum x_{ia}x_{jb}+2\sum_{a\neq b}\sum\sum_{i\neq j\neq l}\sum\sum x_{ia}x_{jb}x_{la}+2\sum_{a\neq b\neq c}\sum\sum\sum_{i\neq j}\sum x_{ia}x_{jb}x_{ic}$$

$$+\sum_{a\neq b\neq c}\sum\sum\sum_{i\neq j\neq l}\sum\sum x_{ia}x_{jb}x_{lc}+2\sum_{a\neq b}\sum\sum_i x_{ia}x_{ib}+4\sum_{a\neq b}\sum\sum_{i\neq j}\sum x_{ia}x_{ib}x_{ja}x_{jb}$$

$$+2\sum_{a\neq b\neq c}\sum\sum\sum_i x_{ia}x_{ib}x_{ic}+\sum_{a\neq b\neq c\neq d}\sum\sum\sum\sum_i x_{ia}x_{ib}x_{ic}x_{id}+4\sum_{a\neq b\neq c}\sum\sum\sum_{i\neq j}\sum x_{ia}x_{ib}x_{j\iota}x_{jc}$$

$$+\sum_{a\neq b\neq c\neq d}\sum\sum\sum\sum_{i\neq j}\sum x_{ia}x_{ib}x_{jc}x_{jd}+4\sum_{a\neq b}\sum\sum_{i\neq j}\sum x_{ia}x_{jb}x_{ib}$$

$$+8\sum_{a\neq b\neq c}\sum\sum\sum_{i\neq j}\sum x_{ia}x_{jb}x_{ic}+2\sum_{a\neq b}\sum\sum_{i\neq j\neq l}\sum\sum x_{ia}x_{jb}x_{la}x_{lb}$$

$$+2\sum_{a\neq b\neq c\neq d}\sum\sum\sum\sum_{i\neq j}\sum x_{ia}x_{jb}x_{ic}x_{id}+4\sum_{a\neq b\neq c}\sum\sum\sum_{i\neq j\neq l}\sum\sum x_{ia}x_{jb}x_{la}x_{lc}$$

$$+\sum_{a\neq b\neq c\neq d}\sum\sum\sum\sum_{i\neq j\neq l}\sum\sum x_{ia}x_{jb}x_{lc}x_{ld}\Big\}\;.$$

Then we get

$$E\Big(\frac{1}{k}\sum_i s_i^2\cdot s_z^2\Big)\doteqdot\frac{N-1}{N^3}\Big\{\sum_i\Big(\zeta_a^3+\frac{3}{k}\zeta_a^3(1-\zeta_a)\Big)+\sum_{a\neq b}\sum\Big(\zeta_a^2+\frac{1}{k}\mu_{20,a}\Big)\zeta_b\Big\}$$

$$-\frac{N-1}{k^3N^4}\{k\sum_a\zeta_a+k(k-1)\sum_a\zeta_a^2+k\sum_{a\neq b}\sum\zeta_{ab}+k(k-1)\sum_{a\neq b}\sum\zeta_a\zeta_b$$

$$+2k(k-1)\sum_a\zeta_a^2+2k(k-1)\sum_{a\neq b}\sum\zeta_{ab}\zeta_a+k(k-1)(k-2)\sum_a\zeta_a^3$$

$$+k(k-1)(k-2)\sum_{a\neq b}\sum\zeta_a^2\zeta_b+2k\sum_{a\neq b}\sum\zeta_{ab}+k\sum_{a\neq b\neq c}\sum\sum\zeta_{abc}+2k(k-1)\sum_{a\neq b}\sum\zeta_{ab}\zeta_a$$

$$+k(k-1)\sum_{a\neq b\neq c}\sum\sum\zeta_{ab}\zeta_c+4k(k-1)\sum_{a\neq b}\sum\zeta_{ab}\zeta_a+2k(k-1)(k-2)\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2$$

$$+2k(k-1)\sum_{a\neq b\neq c}\sum\sum\zeta_{ab}\zeta_c^2+k(k-1)(k-2)\sum_{a\neq b\neq c}\sum\sum\zeta_{ab}\zeta_c^2\}$$

$$-\frac{1}{N^3}\Big\{2\sum_{a\neq b}\sum\Big(\zeta_a^2+\frac{\mu_{20,a}}{k}\Big)\zeta_b+\sum\sum_{a\neq b\neq c}\sum\zeta_a\zeta_b\zeta_c\Big\}+\frac{1}{k^3N^4}\{2k\sum_{a\neq b}\sum\zeta_{ab} \tag{26}$$

$$+2k(k-1)\sum_{a\neq b}\sum\zeta_{ab}\zeta_a+k\sum\sum_{a\neq b\neq c}\sum\zeta_{abc}+k(k-1)\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c$$

$$+4k(k-1)\sum_{a\neq b}\sum\zeta_a\zeta_b+2k(k-1)(k-2)\sum_{a\neq b}\sum\zeta_a^2\zeta_b+2k(k-1)\sum\sum_{a\neq b\neq c}\sum\zeta_{ac}\zeta_b$$

$$+k(k-1)(k-2)\sum\sum_{a\neq b\neq c}\sum\zeta_a\zeta_b\zeta_c+2k\sum_{a\neq b}\sum\zeta_{ab}+4k(k-1)\sum_{a\neq b}\sum\zeta_{ab}^2$$

$$+2k\sum\sum_{a\neq b\neq c}\sum\zeta_{abc}+k\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{abcd}+4k(k-1)\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_{ac}$$

$$+k(k-1)\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{ab}\zeta_{cd}+4k(k-1)\sum_{a\neq b}\sum\zeta_{ab}\zeta_b+8k(k-1)\sum\sum_{a\neq b\neq c}\sum\zeta_{ac}\zeta_b$$

$$+2k(k-1)(k-2)\sum_{a\neq b}\sum\zeta_a\zeta_b\zeta_{ab}+2k(k-1)\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{acd}\zeta_b$$

$$+4k(k-1)(k-2)\sum\sum_{a\neq b\neq c}\sum\zeta_{ac}\zeta_a\zeta_b+k(k-1)(k-2)\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{cd}\zeta_a\zeta_b\}$$

On the other hand, we have

$$E\Big(\frac{1}{k}\sum_i s_i^2\Big)E(s_z^2)=\frac{N-1}{N^3}\Big(\sum_a\zeta_a^3+\sum_{a\neq b}\sum\zeta_a^2\zeta_b\Big)-\frac{1}{N^3}\Big(2\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2$$

$$+\sum\sum_{a\neq b\neq c}\sum\zeta_a^2\zeta_{bc}\Big)-\frac{N-1}{N^4}\Big(\sum_a\zeta_a^3+3\sum_{a\neq b}\sum\zeta_a^2\zeta_b+\sum\sum_{a\neq b\neq c}\sum\zeta_a\zeta_b\zeta_c\Big)$$

$$+\frac{1}{N^4}\Big(2\sum_{a\neq b}\sum\zeta_a^2\zeta_{ab}+\sum\sum_{a\neq b\neq c}\sum\zeta_a^2\zeta_{bc}+2\sum_{a\neq b}\sum\zeta_{ab}\zeta_a\zeta_b+4\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_a\zeta_c \tag{27}$$

$$+\sum\sum_{a\neq b\neq c\neq d}\sum\sum\zeta_{ab}\zeta_c\zeta_d\Big)+\frac{(N-1)^2}{kN^4}\Big(\sum_a\zeta_a^2+\sum_{a\neq b}\sum\zeta_a\zeta_b-\sum_a\zeta_a^3-\sum_{a\neq b}\sum\zeta_a\zeta_b^2\Big)$$

$$-\frac{N-1}{kN^4}\Big(2\sum_{a\neq b}\sum\zeta_{ab}\zeta_a+\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c-2\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2-\sum\sum_{a\neq b\neq c}\sum\zeta_a^2\zeta_{bc}\Big)$$

Hence we can derive the relation

$$\mathrm{cov}\Big(\frac{1}{k}\sum_i s_i^2, s_z^2\Big)\doteq\frac{1}{k}\Big\{\frac{1}{N^2}\Big(4\sum_a\zeta_a^3-3\sum_a\zeta_a^4+\sum_{a\neq b}\sum\mu_{20,a}\zeta_a-\sum_a\zeta_a^2$$

$$-\sum_{a\neq b}\sum\zeta_a\zeta_b+\sum_{a\neq b}\sum\zeta_a\zeta_b^2\Big)+\frac{1}{N^3}\Big(3\sum_a\zeta_a^4-3\sum_{a\neq b}\sum\mu_{20,a}\zeta_b-\sum_a\zeta_a^2$$

$$+\sum_{a\neq b}\sum\zeta_a^2\zeta_b-6\sum_{a\neq b}\sum\zeta_{ab}\zeta_a+4\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2+\sum_{a\neq b}\sum\zeta_a\zeta_b-2\sum_a\zeta_a^3\Big)$$

$$+\frac{1}{N^4}\Big(2\sum_a\zeta_a^2-2\sum_a\zeta_a^3-2\sum_{a\neq b}\sum\zeta_a^2\zeta_b+12\sum_{a\neq b}\sum\zeta_{ab}\zeta_a-4\sum_{a\neq b}\sum\zeta_{ab}\zeta_a^2 \tag{28}$$

$$+11\sum\sum_{a\neq b\neq c}\sum\zeta_{ab}\zeta_c+4\sum_{a\neq b}\sum\zeta_a\zeta_b-6\sum_{a\neq b}\sum\zeta_a^2\zeta_b-3\sum\sum_{a\neq b\neq c}\sum\zeta_a\zeta_b\zeta_c$$

$$+4\sum_{a\neq b}\sum \zeta_{ab}^2+4\sum_{a\neq b\neq c}\sum\sum \zeta_{ab}\zeta_{ac}+\sum_{a\neq b\neq c\neq d}\sum\sum\sum \zeta_{ab}\zeta_{cd}-6\sum_{a\neq b}\sum \zeta_a\zeta_b\zeta_{ab}$$

$$+2\sum_{a\neq b\neq c\neq d}\sum\sum\sum \zeta_{acd}\zeta_b-12\sum_{a\neq b\neq c}\sum\sum \zeta_{ac}\zeta_a\zeta_b-3\sum_{a\neq b\neq c\neq d}\sum\sum\sum \zeta_a\zeta_b\zeta_{cd})\Big\}$$

$$\equiv B/k \text{ (say)}$$

Also we have

$$s_z^4=\frac{1}{N^2}(\sum_a z_a^4+\sum_{a\neq b}\sum z_a^2z_b^2)+\frac{1}{N^4}(\sum_a z_a^4+4\sum_{a\neq b}\sum z_a^3z_b+3\sum_{a\neq b}\sum z_a^2z_b^2$$

$$+6\sum_{a\neq b\neq c}\sum\sum z_a^2z_bz_c+\sum_{a\neq b\neq c\neq d}\sum\sum\sum z_az_bz_cz_d)-\frac{2}{N^3}(\sum_a z_a^4+\sum_{a\neq b}\sum z_a^2z_b^2 \tag{29}$$

$$+2\sum_{a\neq b}\sum z_a^3z_b+\sum_{a\neq b\neq c}\sum\sum z_a^2z_bz_c)$$

As is easily shown for the sample mean m of sample size k

$$\begin{cases} E(m^3)=\mu_1^3+\dfrac{3}{k}\mu_1\mu_2+\dfrac{\mu_3}{k^2} \\ E(m^4)=\mu_1^4+\dfrac{6\mu_1^2\mu_2}{k}+\dfrac{1}{k^2}(4\mu_1\mu_3+3\mu_2^2)+\dfrac{1}{k^3}(\mu_4-3\mu_2^2) \end{cases} \tag{30}$$

hence we have

$$E(z_z^4)=\Big(\frac{1}{N}\sum \zeta_a^2-\overline{\zeta}^2\Big)^2+\frac{1}{N^2k}\Big\{6\sum_a \zeta_a^3(1-\zeta_a)+2\sum_{a\neq b}\sum \zeta_a^2\zeta_b(1-\zeta_b)$$

$$+\frac{1}{N^2}\Big(12\sum_{a\neq b}\sum \zeta_a^2\zeta_b(1-\zeta_a)+6\sum_{a\neq b}\sum \zeta_a^2\zeta_b(1-\zeta_b) \tag{31}$$

$$+6\sum_{a\neq b\neq c}\sum\sum \zeta_a\zeta_b\zeta_c(1-\zeta_a)\Big)-\frac{2}{N}\Big((6\sum_a \zeta_a^3(1-\zeta_a)+2\sum_{a\neq b}\sum \zeta_a\zeta_b^2(1-\zeta_a)$$

$$+6\sum_{a\neq b}\sum \zeta_a^2\zeta_b(1-\zeta_a)+\sum_{a\neq b\neq c}\sum\sum \zeta_a\zeta_b\zeta_c(1-\zeta_a)\Big)\Big\}+O(1/k^2)$$

Consequently from (20) and (31)

$$D^2(s_z^2)=E(s_z^4)-\{E(s_z^2)\}^2=\frac{1}{kN^2}\Big\{\Big(4-\frac{8}{N}-\frac{2}{N^2}\Big)\sum_a \zeta_a^3(1-\zeta_a)$$

$$+\frac{4}{N^2}\sum_{a\neq b}\sum \zeta_a^2\zeta_b(1-\zeta_b)+\Big(\frac{10}{N^2}-\frac{10}{N}\Big)\sum_{a\neq b}\sum \zeta_a^2\zeta_b(1-\zeta_a) \tag{32}$$

$$+\frac{4}{N^2}\sum_{a\neq b\neq c}\sum\sum \zeta_a\zeta_b\zeta_c(1-\zeta_a)+O(1/k^2)\equiv C/k \quad \text{(say)}$$

Therefore, we have

$$D^2(r_{xx})\doteqdot\frac{1}{(k-1)^2}\Big\{\frac{1}{\sigma_z^4}D^2\Big(\frac{1}{k}\sum_i s_i^2\Big)+\frac{(\sum \sigma_i^2)^2}{k^2\sigma_z^8}D^2(s_z^2)-\frac{2\sum_i \sigma_i^2}{k\sigma_z^6}\mathrm{cov}\Big(\frac{1}{k}\sum_i s_i^2, s_z^2\Big)\Big\}$$

$$(33)\qquad =\frac{K^4}{k(k-1)^2\sigma_x^2}\left\{A-2K\left(1-\frac{K-1}{K}\rho_{xx}\right)B+K^2\left(1-\frac{K}{K-1}\rho_{xx}\right)^2C\right\}$$

$$=\frac{k}{\sigma_x^2}\{A-2K(1-\rho_{xx})B+K^2(1-\rho_{xx})^2C\}+O\left(\frac{1}{k^4}\right)$$

4. Spearman-Brown reliability of type-1 sampling

Spearman-Brown reliability coefficient r_s is usually defined as the correlation coefficient of two parallel parts of the test, that is,

$$(34)\qquad r_s=\frac{2r}{1+r}$$

If we put in the type-1 sampling

$$(35)\qquad E(r)=\rho$$

then for large number of examinee, n, we have

$$(36)\qquad r_s=\frac{2\rho}{1+\rho}+\frac{2}{(1+\rho)^2}(r-\rho)-\frac{2}{(1+\rho)^3}(r-\rho)^2+R$$

where R denotes the residual term. Hence we have

$$(37)\qquad E(r_s)=\frac{2\rho}{1+\rho}-\frac{2}{(1+\rho)^3}D^2(r)=\rho_s+O\left(\frac{1}{n}\right)$$

where $D^2(r)$ is of order $1/n$ and in the case of normal distribution it is approximated by $(1-\rho^2)^2/n$.

As for the variance of r_s it is easily shown that

$$(38)\qquad D^2(r_s)\doteqdot E\left(r_s-\frac{2\rho}{1+\rho}\right)^2\doteqdot\frac{4}{(1+\rho)^4}D^2(r)$$

neglecting higher infinitesimal in regard to n and ρ, where $D^2(r)$ is given by well known formula

$$\frac{\rho^2}{4n}\left(\frac{\mu_{40}}{\mu_{20}^2}+\frac{\mu_{04}}{\mu_{02}^2}+\frac{2\mu_{22}}{\mu_{20}\mu_{02}}+\frac{4\mu_{22}}{\mu_{11}^2}-\frac{4\mu_{31}}{\mu_{11}\mu_{20}}-\frac{4\mu_{13}}{\mu_{11}\mu_{02}}\right)$$

5. Spearman-Brown reliability of type-2 sampling

In the type-2 sampling we assume that the $2k$ items in a single sample contain always two parallel test items of equal number k. This assumption is realized by the stratified random sampling from two sub-populations of parallel test items.

In this section we use the following notation similar as that of F. M. Lord:

t_a=the one observed test score of examinee a, obtained by the number of items of stratum 1 answered correctly on a single test.

u_a=the other observed test score of examinee a, obtained by the number of items of stratum 2 answered correctly on a single test.

$\bar{t}$=the mean of the one scores obtained by the N examinees on a single test

$$\bar{t}=\sum_a t_a/N=k\bar{z}$$

$\bar{u}$=the mean of the other scores obtained by the N examinees on a single test

$$\bar{u}=\sum_a u_a/N=k\bar{y}\ .$$

Let ρ be the correlation coefficient of two parallel parts in the whole $2K$ items. Then we have

$$\rho=\frac{\frac{1}{N}\sum_a \zeta_a\eta_a-\bar{\zeta}\,\bar{\eta}}{\sigma_\zeta\sigma_\eta} \tag{39}$$

and for the corresponding sample value r

$$r=\frac{\frac{1}{N}\sum_a t_a u_a-\bar{t}\,\bar{u}}{s_t s_u}=\frac{\frac{1}{N}\sum_a z_a y_a-\bar{z}\bar{y}}{s_z s_y} \tag{40}$$

where

$$\zeta_a=E(z_a),\ \eta_a=E(y_a),\ \bar{\zeta}=\sum_a \zeta_a/N,\ \bar{\eta}=\sum_a \eta_a/N,\ \sigma_\zeta^2=\sum_a \zeta_a^2/N-\bar{\zeta}^2$$

and

$$\sigma_\eta^2=\sum_a \eta_a^2/N-\bar{\eta}^2$$

Because of the parallelism it holds that $\bar{\zeta}=\bar{\eta}$, $\sigma_\zeta^2=\sigma_\eta^2$. By the relation***

$$\operatorname{cov}(m'_{10},\ m'_{01})=\mu_{11}/k$$

we have

*** See M. G. Kendall, loc. cit.

(41) $$E\left(\frac{1}{N}\sum_a z_a y_a - \bar{z}\bar{y}\right) = \frac{1}{N}\sum_a \zeta_a \eta_a - \bar{\zeta}\,\bar{\eta} + \frac{\bar{\mu}_{11}}{k}\left(1-\frac{1}{N}\right)$$

where $\mu_{11,a}$=the covariance between x_{ia} in stratum 1 and x_{ja} in stratum 2 for examinee a,

$$\bar{\mu}_{11} = \sum_a \mu_{11,a}/N$$

On the other hand, from (20), (32) and the relations

(42) $$\begin{cases} E^2(s_z) = E(s_z^2) - D^2(s_z) \\ D^2(s_z) \doteqdot \dfrac{1}{4\sigma_\zeta^2} D^2(s_z^2) \end{cases}$$

we have

(43) $$\begin{aligned} E^2(s_z) \doteqdot \sigma_\zeta^2 + \frac{1}{kN}\left(1-\frac{1}{N}\right)\sum_a \zeta_a(1-\zeta_a) - \frac{1}{4\sigma_\zeta^2 kN^2}\Big\{\left(4-\frac{8}{N}-\frac{2}{N^2}\right)\cdot \\ \cdot\sum_a \zeta_a^3(1-\zeta_a) + \frac{4}{N^2}\sum\sum_{a\neq b} \zeta_a^2\zeta_b(1-\zeta_b) - \frac{10}{N}\left(1-\frac{1}{N}\right)\sum\sum_{a\neq b}\zeta_a^2(1-\zeta_a)\zeta_b \\ + \frac{4}{N^2}\sum\sum\sum_{a\neq b\neq c} \zeta_a(1-\zeta_a)\zeta_b\zeta_c\Big\} \equiv \sigma_\zeta^2\left(1+\frac{\alpha}{k}\right) \qquad \text{(say)} \end{aligned}$$

and similarly

(44) $$E^2(s_y) \equiv \sigma_\eta^2\left(1+\frac{\beta}{k}\right) \qquad \text{(say)}$$

Hence we have

(45) $$\begin{aligned} E(r) &= \frac{E\left(\frac{1}{N}\sum_a z_a y_a - \bar{z}\bar{y}\right)}{E(s_z)E(s_y)} + O\left(\frac{1}{k}\right) \\ &= \frac{\frac{1}{N}\sum_a \zeta_a\eta_a - \bar{\zeta}\,\bar{\eta} + \frac{\bar{\mu}_{11}}{k}\left(1-\frac{1}{N}\right)}{\sigma_\zeta\sigma_\eta\sqrt{1+\alpha/k}\sqrt{1+\beta/k}} + O\left(\frac{1}{k}\right) \\ &\equiv \rho + \frac{F}{k} \qquad \text{(say)} \end{aligned}$$

and

(46) $$E(r_s) = \frac{2\rho}{1+\rho} + O\left(\frac{1}{k}\right) = \rho_s + O\left(\frac{1}{k}\right)$$

We shall next calculate the variance of r_s. As is seen in § 4 we have the similar equation as (37) except the expectation in type-2 sampling. We proceed as before to compute r^2:

$$r^2=\frac{\frac{1}{N^2}(\sum_a z_a^2y_a^2+\sum\sum_{a\neq b} z_ay_az_by_b)+\bar{z}^2\bar{y}^2-\frac{2}{N}\bar{z}\bar{y}\sum_a z_ay_a}{s_z^2s_y^2}. \tag{47}$$

The numerator of r^2 is equal to

$$\begin{aligned}&\frac{1}{N^2}\sum_a z_a^2y_a^2+\frac{1}{N^2}\sum\sum_{a\neq b} z_ay_az_by_b+\frac{1}{N^4}(\sum_a z_a^2y_a^2+\sum\sum_{a\neq b} z_a^2y_b^2+2\sum\sum_{a\neq b} z_az_by_a^2\\&+\sum\sum\sum_{a\neq b\neq c} z_az_by_c^2+2\sum\sum_{a\neq b} y_ay_bz_a^2+\sum\sum\sum_{a\neq b\neq c} y_ay_bz_c^2+2\sum\sum_{a\neq b} z_az_by_ay_b\\&+4\sum\sum\sum_{a\neq b\neq c} z_az_by_as_c+\sum\sum\sum\sum_{a\neq b\neq c\neq d} z_az_by_cy_d)-\frac{2}{N^3}(\sum_a z_a^2y_a^2+\sum\sum_{a\neq b} z_ay_az_by_b\\&+\sum\sum_{a\neq b} z_a^2y_ay_b+\sum\sum_{a\neq b} z_az_by_b^2+\sum\sum\sum_{a\neq b\neq c} z_ay_bz_cy_c)\end{aligned} \tag{48}$$

Since, for mean m'_{10}, m'_{01} and sample size k, we can get the formula

$$\begin{aligned}E(m'_{10}m'_{01})^2=&\mu_{10}^2\mu_{01}^2+\frac{1}{k}(\mu_{02}\mu_{10}^2+\mu_{20}\mu_{01}^2+4\mu_{11}\mu_{10}\mu_{01})\\&+\frac{1}{k^2}(\mu_{20}\mu_{02}+2\mu_{12}\mu_{10}+2\mu_{21}\mu_{01}-4\mu_{11}\mu_{10}\mu_{01}+2\mu_{11}^2)\\&+\frac{1}{k^3}(\mu_{22}-\mu_{20}\mu_{02}+4\mu_{11}\mu_{10}\mu_{01}-2\mu_{11}^2)\end{aligned} \tag{49}$$

$$E\{(m'_{10})^2\cdot m'_{01}\}=\mu_{10}^2\mu_{01}+\frac{1}{k}(\mu_{20}\mu_{01}+2\mu_{11}\mu_{10})+\frac{1}{k^2}\mu_{31} \tag{50}$$

we have

$$\begin{aligned}E(\text{numerator of } r^2)=&\left(\frac{1}{N}\sum_a \zeta_a\eta_a-\bar{\zeta}\bar{\eta}\right)^2+\frac{1}{kN^4}\Big\{(N-1)^2\sum_a(\mu_{02,a}\zeta_a^2\\&+\mu_{20,a}\eta_a^2+4\mu_{11,a}\zeta_a\eta_a)+(N^2-2N+2)\sum\sum_{a\neq b}(\zeta_b\eta_b\mu_{11,a}\\&+\zeta_a\eta_a\mu_{11,b})+2\sum\sum_{a\neq b}\zeta_a\zeta_b^2(1-\zeta_a)-2(N-1)\sum\sum_{a\neq b}\zeta_b(\mu_{02,a}\zeta_a\\&+2\mu_{11,a}\eta_a)-2(N-1)\sum\sum_{a\neq b}\eta_b(\mu_{20,a}\eta_a+2\mu_{11,a}\zeta_a)\\&+\sum\sum\sum_{a\neq b\neq c}(\zeta_a\zeta_b\eta_c(1-\eta_c)+\eta_a\eta_b\zeta_c(1-\zeta_c))-2(N-2)\sum\sum\sum_{a\neq b\neq c}\zeta_b\eta_c\mu_{11,a}\Big\}\\\equiv&\left(\frac{1}{N}\sum_a \zeta_a\eta_a-\bar{\zeta}\bar{\eta}\right)^2+\frac{\gamma}{k}\qquad\text{(say)}\end{aligned} \tag{51}$$

Therefore we get from (47), (51) and (20)

$$E(r^2)=\frac{\left(\frac{1}{N}\sum_a \zeta_a\eta_a-\bar{\zeta}\bar{\eta}\right)^2+\frac{\gamma}{k}}{(\sigma_\zeta^2+\delta/k)(\sigma_\eta^2+\varepsilon/k)}+O\left(\frac{1}{k}\right)\equiv\rho^2+\frac{G}{k}\qquad\text{(say)} \tag{52}$$

where we put

(53)
$$\begin{cases} \delta=\dfrac{1}{N}\left(1-\dfrac{1}{N}\right)\sum\limits_{a}\zeta_a(1-\zeta_a) \\ \varepsilon=\dfrac{1}{N}\left(1-\dfrac{1}{N}\right)\sum\limits_{a}\eta_a(1-\eta_a) \end{cases}$$

Thus we finally get the variance of r_s

(54)
$$D^2(r_s)\doteqdot\frac{4}{(1+\rho)^4}D^2(r)=\frac{4}{(1+\rho)^4}\left\{\rho^2+\frac{G}{k^2}-\left(\rho+\frac{F}{k}\right)^2\right\}$$
$$=\frac{4(G-2\rho F)}{k(1+\rho)^4}+O\left(\frac{1}{k^2}\right)$$

As is easily shown it holds

$$G-2\rho F=\rho^2 H$$

so that we have

(55)
$$D^2(r_s)=\frac{\rho_s^2 H}{k(1+\rho)^2}+O\left(\frac{1}{k^2}\right)$$

where with the relation $\sigma_\eta=\sigma_\zeta$ for large N

(56)
$$H\doteqdot\frac{\gamma}{\{\text{cov}\,(\eta,\,\zeta)\}^2}-\frac{2\bar{\mu}_{11}}{\text{cov}\,(\eta,\,\zeta)}$$

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REFERENCE

[1] F.M. Lord, Sampling fluctuations resulting from the sampling of test items, *Psychometrika*, vol. 20, No. 1, 1955.