ON THE EVALUATION OF THE SAMPLING ERROR OF A CERTAIN DETERMINANT

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(Received Feb. 25, 1956)

1. Introduction

It happens often that we have to treat the sampling error of a determinant, all elements of which are submitted to measurement error or sampling error. In this paper we shall consider the evaluation of the maximum variance of a determinant under a certain condition.

2. Main results.

Let $X=|x_{ij}|$ be a determinant of k-th order, and x_{ij} the (i, j)-element. We assume that the following conditions are satisfied.

Condition (A):

(i) x_{ij} are independent random variables with non-negative real values,

(ii) the expectation and the variance of x_{ij} are respectively

$$E(x_{ij})=a_{ij},$$

$$D^2(x_{ij})=\sigma_{ij}^2=\sigma^2 \quad \text{for all} \quad i,\ j\,,$$

(iii) $\max\limits_{i,j}(a_{ij})=M$ and $\min\limits_{i,j}(a_{ij})=m$.

The condition (i) is only for the sake of simplicity, but in general we shall be able to get a smaller bound of the sampling error of X (see § 3).

Expanding X we have

$$X=\sum \operatorname{sgn}(i)\, x_{1i}\, x_{2i_2} \cdots x_{ki_k} \tag{1}$$

where 'sgn (i)' means $+$ or $-$ according to whether the permutation $\begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$ is even or odd.

Now, as for the expectation of X, we have

$$E(X)=\sum \operatorname{sgn}(i)\, a_{1i_1} \cdots a_{ki_k}=|a_{ij}| \tag{2}$$

and as for the variance of X we have

$$D^2(X)=E(X^2)-\{E(X)\}^2 \tag{3}$$

where

$$X^2=\sum_i x_{1i}^2 \cdots x_{ki_k}^2+2\sum_{i<j}\sum \operatorname{sgn}(i)\operatorname{sgn}(j)x_{1i_1}\cdots x_{ki_k}x_{1j_1}\cdots x_{kj_k} \tag{4}$$

The computation in taking the expectation of the second term of (4) is somewhat complicated. We calculate it in several steps as follows.

At first we introduce the notation $\begin{bmatrix} p_1 & p_2 & \cdots & p_k \\ q_1 & q_2 & \cdots & q_k \end{bmatrix}$ which means the product of two permutations $\begin{pmatrix} 1 & 2 & \cdots & k \\ p_1 & p_2 & \cdots & p_k \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & \cdots & k \\ q_1 & q_2 & \cdots & q_k \end{pmatrix}$. Further $\begin{bmatrix} p_1 & p_2 & \cdots & p_k \\ p_1 & q_2 & \cdots & q_k \end{bmatrix}$ is denoted by $\begin{bmatrix} p_1^2 & p_2 & \cdots & p_k \\ & q_2 & \cdots & q_k \end{bmatrix}$.

For example, $\begin{bmatrix} 1^2 & 2 & 3 & 4 \\ & 3 & 4 & 2 \end{bmatrix}$ means $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}$.

With these notations we can express the equation (4) for $k=3$ as follows.

$$
\begin{aligned}
X^2 = & [1^2\, 2^2\, 3^2] + [2^2\, 3^2\, 1^2] + [3^2\, 1^2\, 2^2] + [1^2\, 3^2\, 2^2] + [2^2\, 1^2\, 3^2] + [3^2\, 2^2\, 1^2] \\
& -2\left(\begin{bmatrix} 1^2 & 2 & 3 \\ & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2^2 & 3 & 1 \\ & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3^2 & 1 & 2 \\ & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3^2 \\ 2 & 1 & \end{bmatrix} + \begin{bmatrix} 1 & 2^2 & 3 \\ 3 & & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3^2 & 1 \\ 1 & & 2 \end{bmatrix} \right. \\
& \left. + \begin{bmatrix} 2 & 3 & 1^2 \\ 3 & 2 & \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2^2 \\ 1 & 3 & \end{bmatrix} + \begin{bmatrix} 3 & 1^2 & 2 \\ 2 & & 3 \end{bmatrix} \right) + 2\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \right. \\
& \left. + \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right). \qquad (5)
\end{aligned}
$$

From this equation (5) we have

$$
\begin{aligned}
E(X^2) = & \sum (a_{11}^2 + \sigma^2)(a_{22}^2 + \sigma^2)(a_{33}^2 + \sigma^2) - 2\sum (a_{11}^2 + \sigma^2) a_{22} a_{23} a_{33} a_{32} \\
& + 2\sum a_{11} a_{12} a_{22} a_{23} a_{33} a_{31}.
\end{aligned}
$$

Thus, in order to evaluate $D^2(X)$, we need only to calculate the number of terms with the type $\begin{bmatrix} p_1 & \cdots & p_k \\ q_1 & \cdots & q_k \end{bmatrix}$. Let us say that a permutation $\begin{pmatrix} 1 & 2 & \cdots & k \\ p_1 & p_2 & \cdots & p_k \end{pmatrix}$ has a different type from $\begin{pmatrix} 1 & 2 & \cdots & k \\ q_1 & q_2 & \cdots & q_k \end{pmatrix}$ when $p_i \neq q_i$ for all i.

LEMMA 1. Suppose $n(\geqq 2)$ different numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are given. For a fixed even (odd) permutation, we denote $f(n)$ the number of even (odd) permutations with different type from that fixed permutation. Similarly, we donote $g(n)$ the number of odd (even) permutation with different type from that fixed permutation. Then we have

$$
f(n) = (n-1)\{g(n-1) + g(n-2)\}, \qquad n \geqq 2 \qquad (6)
$$

PROOF. For small n we can easily prove this relation by the representation of cycles.

For example, when $n=3$, we have $f(3)=2$, $g(3)=0$.

Even permutation	Representation by cycles	Odd permutation	Representation by cycles
$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	e (identity)	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	(1 2)
$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	(1 2 3)	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	(2 3)
$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	(1 3 2)	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	(1 3)

When we fix the permutation $\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$, then even permutations of different types from that permutation are $\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$ and $\begin{pmatrix}1&2&3\\3&1&1\end{pmatrix}$, because $\begin{bmatrix}2&3&1\\1&2&3\end{bmatrix}$ and $\begin{bmatrix}2&3&1\\3&1&2\end{bmatrix}$ have different type from each other. But the odd permutation does not exist, because $\begin{bmatrix}2&3&1\\2&1&3\end{bmatrix}$, $\begin{bmatrix}2&3&1\\1&3&2\end{bmatrix}$ and $\begin{bmatrix}2&3&1\\3&2&1\end{bmatrix}$ are not different as they have the same element 2^2, 3^2 and 1^2, respectively. Similarly, we have the following table.

n	2	3	4	5	6	7	8
$f(n)$	0	2	3	24	130	930	7413
$g(n)$	1	0	6	20	135	924	7420

In order to prove (6) for the general case we use the mathematical induction.

Suppose we have proved (6) for n not greater than n_0. Then we have

$$f(n)=(n-1)\{g(n-1)+g(n-2)\} \quad \text{for } n\leqq n_0$$

At first we represent every permutation by means of cycles. In order to compute $f(n_0+1)$ we pick up anyone of $g(n_0)$ permutations. If this permutation is represented by the cycle $(\alpha_1, \alpha_2, \cdots, \alpha_{n_0})$, then the number of new permutations such as $(\alpha_1, \alpha_{n_0+1}, \alpha_2, \cdots, \alpha_{n_0})$, $(\alpha_1, \alpha_2, \alpha_{n_0+1}, \cdots, \alpha_{n_0})$, etc. is n_0. Similarly, in the case of the cycle (α_1, α_2) (α_3, α_4) $\cdots(\alpha_{n_0-1}, \alpha_{n_0})$ we have new n_0 permutations $(\alpha_1, \alpha_{n_0+1}, \alpha_2)$ $(\alpha_3, \alpha_4)\cdots$ $(\alpha_{n_0-1}, \alpha_{n_0})$, $(\alpha_1, \alpha_2, \alpha_{n_0+1})$ (α_3, α_4) $\cdots(\alpha_{n_0-1}, \alpha_{n_0})$, etc.

Thus we can always make the new permutations of order n_0+1 from $g(n_0)$ permutations of order n_0 and in consequence we have $n_0 g(n_0)$ permutations. Besides of these new permutations we can make new permutations of order n_0+1, which do not coincide with those derived from $g(n_0)$ permutations of order n_0, from $g(n_0-1)$ permutations of order n_0-1.

For example we get n_0 new permutations $(\alpha_1, \alpha_2, \cdots, \alpha_{n_0-1})(\alpha_{n_0}, \alpha_{n_0+1})$, $(\alpha_{n_0}, \alpha_2, \cdots, \alpha_{n_0-k})(\alpha_1, \alpha_{n_0+1})$, $\cdots$, and $(\alpha_1, \alpha_2, \cdots, \alpha_{n_0-2}, \alpha_{n_0})(\alpha_{n_0-1}, \alpha_{n_0})$ which are not found in the previous construction. Similarly, from

$(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)\cdots(\alpha_{n_0-2}, \alpha_{n_0-1})$ we get new n_0 permutations $(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)\cdots(\alpha_{n_0-2}, \alpha_{n_0-1})(\alpha_{n_0}, \alpha_{n_0+1})$, $(\alpha_{n_0}, \alpha_2)(\alpha_3, \alpha_4)\cdots(\alpha_{n_0-2}, \alpha_{n_0-1})(\alpha_1, \alpha_{n_0+1})$, $(\alpha_1, \alpha_{n_0})(\alpha_3, \alpha_4)\cdots(\alpha_{n_0-2}, \alpha_{n_0-1})(\alpha_2, \alpha_{n_0+1})$, $\cdots$, and $(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)\cdots(\alpha_{n_0-2}, \alpha_{n_0})(\alpha_{n_0-1}, \alpha_{n_0+1})$.

Thus we have in general

$$f(n_0+1)=n_0\{g(n_0)+g(n_0-1)\}$$

which proves our lemma 1.

n	Type of cycles in $f(n)$ permutations	Type of cycles in $g(n)$ permutations
3	(123)	—
4	(12)(34)	(1234)
5	(12345)	(12)(345)
6	(12)(3456) (123)(456)	(123456) (12)(34)(56)
7	(1234567) (12)(34)(567)	(12)(34567) (123)(4567)
8	(12)(345678) (123)(45678) (1234)(5678) (12)(34)(56)(78)	(12345678) (12)(34)(5678) (12)(345)(678)

LEMMA 1′. It holds that for $n\geqq 2$

$$g(n)=(n-1)\{f(n-1)+f(n-2)\} \tag{7}$$

LEMMA 2. We have

$$g(n)={}_nP_{n-2}\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots\pm\frac{1}{(n-2)!}\right), \quad \text{for} \quad n\geqq 4 \tag{8}$$

where ${}_nP_{n-2}$ is the number of permutations of $n-2$ numbers taken out of n numbers at a time, and sign $+$ holds for even n, sign $-$ holds for odd n.

PROOF. From lemmas 1 and 1′ we get

$$g(n)=(n-1)(n-2)\{g(n-2)+g(n-3)\}+(n-1)(n-3)\{g(n-3)+g(n-4)\}$$

Applying the mathematical induction we can easily prove the lemma.

LEMMA 3. It holds that for $n\geqq 2$

$$g(n)+\binom{n}{1}g(n-1)+\binom{n}{2}g(n-2)+\cdots+\binom{n}{n-2}g(2)={}_nP_{n-2} \tag{9}$$

PROOF. Inserting the equation $g(n)$ of lemma 2 into the left side of the equation (9) we can prove this lemma.

Now we can prove that following theorem.

THEOREM. *Under the condition (A) we have*

$$\operatorname{Max} D^2(X)=k!\left\{\sigma^{2k}+k\sigma^{2(k-1)}M^2+{}_kP_{k-2}\sum_{r=2}^{k-1}\frac{\sigma^{2(k-r)}}{(k-r)!}(M^{2r}-m^{2r})\right\} \tag{10}$$

PROOF. Expanding X^2 in terms of type $\begin{bmatrix} p_1 \cdots p_k \\ q_1 \cdots q_k \end{bmatrix}$, we have

$$\begin{aligned} X^2=&\sum\begin{bmatrix} p_1^2p_2^2\cdots p_k^2 \end{bmatrix}+\sum\begin{bmatrix} p_1^2p_2^2\cdots p_{k-2}^2 & p_{k-1}\,p_k \\ & q_{k-1}\,q_k \end{bmatrix} \\ &+\sum\begin{bmatrix} p_1^2p_2^2\cdots p_{k-4}^2 & p_{k-3}\,p_{k-2}\,p_{k-1}\,p_k \\ & q_{k-3}\,q_{k-2}\,q_{k-1}\,q_k \end{bmatrix} \\ &+\cdots+\sum\begin{bmatrix} p_1^2 & p_2\,p_3\cdots p_k \\ & q_2\,q_3\cdots q_k \end{bmatrix}+\sum\begin{bmatrix} p_1\,p_2\cdots p_k \\ q_1\,q_2\cdots q_k \end{bmatrix} \end{aligned} \tag{11}$$

Then we obtain

$$\begin{aligned} E(X^2)=&\sum(a_{1p_1}^2+\sigma^2)(a_{2p_2}^2+\sigma^2)\cdots(a_{kp_k}^2+\sigma^2) \\ &\pm\sum(a_{1p_1}^2+\sigma^2)\cdots(a_{k-2,\,p_{k-2}}^2+\sigma^2)\,a_{k-1,\,p_{k-1}}\,a_{k-1,\,q_{k-1}}\,a_{k,\,p_k}\,a_{k,\,q_k} \\ &\pm\cdots\cdots\pm\sum a_{1p_1}\,a_{1q_1}\,a_{2p_2}\,a_{2q_2}\cdots a_{k,\,p_k}\,a_{k,\,q_k} \end{aligned} \tag{12}$$

Putting a_{ij} equal to m for negative terms and M for positive terms, we have the upper bound of $D^2(X)$:

$$\max D^2(X)=k!\{\sigma^{2k}+c_1\sigma^{2(k-1)}M^2+\sum_{r=2}^{k-1}c_r\sigma^{2(k-r)}(M^{2r}-m^{2r})\}\ \text{(say)} \tag{13}$$

From the equation (12) we have easily

$$c_1 = k \tag{14}$$

and for $r \geqq 2$

$$c_r = g(r)\binom{k}{k-r} + g(r-1)\binom{k}{k-r+1}\binom{k-r+1}{1} + g(r-2)\binom{k}{k-r+2}\binom{k-r+2}{2}$$

$$+ \cdots\cdot + g(2)\binom{k}{k-2}\binom{k-2}{r-2}$$

$$= \binom{k}{k-r}\left\{ g(r) + \binom{r}{1} g(r-1) + \binom{r}{2} g(r-2) + \cdots\cdot + \binom{r}{r-2} g(2) \right\}$$

From lemma 3 we get

$$c_r = {}_rP_{r-2}\binom{k}{k-r} = \frac{{}_kP_{k-2}}{(k-r)!}$$

Thus we have proved the theorem.

3. Comparison with other result and some remark.

1. If we calculate the measuring error by Hadamard's theorem, we have

$$|\delta X| \leqq 3k^2(k-1)^{(k-1)/2} M^{k-1} \sigma \tag{15}$$

where $|\delta X|$ denotes the absolute value of the error of X derived from the error 3σ of x_{ij} (see [1]). For $M = 10\sigma$, $m = 3\sigma$, for instance, we have from (15)

$$|\text{maximum error}| = 3 \cdot 10^{k-1} \cdot k^2(k-1)^{(k-1)/2} \sigma^k \tag{15}$$

Comparing this bound and $3\sqrt{\max D^2(X)}$ for $k=3, 4, \cdots\cdot, 10$, we have the following table.

k	Hadamard (16)/σ^k	$3\sqrt{\max D^2(X)}/\sigma^k$
3	5400	1274.0
4	2.494×10^5	5.102×10^4
5	1.200×10^7	2.552×10^6
⋮	⋮	⋮
10	5.905×10^{15}	7.717×10^{15}

2. The estimation of the upper bound of the error in our theorem

is very crude. But in usual calculation we shall get smaller bound than that. For in the equation (13) the term $\sigma^{2(k-r)}(M^{2r}-m^{2r})$ comes from $E(x_{1,p_1}x_{1,q_1}\cdot\cdot x_{r,p_r}x_{r,q_r})-E(x_{1,p_1'}x_{1,q_1'}\cdot\cdot x_{r,p_r'}x_{r,q_r'})$. On the average we have approximately

$$\varepsilon E(x_{1p_1}x_{1q_1}\cdot\cdot x_{r,p_r}x_{rq_r})=\varepsilon(a_{1p_1}a_{1q_1}\cdot\cdot a_{r,q_r})=A_{2r}$$

where A_{2r} denotes the expected value of the product of different $2r$ factors of a_{ij} sampled from all a_{ij}. Therefore the principal part of (10) on the average is reduced to

$$k!\{\sigma^{2k}+k\sigma^{2(k-1)}M^2\} \tag{17}$$

and $3\sqrt{\max \varepsilon D^2(X)}/\sigma^k$ is calculated as follows.

k	$3\sqrt{\max \varepsilon D^2(X)}/\sigma^k$
3	127.5
4	294.3
5	735.6
⋮	⋮
10	1.808×10^5

3. Under the condition (B) that the random variables x_{ij} are not always non-negative, we shall get the equation (17) as the principal part of the upper bound of $D^2(X)$ by the similar procedure as that in 2.

4. In problems of quantification we cannot get the exact formulas of sampling errors, but we can estimate the upper bounds of errors of order $1/\sqrt{n}$ by this theorem. As for the details refer to the author's paper [1].

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REFERENCE

[1] H. Aoyama, On Sampling Errors in Certain Problems of Quantifications, *The Proceedings of the Institute of Statistical Mathematics*, vol. 2 No. 2, 1955 (in Japanese).