# NOTE ON PREFERENCE AND AXIOMS OF CHOICE

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In this paper we shall show that the preference relation can be deduced from the plausible axioms concerning the choice function. This Cournot-type approach to the problem of choice has been pointed out by Arrow in his book, *Social Choice and Individual Volumes*, p. 11. We are persuing the idea expressed there (sections 1–31) and especially interested in applying the method to the theory of consumer's choice or to the theory of games, which is reduced to the problem of preference on a topological space (section 4).

## 1. Preference relation

We shall denote the alternatives by $x$, $y$, $\cdots$, and by $\Omega$ the set of all alternatives which are conceivably presented to the chooser. The set $\Omega$ need not be a finite set nor a set of vectors, but it may be an arbitrary set of elements.

A binary relation $P$ on $\Omega$ is said to be a *preference relation* if the following axioms are fulfilled:

P 1. For all $x \in \Omega$, $\overline{xPx}$.

P 2. For all $x$, $y$ and $z \in \Omega$, $xPy$ and $yPz$ imply $xPz$.

P 1 and P 2 are equivalent to P 1 and P 2′ :

P 2′. For all $x$ and $y \in \Omega$, $xPy$ implies $\overline{yPx}$.

A binary relation $R$ on $\Omega$ is said to be a *weak ordering* if the following axioms are fulfilled :

R 1. For all $x$ and $y \in \Omega$, $xRy$ or $yRx$.

R 2. For all $x$, $y$ and $z \in \Omega$, $xRy$ and $yRz$ imply $xRz$.

For a weak ordering $R$, we define a relation $P$ as follows :

For all $x$ and $y \in \Omega$, $xPy$, when $\overline{yRx}$.

Then the relation $P$ is a preference and is called the *preference associated with* a weak ordering $R$.

For a weak ordering $R$ and the associated preference $P$ we have

(a) $xIy$ if and only if $\overline{xPy}$ and $\overline{yPx}$,

(b) $xRy$ if and only if $xPy$ or $xIy$, where the relation $I$ is defined as follows:

For all $x$ and $y \in \Omega$, $xIy$, when $xRy$ and $yRx$.

When there is a preference $P$, we can define the relation $I$ and $R$ according to (a) and (b). The relation $R$, however, is not necessarily a weak ordering. $R$ is a weak ordering if and only if the relation $I$ is transitive:

For all $x$, $y$ and $z \in \Omega$, $xIy$ and $yIz$ imply $xIz$.

## 2. Choice function

Let $\mathfrak{B}$ be a class of non-empty sets of alternatives in $\Omega$, and $C$ a rule, which associates a non-empty subset $C(X)$ of $X$ with any $X \in \mathfrak{B}$. Such a rule is called a *choice function* on $\mathfrak{B}$. If, for any $X \in \mathfrak{B}$, $C(X)$ is a one-point set, the choice function $C$ is said to be *univalent*.

Suppose that there is a preference $P$ on $\Omega$, and let $\mathfrak{B}$ be a class of sets of alternatives in $\Omega$ so that, for any $X \in \mathfrak{B}$, the set

$$C(X) = \{x^\circ ; x^\circ \in X \text{ and } \overline{xPx^\circ} \text{ for all } x \in X\}$$

is non-empty. Then the choice function $C$ on $\mathfrak{B}$ is said to be *derived* from preference relation $P$.

For a choice function $C$ on a class $\mathfrak{B}$, we shall define binary relations $\tilde{P}$ and $P^*$ as follows:

(a) For all $x$ and $y \in \Omega$, $x\tilde{P}y$, when there exists a set $X$ in $\mathfrak{B}$ such that

$$x \in C(X) \quad \text{and} \quad y \in X - C(X).$$

(b) For all $x$ and $y \in \Omega$, $xP^*y$, when there exists a finite sequence of alternatives

$$x^1, \cdots, x^r \text{ such that}$$

$$x\tilde{P}x^1, x^1\tilde{P}^2, \cdots, x^{r-1}\tilde{P}^r, x^r\tilde{P}y.$$

$P^*$ is transitive, but not necessarily a preference relation.

A choice function $C$ on $B$ is said to be *rational*—the terminology used only in this paper—if the following axiom is satisfied:

C 1. For any $x$ and $y \in \Omega$, if there is a set $X \in \mathfrak{B}$ such that $x, y \in C(X)$, then $\overline{xP^*y}$.

THEOREM 1. *Let $P$ be the associated preference of a weak ordering $R$ and $C$ the derived choice function. Then $C$ is a rational choice and the relation $P^*$ has the following property:*

*For all $x$ and $y \in \Omega$, $xP^*y$ implies $xPy$.*

PROOF: For any $x$ and $y$ such that $x \in C(X)$ and $y \in X, y \in X - C(X)$ if and only if $xPy$. Because under the asumptions of the theorem we have

$$C(X) = \{x^\circ ; x^\circ \in X \text{ and } x^\circ Rx \text{ for all } x \in X\} .$$

Therefore, $x\tilde{P}y$ implies $xPy$. Since the relation $P$ is transitive, $xP^*y$ implies $xPy$.

To prove the rationality of $C$, assume that there existed a set $X \in \mathfrak{B}$ and alternatives $x$ and $y$ such that $x, y \in C(X)$ and $xP^*y$. Then we would have $xPy$ and $y \in X - C(X)$, which is a contradiction, q.e.d.

THEOREM 2. *If a choice function $C$ on a class $\mathfrak{B}$ is rational, then the relation $P^*$ is a preference relation and the choice function $C$ is the one derived from the preference $P^*$:*

$$C(X) = \{x^\circ ; x^\circ \in X \text{ and } \overline{xPx^\circ} \text{ for all } x \in X\}$$

for all $X \in \mathfrak{B}$.

PROOF: (1) Assume that $P^*$ does not satisfy P 1. Then there will exist an alternative $x \in \Omega$ such that $xP^*x$, consequently there must be a set $X \in \mathfrak{B}$ such that $x \in C(X)$, which contradicts to the rationality of $C$.

(2) The transitivity of $P^*$ is obvious.

(3) If $x^\circ \notin C(X)$ for $X \in \mathfrak{B}$, there will exist $x \in C(X)$ such that $xP^*x^\circ$. On the other hand, if $x^\circ \in C(X)$ and there existed $x^1 \in X$ such that $x^1P^*x^\circ$, then there would exist $x^2C(x)$ such that $x^2P^*x^\circ$, which is a contradiction, q.e.d.

If, for a preference $P$ and the derived choice function $C$ on a class $\mathfrak{B}$, we have

$$P = P^*$$

i.e. $xPy$ if and only if $xP^*y$, then the class $\mathfrak{B}$ is said to be *complete with respect to* preference $P$.

### 3. A special case

Suppose that a class $\mathfrak{B}$ contains all the sets of finite alternatives in $\Omega$. In this case, we can prove the following theorem.

THEOREM 3. *If $\mathfrak{B}$ contains all the sets of finite alternatives in $\Omega$, then $\mathfrak{B}$ is complete with respect to any preference relation $P$.*

PROOF: (1) We prove that $xP^*y$ if and only if $C(\{x, y\})=\{x\}$. It is evident that $C(\{x, y\})=\{x\}$ implies $xP^*y$. On the other hands, since $C$ is rational, $C(\{x, y\})=\{x\}$ implies $xP^*y$.

(2) To show $P=P^*$, it will be sufficient to prove that, for any $x$ and $y \in \Omega$, $xPy$ implies $xP^*y$. If $xPy$, then $C(\{x, y\})=\{x\}$. Therefore, $x\tilde{P}y$, and a fortiori, $xP^*y$, q.e.d.

THEOREM 4. *Let $\mathfrak{B}$ be a class which contains all the sets of finite alternatives in $\Omega$, and $C$ be an univalent choice function on $\mathfrak{B}$. Then $C$ is derived from a preference relation if and only if the following property is satisfied:*

C 2. *For all $X$ and $Y \in \mathfrak{B}$ such that $X \subset Y$,*

$$X-C(X) \subset Y-C(Y).$$

This shows that, if an alternative in $X$ is not chosen in $X$, it must not be chosen in any environment which contains $X$. It may be considered that the choice which violates this condition is extremely irrational in any sense.

PROOF: (1) It is evident that the derived choice function has the property C 2.

(2) Since, for any $x, y \in \Omega$, $xP^*y$ implies $x\tilde{P}y$, it is easily seen that the choice function $C$ which satisfies C2 must satisfy C 1, q.e.d.

## 4. Preference on a topological space

If $\Omega$ is a topological space, then a preference relation $P$ on $\Omega$ is said to be *compatible* with the topology on $\Omega$ if, for any alternative $x^\circ \in \Omega$, the sets

$$\{x; x \in \Omega \text{ and } xPx^\circ\} \text{ and } \{x; x \in \Omega \text{ and } x^\circ Px\}$$

are open in $\Omega$.

For any preference $P$ on $\Omega$, there is at least a topology on $\Omega$ such that the preference $P$ is compatible with that topology.

If $P$ is compatible, for any alternative $x^\circ \in \Omega$, the sets

$$\{x; x \in \Omega \text{ and } \overline{xPx^\circ}\} \text{ and } \{x; x \in \Omega \text{ and } \overline{x^\circ Px}\}$$

are closed in $\Omega$.

THEOREM 5. *Let $P$ be a compatible preference on a topological space which is derived from a weak ordering $R$. Then, for any non-empty and compact set $X$, $C(X)$ is non-empty and compact.*

PROOF: (1) Set

$$Z_x = \{z;\ z \in X,\ \overline{xPz}\}$$

for any $x \in X$.

For any finite set of alternatives $x^1, \cdots, x^s \in X$, we have

$$x^i \in Z_{x^1} \cap \cdots \cap Z_{x^s}$$

for at least one index $i$.

We shall prove this by the induction with respect to the numbers of alternatives. For $s=1$, $\overline{x^1Px^1}$, hence $x^1 \in Z_{x^1}$. We assume that this holds for any set of $s-1$ alternatives. Then there is an index $i$ such that

$$1 \leqq i \leqq s-1,$$

$$x^i \in Z_{x^1} \cap \cdots \cap Z_{x^{s-1}},$$

i.e.

$$\overline{x^1Px^i}, \cdots, \overline{x^{s-1}Px^i}.$$

When $\overline{x^sPx^i}$, we have $x^i \in Z_{x^1} \cap \cdots \cap Z_{x^s}$.

When $x^sPx^i$, we must have $x^s \in Z_{x^1} \cap \cdots \cap Z_{x^s}$. Because, if there existed an index $j$ such that

$$x^jPx^s,$$

then $1 \leqq j \leqq s-1$, and $x^jPx^i$, which would contradict $\overline{x^jPx^i}$.

(2) Since, for any $x \in X$, $Z_x$ is a closed subset of a compact set $X$, and

$$Z_{x^1} \cap \cdots \cap Z_{x^s} \neq \phi$$

for any finite set of alternatives $x^1, \cdots, x^s \in X$, we have

$$\bigcap_{x \in X} Z_x \neq \phi.$$

Therefore, there exists an alternative $x^\circ \in X$ such that

$$\overline{xPx^\circ} \quad \text{for all } x \in X.$$

This shows $x^\circ \in C(X)$.

(3)

$$X - C(X) = \bigcup_{x \in X} N_x,$$

where $N_x = \{z; z \in X, xPz\}$. Because, for $x^\circ \in X$, $x^\circ \notin C(X)$ if and only if there exists an alternative $x$ such that

$$xPx^\circ, \text{ i.e. } x^\circ \in N_x .$$

(4) Since, for any $x \in X$, $N_x$ is open in $X$, $X - C(X)$ is open. Hence $C(X)$ is closed in $X$. From the compactness of $X$, $C(X)$ is compact, q.e.d.

THEOREM 5. *Let $P$ be a compatible preference on a topological space. If, for any alternatives $x$ and $x^\circ$ such that*

$$\overline{xPx^\circ} \text{ and } \overline{x^\circ Px} ,$$

*every neighborhood of $x$ contains alternatives $y$ and $z$ so that $x^\circ Py$ and $zPx^\circ$, then $P$ is the associated preference of a weak ordering.*

PROOF: It will be sufficient to prove the transitivity of the relation $I$, where $xIy$ means $\overline{xPy}$ and $\overline{yPx}$. For that purpose we assume the there are three alternatives $x$, $y$ and $z$ such that

$$xIy,\ yIz \text{ and } \overline{xIz} .$$

Since $xPz$ or $zPx$ and the relation $I$ is symmetric, we may assume that $xPz$. The set

$$\{u; u \in \Omega \text{ and } xPu\}$$

is open, and then there exists a neighborhood $V$ of $z$ such that

$$V \subset \{u; x \in \Omega,\ xPv\} .$$

Then there is at least one alternative $u$ in $v$ so that $uPy$. Since $xPu$, we must have $xPy$, which contradicts $xIy$, q.e.d.