ON THE DISTRIBUTION OF THE PRODUCT OF TWO Γ-DISTRIBUTED VARIABLES

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1. In this note the distribution function for the product of two variable X_1, X_2, which are distributed independently according to $\frac{\alpha_i^{\lambda_i}}{\Gamma(\lambda_i)} e^{-\alpha_i x} x^{\lambda_i - 1}\, dx (i=1, 2)$, is obtained. It takes a very simple form when $\lambda_2 - \lambda_1 = n + \frac{1}{2}$ holds for some integer $n \geq 0$. This distribution can be applied to solve some problems on microwave transmission.

2. When the two independent random variables X_1, X_2 follow the distribution $p_{x_1}(x)dx$, $p_{x_2}(x)dx$ respectively, the distribution $p_{x_1x_2}(x)dx$ for the product X_1X_2 is given by $p_{x_1x_2}(x) = \int p_{x_1}(\xi) p_{x_2}\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}$. Thus for $p_{x_i}(x)\,dx = \frac{\alpha_i^{\lambda_i}}{\Gamma(\lambda_i)} e^{-\alpha_i x} x^{\lambda_i - 1}\, dx$, we have

$$
\begin{aligned}
p_{x_1x_2}(x) &= \int_0^\infty \frac{\alpha_1^{\lambda_1}}{\Gamma(\lambda_1)} e^{-\alpha_1 \xi}\, \xi^{\lambda_1 - 1} \frac{\alpha_2^{\lambda_2}}{\Gamma(\lambda_2)} e^{-\alpha_2 (x/\xi)} \left(\frac{x}{\xi}\right)^{\lambda_2 - 1} \frac{d\xi}{\xi} \\
&= \frac{\alpha_1^{\lambda_1}\alpha_2^{\lambda_2} x^{\lambda_2 - 1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty e^{-\alpha_1\xi - \alpha_2(x/\xi)} \frac{d\xi}{\xi^{\lambda_2 - \lambda_1 + 1}} \\
&= \frac{\alpha_1^{\lambda_1}\alpha_2^{\lambda_2} x^{\lambda_2 - 1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty e^{-\alpha_1\xi - (\alpha_1\alpha_2 x/\alpha_1\xi)} \frac{\alpha_1^{\lambda_2 - \lambda_1}}{(\alpha_1\xi)^{\lambda_2 - \lambda_1 + 1}} d(\alpha_1\xi) \\
&= \frac{\alpha_1^{\lambda_2}\alpha_2^{\lambda_2} x^{\lambda_2 - 1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty e^{-t - (\alpha_1\alpha_2 x/t)} \frac{dt}{t^{\lambda_2 - \lambda_1 + 1}}.
\end{aligned}
$$

It is well known in the theory of Bessel functions that the following formula holds:

$$
\int_0^\infty e^{-t - (z^2/4t)} \frac{dt}{t^{\nu+1}} = 2\left(\frac{z}{2}\right)^{-\nu} K_\nu(z) \qquad \cdots\cdots\cdots \text{(A)}
$$

where $K_\nu(z)$ is the modified Bessel function of the second kind of order ν (see [1]). Then we have

$$
p_{x_1x_2}(x) = \frac{2}{\Gamma(\lambda_1)\Gamma(\lambda_2)} (\alpha_1\alpha_2)^{(\lambda_1+\lambda_2)/2} (x)^{(\lambda_1+\lambda_2)/2 - 1} K_{\lambda_2 - \lambda_1}(2\sqrt{\alpha_1\alpha_2 x}).
$$

As for the distribution function of X_1X_2

$$\int_0^x p_{x_1x_2}(t)\,dt=\int_0^x \frac{2}{\Gamma(\lambda_1)\,\Gamma(\lambda_2)}(\alpha_1\alpha_2 t)^{(\lambda_1+\lambda_1)/2-1}K_{\lambda_2-\lambda_2}(2\sqrt{\alpha_1\alpha_2 t})\,d(\alpha_1\alpha_2 t),$$

we have, putting $z=2\sqrt{\alpha_1\alpha_2 t}$,

$$=\frac{2^{2-\lambda_1-\lambda_2}}{\Gamma(\lambda_1)\,\Gamma(\lambda_2)}\int_0^{2\sqrt{\alpha_1\alpha_2 x}} z^{\lambda_1+\lambda_2-1}K_{\lambda_2-\lambda_1}(z)dz.$$

By formula (12) in p. 80 of [1]

$$K_{n+1/2}(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}\sum_{r=0}^{n}\frac{(\mathrm{n}+r)!}{r!(n-r)!(2z)^r}, \qquad \cdots\cdots\cdots \text{(B)}$$

we have for the case $\lambda_2-\lambda_1=\mathrm{n}+1/2$

$$\int_0^x p_{x_1x_2}(t)\,dt=\frac{2^{2-\lambda_1-\lambda_2}}{\Gamma(\lambda_1)\,\Gamma(\lambda_2)}\sqrt{\frac{\pi}{2}}\sum_{r=0}^{n}\frac{(\mathrm{n}+r)!}{r!(\mathrm{n}-r)!}\left(\frac{1}{2}\right)^r\int_0^{2\sqrt{\alpha_1\alpha_2 x}} z^{\lambda_1+\lambda_2-1/2-r-1}e^{-z}\,dz$$

$$=\frac{2^{3/2-\lambda_1-\lambda_2}\sqrt{\pi}}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\sum_{r=0}^{n}\frac{(\mathrm{n}+r)!}{r!(n-r)!}\left(\frac{1}{2}\right)^r\Gamma\left(\lambda_1+\lambda_2-\frac{1}{2}-r\right)\times$$

$$\times I\left(2\sqrt{\alpha_1\alpha_2 x},\ \lambda_1+\lambda_2-\frac{1}{2}-r-1\right),$$

where $I(v,\,p)$ represents the incomplete Γ-function

$$I(v,\,p)=\int_0^v e^{-t}t^p\,dt\Big/\int_0^\infty e^{-t}t^p\,dt.$$

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REFERENCE

[1] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge, (1922), p. 183 formula (15).