ON THE DISTRIBUTIONS OF THE HOTELLING'S T^2-STATISTICS

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1. Summary and introduction. The problems considered here are related to the multivariate generalization of the analysis of variance of R. A. Fisher based on the Hotelling's T^2-statistics. These were initially considered by H. Hotelling in connection with a wartime problem of air testing sample bombsights ([2], [3], [4]). Let $x_1, x_2, \cdots, x_p$ are normally correlated variates with zero means and variance-covariance matrix $\Lambda=(\lambda_{ij})$. If we know beforehand Λ, the statistic

$$\chi^2=\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^{ij}x_i x_j \tag{1}$$

can be used for a multivariate normal distribution as the appropriate statistic, where $(\lambda^{ij})=\Lambda^{-1}$ is the inverse of matrix Λ. But since Λ must in almost all practical cases be estimated from a old sample with (say) n degrees of freedom, we should use the Hotelling's generalized Student statistic

$$T^2=\sum_{i=1}^{p}\sum_{j=1}^{p}l^{ij}x_i x_j \tag{2}$$

in place of χ^2, where $L^{-1}=(l^{ij})$ is the inverse of the matrix $L=(l_{ij})$ of variance-convariance estimates derived from the old sample, and T^2 has the distribution determined by

$$\frac{1}{B\left(\frac{n-p+1}{2},\ \frac{p}{2}\right)}\cdot\frac{\left(\frac{T^2}{n}\right)^{(p-2)/2}}{\left(1+\frac{T^2}{n}\right)^{(n+1)/2}}\,d\left(\frac{T^2}{n}\right). \tag{3}$$

Suppose now that we take a new sample of N observations, and let $x_{i\alpha}$ be the α-th value on x_i among these N observations. Then the statistic (2) for the α-th observation may be written as

$$T_\alpha^2=\sum_{i=1}^{p}\sum_{j=1}^{p}l^{ij}x_{i\alpha}x_{j\alpha}. \tag{4}$$

Hotelling has considered the division of the sum over the new sample of T_α^2, e.i. $\sum_{\alpha=1}^{N}T_\alpha^2\equiv T_0^2$, into 'conditionally independent' components meaningful with respect to the causual system. By conditionally independent components we mean the ones which are mutually independent for fixed variance-covariance estimates. A simple example of the division of T_α^2 is given by

$$
\begin{aligned}
T_0^2 &= \sum_{\alpha=1}^{N} T_\alpha^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\,(\sum_{\alpha=1}^{N} x_{i\alpha}x_{j\alpha}) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\{\sum_{\alpha=1}^{N}(x_{i\alpha}-\bar{x}_i)(x_{j\alpha}-\bar{x}_j)\} + N\sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\,\bar{x}_i\bar{x}_j \qquad (5) \\
&= T_A^2 + T_B^2,
\end{aligned}
$$

where $\bar{x}_i = \sum_{\alpha=1}^{N} x_{i\alpha}/N$, $(i=1,\ 2,\ \cdot\cdot,\ p)$. Let $V=(v_{ij})$ be the matrix of v_{ij} which are the sums over the new sample of products of deviations of $x_{i\alpha}$ and $x_{j\alpha}$ from their respective least-square regression values upon a common set of independent variables, and let m be the degrees of freedom of v_{ij}. Then T_0^2, T_A^2 and T_B^2 in the above example are expressible in general form

$$
T^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\,v_{ij} \qquad (6)
$$

with $m=N$, $N-1$, and 1, respectively. Though various kinds of the partition of T_0^2 are possibly considered, we shall, in this paper, discuss the case where each component of T_0^2 has the form (6). It should be noted that $s_{ij}=v_{ij}/m$ is a unbiased estimate of λ_{ij} and that, if $m>p-1$, the joint distribution of s_{ij} or v_{ij} is the Wishart distribution with m degrees of freedom. If we know Λ exactly, we can obtain the general form

$$
\chi^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} \lambda^{ij}\,v_{ij} \qquad (7)
$$

and it has the well known χ^2 distribution with mp degrees of freedom.

In this paper we shall consider the sampling distributions of T^2-statistic in (6) and also of the ratio of two T^2's which are conditionally independent for fixed variance covariance estimates. For $m=1$, the distribution of T^2 is reduced to that described in (3) for $n>p$. For $p=2$, Hotelling has obtained the exact distribution of T^2 when $m>1$ and $n>2$, [4]. This is

$$
\begin{aligned}
P\{T^2>T'^2\} &= 1-I_w(m-1,\ n) \qquad (8) \\
&+\sqrt{\pi}\,\frac{\Gamma\left(\frac{m+n-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{1-w}{1+w}\right)^{(n-1)/2} I_{w^2}\left(\frac{m-1}{2},\ \frac{n+1}{2}\right),
\end{aligned}
$$

where $w=T'^2/(2n+T'^2)$ and $I_x(a, b)$ is the incomplete beta function [7]. We shall here study the distributions for any p and m when $n>p$, but these are not exact and for any probability level η, we shall obtain the $T^2(\eta)$ and $F(\eta)$ as the expanded forms in n^{-1} such that $P\{T^2>T^2(\eta)\}=\eta$ and $P\left\{\frac{T_1^2}{T_2^2}\frac{m_2}{m_1}>F(\eta)\right\}=\eta$, respectively.

2. $T^2(\eta)$; percentage point of the distribution of T^2. If population variance-covariance matrix Λ is known, the statistic (7), i.e, $\chi^2=\sum_{i=1}^{p}\sum_{i=1}^{p}\lambda^{ij}v_{ij}$, has χ^2-distribution with mp degrees of freedom. Hence we can write

$$P\{\chi^2=\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^{ij}v_{ij}\leqq\chi^2_{mp}(\eta)\}=G_\rho(\xi)=1-\eta, \tag{9}$$

where $\chi^2_{mp}(\eta)$ is the $\eta\times 100$ percentage point of χ^2-distribution with mp degrees of freedom and $\xi=\chi^2_{mp}(\eta)/2$, $\rho=mp/2$ and $G_\rho(\xi)=[\Gamma(\rho)]^{-1}\int_0^\xi t^{\rho-1}e^{-t}dt$. Since Λ is not actually known, we try to find $T^2(\eta)$ such that

$$P\{T^2=\sum_{i=1}^{p}\sum_{j=1}^{p}l^{ij}v_{ij}\leqq T^2(\eta)\}=G_\rho(\xi)=1-\eta. \tag{10}$$

The author has evaluated this percentage point in [8] by using the method which G. S. James devised for testing the multivariate linear hypothesis [6]. $T^2(\eta)$ obtained in this method can be written as an asymptotic series in n^{-1}, where n is the number of degrees of freedom for estimating Λ by an old sample and can be used for moderate values of n. The result*) is written as follows:

$$\begin{aligned}T^2(\eta)=\chi^2&+\frac{m}{2n}\Big[p(p+1)(\chi_4+\chi_2)+mp(\chi_4-\chi_2)\Big]\\&+\frac{m}{n^2}\Big\{\frac{m}{16}\Big(1-\frac{mp-2}{\chi^2}\Big)\Big[p(p+1)(\chi_4+\chi_2)+mp(\chi_4-\chi_2)\Big]^2\\&-\frac{m}{8}\Big[p(p+1)(\chi_4+\chi_2)+mp(\chi_4-\chi_2)\Big]\Big[p(p+1)(\chi_4-1)+mp(\chi_4-2\chi_2+1)\Big]\\&-\frac{1}{3}\Big[p(p^2+3p+4)(\chi_6+\chi_4+\chi_2)\\&+3mp(p+1)(\chi_6-\chi_2)+m^2p(\chi_6-2\chi_4+\chi_2)\Big]\end{aligned} \tag{11}$$

*) Dr. J. Ogawa informed me that this result coincides with that obtained by Mr. K. Ito of Nanzan University.

$$+\frac{1}{16}\Big[4p(2p^2+5p+5)(\chi_8+\chi_6+\chi_4+\chi_2)+16mp(p+1)(\chi_8-\chi_2)$$
$$+mp(p^3+2p^2+5p+4)(\chi_8+\chi_6-\chi_4-\chi_2)$$
$$+2m^2p(p^2+p+4)(\chi_8-\chi_6-\chi_4+\chi_2)+m^3p^2(\chi_8-3\chi_6+3\chi_4-\chi_2)\Big]\Big\}$$
$$+0\,(n^{-3}),$$

where $\chi^2 \equiv \chi^2_{mp}(\eta)$ and $\chi_{2s} \equiv \chi^{2s}_{mp}(\eta)/mp(mp+2)\cdots(mp+2s-2)$. If we put $m=1$, then $T^2(\eta)$ is reduced to the known formula [1],

$$T^2(\eta)=\chi^2\left\{1+\frac{p+\chi^2}{2n}+\frac{7p^2-4+(13p-2)\chi^2+4\chi^4}{24n^2}\right\}+0\,(n^{-3}). \tag{12}$$

In order to see the degree of approximation, we compare our $T^2(\eta)$ with the $\eta\times 100$ percentage point calculated by Hotelling's exact distribution function (8) when $p=2$. Table A shows this comparison for $\eta=0.05$.

The approximation for $n\geq 20$ is good enough to be used in practical problems.

Table A, Comparison. The upper figure (in bold type) is for (8) and the lower is $T^2\,(0.05)$

n \ m	2	5	10	20
10	**15.82** 15.20	**32.85** 31.19	**59.84** 56.71	**112.86** 104.82
20	**12.04** 11.97	**24.05** 23.86	**42.60** 42.27	**78.45** 77.46
29	**11.15** 11.12	**21.96** 21.95	**38.59** 38.22	**70.46** 70.12
39	**10.67** 10.67	**20.93** 20.93	**36.55** 36.54	**66.33** 66.18

3. Preparation for the next section. Let $\mathbf{y}_\alpha$ $(\alpha=1, \cdots, m)$ be m column vectors $\{y_{1\alpha}, y_{2\alpha}, \cdots, y_{p\alpha}\}$ which are independently distributed according to the same p-variate normal distribution which has null vector as mean and as variance-covariance matrix. Then the statistic (7), i.e. $\chi^2=\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^{ij}v_{ij}$, can be written as

$$\chi^2=\sum_{\alpha=1}^{m}\mathbf{y}'_\alpha\Lambda^{-1}\mathbf{y}_\alpha \tag{13}$$

To prove the above statements we first note that v_{ij} are the sums of products of deviations from the regression values by definition and may be expressible as $v_{ij}=\sum_{\alpha=1}^{m}y_{i\alpha}y_{j\alpha}$ by means of an orthogonal transformation, where $y_{i\alpha}$ is th i-th elements of $\mathbf{y}_\alpha$. Hence

$$\chi^2=\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^{ij}v_{ij}=\sum_{\alpha=1}^{m}\Big(\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^{ij}y_{i\alpha}v_{j\alpha}\Big)$$
$$=\sum_{\alpha=1}^{m}\boldsymbol{y}_\alpha'\boldsymbol{\Lambda}^{-1}\boldsymbol{y}_\alpha$$

which complets the proof.

Now we shall derive the expression of a distribution which is needed in the next section. Consider the two independent sets of the p-dimensional column vector variates, $\boldsymbol{t}_\alpha$ ($\alpha=1, 2, \cdots, m_1$) and $\boldsymbol{w}_\beta$ ($\beta=1, \cdots, m_2$), which are distributed independently according to the same p-dimensional normal distribution with the mean vector 0 and the variance-covariance matrix $2^{-1}(\boldsymbol{I}_p-\boldsymbol{\gamma})^{-1}$, where $\boldsymbol{I}_p$ denotes the unit matrix of degree p, $\boldsymbol{\gamma}=\mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$ and $|\gamma_i|<1$ for all i. It is necessary to obtain a simple form of

$$J=P\left\{\frac{\sum_{\alpha=1}^{m_1}\boldsymbol{t}_\alpha'\boldsymbol{t}_\alpha}{\sum_{\beta=1}^{m_2}\boldsymbol{w}_\beta'\boldsymbol{w}_\beta}\leqq\xi\right\} \tag{14}$$
$$=\pi^{-(\rho_1+\rho_2)}|\boldsymbol{I}_p-\boldsymbol{\gamma}|^{\frac{1}{2}(m_1+m_2)}\int_{R'}\exp\Big\{-\sum_{\alpha=1}^{m_1}\boldsymbol{t}_\alpha'(\boldsymbol{I}_p-\boldsymbol{\gamma})\boldsymbol{t}_\alpha$$
$$-\sum_{\beta=1}^{m_2}\boldsymbol{w}_\beta'(\boldsymbol{I}_p-\boldsymbol{\gamma})\boldsymbol{w}_\beta\Big\}\prod_\alpha d\boldsymbol{t}_\alpha\prod_\beta d\boldsymbol{w}_\beta,$$

where

$$\rho_1=\frac{1}{2}m_1p,\quad \rho_2=\frac{1}{2}m_2p\quad\text{and}\quad R':\ \sum_{\alpha=1}^{m_1}\boldsymbol{t}_\alpha'\boldsymbol{t}_\alpha\Big/\sum_{\beta=1}^{m_2}\boldsymbol{w}_\beta'\boldsymbol{w}_\beta\leqq\xi.$$

To do this, consider the distribution function of $\frac{1}{2}\sum_{\alpha=1}^{m_1}\boldsymbol{t}_\alpha'\boldsymbol{t}_\alpha$, i.e. $G_1(\xi')=P[\frac{1}{2}\sum_{\alpha=1}^{m_1}\boldsymbol{t}_\alpha'\boldsymbol{t}_\alpha\leqq\xi']$. According to James' calculation [5, p. 327], we obtain

$$G_1(\xi')=\pi^{-\rho_1}|\boldsymbol{I}_p-\boldsymbol{\gamma}|^{m_1/2}\sum_{\nu_{11},\nu_{12},\ldots,\nu_{pm_1}=0}^{\infty}\frac{\prod_{i=1}^{p}\gamma_i^{\sum_\alpha\nu_{i\alpha}}}{\prod_{i=1}^{p}\prod_{\alpha=1}^{m_1}\nu_{i\alpha}!}\frac{\prod_{i=1}^{p}\prod_{\alpha=1}^{m_1}\Gamma(\nu_{i\alpha}+\frac{1}{2})}{\Gamma(\sum_{i=1}^{p}\sum_{\alpha=1}^{m_1}\nu_{i\alpha}+\rho_1)}\times$$
$$\times\int_0^{\xi'}u^{\sum_i\sum_\alpha\nu_{i\alpha}+\rho_1-1}e^{-u}du. \tag{15}$$

Hence the frequency function $g(\xi')$ of $\frac{1}{2}\sum_{\alpha=1}^{m_1}\boldsymbol{t}_\alpha'\boldsymbol{t}_\alpha$ is

$$g_1(\xi')=\pi^{-\rho_1}|\boldsymbol{I}_p-\boldsymbol{\gamma}|^{m_1/2}\sum_{\nu_{11},\nu_{12},\ldots,\nu_{pm_1}=0}^{\infty}\frac{\prod_{i=1}^{p}\gamma_i^{\sum_\alpha\nu_{i\alpha}}}{\prod_{\xi=1}^{p}\prod_{\alpha=1}^{m_2}\nu_{i\alpha}!}\frac{\prod_{i=1}^{p}\prod_{\alpha=1}^{m_1}\Gamma(\nu_{i\alpha}+\frac{1}{2})}{\Gamma(\sum_{i=1}^{p}\sum_{\alpha=1}^{m_1}\nu_{i\alpha}+\rho_1)}\times$$
$$\times\xi'^{\sum_i\sum_\alpha\nu_{i\alpha}+\rho_1-1}e^{-\xi'}. \tag{16}$$

Similarly, the frequency function $g_2(\xi'')$ of $\frac{1}{2}\sum_{\beta=1}^{m_2}\boldsymbol{w}_\beta'\boldsymbol{w}_\beta$ can be obtained and seen to have the same form with $g_1(\xi')$ putting ξ'', ρ_2, m_2, μ and β in place of ξ' p_1, m_1, v and α, respectively. Since $\frac{1}{2}\sum_{\alpha=1}^{m_1} t'_\alpha t_\alpha$ and $\frac{1}{2}\sum_{\beta=1}^{m_2} w'_\beta w_\beta$ are independent, their joint frequency function is $g_1(\xi')g_2(\xi'')$. After making transformations, $\xi=\xi'/\xi''$ and $\zeta=\xi''$ and integrating out ζ, we have, as frequency function of $\sum_{\alpha=1}^{m_1} t'_\alpha t_\alpha / \sum_{\beta=1}^{m_2} w'_\beta w_\beta$,

$$g(\xi)=\pi^{-(\rho_1+\rho_2)}|\boldsymbol{I}_p-\boldsymbol{\gamma}|^{\frac{1}{2}(m_1+m_2)} \sum_{\nu_{11},\,\nu_{12},\,\dots,\,\nu_{pm_1}=0}^{\infty} \sum_{\mu_{11},\,\mu_{12},\,\dots,\,\mu_{pm_2}=0}^{\infty} \times$$

$$\times \frac{\prod_{i=1}^{p} \gamma_i^{\sum_\alpha \nu_{i\alpha}}}{\prod_{i=1}^{p}\prod_{\alpha=1}^{m_1}\nu_{i\alpha}!} \frac{\prod_{i=1}^{p} \gamma_i^{\sum_\beta \mu_{i\beta}}}{\prod_{i=1}^{p}\prod_{\beta=1}^{m_2}\mu_{i\beta}!} \frac{\prod_{i=1}^{p}\prod_{\alpha=1}^{m_1}\Gamma(\nu_{i\beta}+\frac{1}{2})}{\Gamma(\sum_{i=1}^{p}\sum_{\alpha=1}^{m_1}\nu_{i\alpha}+\rho_1)} \frac{\prod_{i=1}^{p}\prod_{\beta=1}^{m_2}\Gamma(\mu_{i\beta}+\frac{1}{2})}{\Gamma(\sum_{i=1}^{p}\sum_{\beta=1}^{m_2}\mu_{i\beta}+\rho_2)} \times$$

$$\times \Gamma(\textstyle\sum_{i=1}^{p}\sum_{\alpha=1}^{m_1}\nu_{i\alpha}+\sum_{i=1}^{p}\sum_{\beta=1}^{m_2}\mu_{i\beta}+\rho_1+\rho_2)\times$$

(17)

$$\times \frac{\xi^{\sum_i\sum_\alpha \nu_{i\alpha}+\rho_1-1}}{(1+\xi)^{\sum_i\sum_\beta \nu_{i\alpha}+\sum_i\sum_\beta \mu_{i\beta}+\rho_1+\rho_2}}$$

$$=|\boldsymbol{I}_p-\boldsymbol{\gamma}|^{\frac{1}{2}(m_1+m_2)} \left\{\prod_{i=1}^{p}\prod_{\alpha=1}^{m_1}\sum_{\nu_{i\alpha}=0}^{\infty} \frac{\frac{1}{2}\,\frac{3}{2}\cdots\left(\nu_{i\alpha}-\frac{1}{2}\right)}{\nu_{i\alpha}!}\gamma_i^{\nu_{i\alpha}}\right\}\times$$

$$\times\left\{\prod_{i=1}^{p}\prod_{\beta=1}^{m_2}\sum_{\mu_{i\beta}=0}^{\infty} \frac{\frac{1}{2}\,\frac{3}{2}\cdots\left(\mu_{i\beta}-\frac{1}{2}\right)}{\mu_{i\beta}!}\gamma_i^{\mu_{i\beta}}\right\}\times$$

$$\times\beta\,(\xi;\ \sum_{i=1}^{p}\sum_{\alpha=1}^{m_1}\nu_{i\alpha}+\rho_1,\ \sum_{i=1}^{p}\sum_{\beta=1}^{m_2}\mu_{i\beta}+\rho_2)$$

where

$$\beta(\xi;\ s,\ t)=[B(s,\ t)]^{-1}\,\xi^{s-1}/(1+\xi)^{s+t}.$$

If we define two operators, E_1, E_2, for a function $f(\rho_1, \rho_2)$ of ρ_1 and ρ_2, such that

(18) $$E_1^a f(\rho_1,\ \rho_2)=f(\rho_1+a,\ \rho_2),$$

(19) $$E_2^b f(\rho_1,\ \rho_2)=f(\rho_1,\ \rho_2+b),$$

(where a and b denote any positive integers), we can express $g(\xi)$ in the simple form

(20) $$g(\xi)=\left\{\frac{|\boldsymbol{I}_p-\boldsymbol{\gamma}E_1|}{|\boldsymbol{I}_p-\boldsymbol{\gamma}|}\right\}^{-m_1/2}\left\{\frac{|\boldsymbol{I}_p-\boldsymbol{\gamma}E_2|}{|\boldsymbol{I}_p-\boldsymbol{\gamma}|}\right\}^{-m_2/2}\beta(\xi;\ \rho_1,\ \rho_2),$$

since $|\gamma_i|<1$. Hence we obtain the expession for J as

(21) $$J=\left\{\frac{|\boldsymbol{I}_p-\boldsymbol{\gamma}E_1|}{|\boldsymbol{I}_p-\boldsymbol{\gamma}|}\right\}^{-m_1/2}\left\{\frac{|\boldsymbol{I}_p-\boldsymbol{\gamma}E_2|}{|\boldsymbol{I}_p-\boldsymbol{\gamma}|}\right\}^{-m_2/2}B(\xi;\ \rho_1,\ \rho_2),$$

where

$$B(\xi;\ \rho_1,\ \rho_2)=\int_0^{\xi}\beta(u;\ \rho_1,\ \rho_2)\,du.$$

4. $F(\eta)$; percentage point of the distribution of the ratio of Two T^2-statistics.

Let

$$T_1^2=\sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\, v_{ij}^{(1)}=m_1\sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\, s_{ij}^{(1)}$$

and

$$T_2^2=\sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\, v_{ij}^{(2)}=m_2\sum_{i=1}^{p}\sum_{j=1}^{p} l^{ij}\, s_{ij}^{(2)}$$

which are independent for fixed $\boldsymbol{L}=(l_{ij})$. In this section, we shall evaluate the $\eta\times 100$ percentage point $\boldsymbol{F}(\eta)$ of the distribution of the ratio, $\dfrac{T_1^2}{m_1 p}\Big/\dfrac{T_2^2}{m_2 p}=\dfrac{T_1^2}{T_2^2}\dfrac{m_2}{m_1}$. In the matrix notation, T_1^2 and T_2^2 may be written as

$$T_1^2=\mathrm{tr}\,\boldsymbol{L}^{-1}\boldsymbol{V}^{(1)},\quad T_2^2=\mathrm{tr}\,\boldsymbol{L}^{-1}\boldsymbol{V}^{(2)}$$

If Λ is known exactly, we use Λ in place of $\boldsymbol{L}$ and the statistic

$$\mathfrak{F}=\frac{\chi_1^2}{\chi_2^2}=\frac{\mathrm{tr}\,\Lambda^{-1}\boldsymbol{V}^{(1)}}{\mathrm{tr}\,\Lambda^{-1}\boldsymbol{V}^{(2)}} \tag{22}$$

has the frequency function

$$\beta\left(x;\frac{m_1 p}{2},\ \frac{m_2 p}{2}\right)=\frac{1}{B\left(\dfrac{m_1 p}{2},\ \dfrac{m_2 p}{2}\right)}\frac{x^{(m_1 p/2)-1}}{(1+x)^{p(m_1+m_2)/2}}. \tag{23}$$

Hence

$$P_r\left\{\mathfrak{F}=\frac{\mathrm{tr}\,\Lambda^{-1}\boldsymbol{V}^{(1)}}{\mathrm{tr}\,\Lambda^{-1}\boldsymbol{V}^{(2)}}\leqq\xi\right\}=B(\xi;\ \rho_1,\ \rho_2), \tag{24}$$

where $\dfrac{m_2}{m_1}\xi$ is the tabled value of F-distribution with $m_1 p$ and $m_2 p$ degrees of freedom for a particular probability, $\rho_1=\dfrac{1}{2}m_1 p$, $\rho_2=\dfrac{1}{2}m_2 p$ and

$$B(\xi;\ \rho_1,\ \rho_2)=\int_0^{\xi}\beta(t;\ \rho_1,\ \rho_2)\,dt.$$

We do not know Λ, so we set the problem to determine a function $h(l)$ of the elements l_{ij} of the matrix $\boldsymbol{L}$ and ξ, which is such that

$$P_r\left\{F=\frac{\mathrm{tr}\,\boldsymbol{L}^{-1}\boldsymbol{V}^{(1)}}{\mathrm{tr}\,\boldsymbol{L}^{-1}\boldsymbol{V}^{(2)}}\leqq h(l)\right\}=B(\xi;\ \rho_1,\ \rho_2) \tag{25}$$

as in Section 2. The solution can be derived by the analogous method of James' and expressible as a series in n^{-1}. In large samples $h(l)$ will approach ξ. Let

$$h(l)=\xi+h_1(l)+h_2(l)+\cdots, \tag{26}$$

where $h_\nu(l)$ is of order $n^{-\nu}$. Then, following the James' method, we obtain, for the equations giving $h_1(\lambda)$ and $h_2(\lambda)$, which are the functions putting λ_{ij} in place of l_{ij}, (see [6], p. 35)

$$\left[h_1(\lambda)D+\frac{1}{n}\sum_{rstu}\lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu}\right]P_r\left\{\frac{\operatorname{tr}\Lambda^{-1}V^{(1)}}{\operatorname{tr}\Lambda^{-1}V^{(1)}}\leqq\xi\right\}=0, \tag{27}$$

$$\begin{aligned}\Big[h_2(\lambda)D&+\frac{1}{2}h_1^2(\lambda)D^2+\frac{1}{n}\sum_{rstu}\lambda_{ur}\lambda_{st}\left(h_1^{(rs,tu)}(\lambda)D+2h_1^{(rs)}(\lambda)\partial_{tu}D+h_1(\lambda)\partial_{rs}\partial_{tu}D\right)\\&+\frac{4}{3}\frac{1}{n^2}\sum_{rstuvw}\lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw}\\&+\frac{1}{2}\frac{1}{n^2}\sum_{rstuvwxy}\lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\Big]\times\\&\times P_r\left\{\frac{\operatorname{tr}\Lambda^{-1}V^{(1)}}{\operatorname{tr}\Lambda^{-1}V^{(1)}}\leqq\xi\right\}=0,\end{aligned} \tag{28}$$

where

$$D\equiv\frac{\partial}{\partial\xi},\qquad \partial_{rs}\equiv\frac{1}{2}(1+\delta_{rs})\frac{\partial}{\partial\lambda_{rs}}\quad(\delta_{rs}\text{ is Kronecker's }\delta)$$

$$h_1^{(rs)}(\lambda)=\partial_{rs}h_1(\lambda),\ \cdots$$

and the sums are over $r, s\cdots=1, 2, \cdots, p$. In order to obtain $h_1(\lambda)$, $h_2(\lambda)$, and thence $h_1(l)$, $h_2(l)$, we must evaluate the derivatives in the above equations. To do this, we consider

$$J=P_r\left\{\frac{\operatorname{tr}(\Lambda+\varepsilon)^{-1}V^{(1)}}{\operatorname{tr}(\Lambda+\varepsilon)^{-1}V^{(2)}}\leqq\xi\right\}, \tag{29}$$

where ε is a symmetric matrix composing of the small increments ε_{ij} to λ_{ij}. Then, by Taylor's theorem,

$$J=\left[1+\sum_{rs}\varepsilon_{rs}\partial_{rs}+\frac{1}{2}\sum_{rstu}\varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu}+\cdots\right]P_r\left\{\frac{\operatorname{tr}\Lambda^{-1}V^{(1)}}{\operatorname{tr}\Lambda^{-1}V^{(1)}}\leqq\xi\right\}. \tag{30}$$

But we can also express J in the form

$$(31)\qquad J=\frac{1}{(2\pi)^{p_1+p_2}|\Lambda|^{(m_1+m_2)/2}}\times$$
$$\times\int_R \exp\left\{-\frac{1}{2}\sum_{\alpha=1}^{m_1} y'_\alpha \Lambda^{-1} y_\alpha-\frac{1}{2}\sum_{\beta=1}^{m_2} z'_\beta \Lambda^{-1} z_\beta\right\}\prod_\alpha dy_\alpha \prod_\beta dz_\beta$$

as obtained in § 3, where

$$(32)\qquad R;\quad \frac{\mathrm{tr}(\Lambda+\varepsilon)^{-1}V^{(1)}}{\mathrm{tr}(\Lambda+\varepsilon)^{-1}V^{(2)}}=\frac{\sum_{\alpha=1}^{m_1} y'_\alpha(\Lambda+\varepsilon)^{-1}y_\alpha}{\sum_{\beta=1}^{m_2} z'_\beta(\Lambda+\varepsilon)^{-1}z_\beta}\leqq \xi.$$

Now in (31) we consider the non-singular linear transformations

$$(33)\qquad y_\alpha=Ct_\alpha,\quad \alpha=1,\ \cdots,\ m_1,$$
$$z_\beta=Cw_\beta,\quad \beta=1,\ \cdots,\ m_2,$$

such that

$$(34)\qquad \frac{1}{2}C'(\Lambda+\varepsilon)^{-1}C=I_p\ ,$$

$$(35)\qquad \frac{1}{2}C'\Lambda^{-1}C=I_p-\gamma,$$

where I_p is the unit matrix of degree p and γ is a diagonal matrix, diag $\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$. It is obvious that $|\gamma_i|<1$ for all i, as we can choose the elements of ε sufficiently small. Under these transformations J becomes

$$(36)\qquad J=\pi^{-(p_1+p_2)}|I_p-\gamma|^{\frac{1}{2}(m_1+m_2)}\int_{R'}\left\{\exp-\sum_{\alpha=1}^{m_1}t'_\alpha(I_p-\gamma)t_\alpha\right.$$
$$\left.-\sum_{\beta=1}^{m_2} w'_\beta(I_p-\gamma)w_\beta\right\}\Pi dt_\alpha \Pi dw_\beta,$$

$$(37)\qquad R';\ \sum_{\alpha=1}^{m_1} t'_\alpha t_\alpha\Big/\sum_{\beta=1}^{m_2} w'_\beta w_\beta\leqq\xi.$$

Then we can express J in the simple form which is obtained in the last section, that is,

$$(38)\qquad J=\left\{\frac{|I_p-\gamma E_1|}{|I_p-\gamma|}\right\}^{-m_1/2}\left\{\frac{|I_p-\gamma E_2|}{|I_p-\gamma|}\right\}^{-m_2/2}B(\xi;\ p_1,\ p_2),$$

where E_1 and E_2 are the same operators as defined before. Since, from (34) and (35),

$$(39)\qquad \frac{|I_p-\gamma E_s|}{|I_p-\gamma|}=|I_p-[(\Lambda+\varepsilon)^{-1}\Lambda-I_p](E_s-1)|$$
$$=|I_p-X\Delta_s|,\qquad (s=1,\ 2)$$

we can express

$$J-\left\{|\boldsymbol{I}_p-\boldsymbol{X}\Delta_1|\right\}^{-m_1/2}\left\{|\boldsymbol{I}_p-\boldsymbol{X}\Delta_2|\right\}^{-m_2/2}B(\xi;\,\rho_1,\,\rho_2), \tag{40}$$

where

$$\Delta_s=E_s-1 \quad \text{and} \quad \boldsymbol{X}=(\boldsymbol{\Lambda}+\varepsilon)^{-1}\boldsymbol{\Lambda}-\boldsymbol{I}_p.$$

Then we can carry out the expansion of the above J in powers of ε_{rs} in the analogus way as [6] or [8]. Comparing this resulting expansion of J with that in (30), we can obtain the derivatives $\partial_{rs}P_r[\cdot\cdot]$, $\partial_{rs}\partial_{tu}P_r[\cdot\cdot]$, etc., after a good deal of algebra. Let us use the abbreviated notations

$$(rs)=\lambda^{rs}, \quad \boldsymbol{\Lambda}_{rs}=\partial_{rs}\boldsymbol{\Lambda}=\frac{1}{2}\Big(1+\delta_{rs}\Big)\frac{\partial}{\partial\lambda_{rs}}\boldsymbol{\Lambda},$$

$$[rs]=\operatorname{tr}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs}=(rs) \tag{41}$$

$$[rs|tu]=\operatorname{tr}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{tu}=\frac{1}{2}\Big((ur)(st)+(us)(rt)\Big) \tag{42}$$

$$\begin{aligned}[rs|tu|vw]=\operatorname{tr}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{tu}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{vw}=\frac{1}{8}\Big\{&(wr)(st)(uv)+(wr)(su)(tv)\\ &+(ws)(rt)(uv)+(ws)(ru)(tv)+(vr)(st)(uw)\\ &+(vr)(su)(tw)+(vs)(rt)(uw)+(vs)(ru)(tw)\Big\}\end{aligned} \tag{43}$$

....

Then we have

$$\partial_{rs}P_r\left\{\frac{\operatorname{tr}\boldsymbol{\Lambda}^{-1}\boldsymbol{V}^{(1)}}{\operatorname{tr}\boldsymbol{\Lambda}^{-1}\boldsymbol{V}^{(2)}}\leqq\xi\right\}=-\frac{1}{2}[rs](m_1\Delta_1+m_2\Delta_2)B(\xi;\,\rho_1,\,\rho_2), \tag{44}$$

$$\begin{aligned}\partial_{rs}\partial_{tu}P_r\{\cdot\cdot\}=-\frac{1}{2}\Big[&[rs|tu]\{m_1(2\Delta_1+\Delta_1^2)+m_2(2\Delta_2+\Delta_2^2)\}\\ &+\frac{1}{2}[rs][tu](m_1\Delta_1+m_2\Delta_2)^2\Big]B(\xi;\,\rho_1,\,\rho_2)\end{aligned} \tag{45}$$

$$\begin{aligned}\partial_{rs}\partial_{tu}\partial_{vw}P_r\{\cdot\cdot\}=-\Big[&[rs|tu|vw]\{m_1(3\Delta_1+3\Delta_1^2+\Delta_1^3)+m_2(3\Delta_2+3\Delta_2^2+\Delta_2^3)\}\\ &+\frac{3}{4}[rs][tu|vw]\{m_1^2(2\Delta_1^2+\Delta_1^3)+m_2^2(2\Delta_2^2+\Delta_2^3)+m_1m_2\Delta_1\Delta_2(4+\Delta_1+\Delta_2)\}\\ &+\frac{1}{8}[rs][tu][vw]\{m_1^3\Delta_1^3+m_2^3\Delta_2^3+3m_1m_2\Delta_1\Delta_2(m_1\Delta_1+m_2\Delta_2)\}\Big]B(\xi;\,\rho_1,\,\rho_2),\end{aligned} \tag{46}$$

$$\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}P_r\{\cdot\cdot\}=\Big[(2[rs|tu|vw|xy]+[rs|vw|tu|xy])\,\{m_1(4\Delta_1+6\Delta_1^2+4\Delta_1^3+\Delta_1^4)$$

$$+m_2(4\Delta_2+6\Delta_2^2+4\Delta_2^3+\Delta_2^4)\Big]$$

$$+2[rs][tu|vw|xy]\Big\{m_1^2(3\Delta_1^2+3\Delta_1^3+\Delta_1^4)+m_2^2(3\Delta_2^2+3\Delta_2^3+\Delta_2^4)$$

$$+m_1m_2(6\Delta_1\Delta_2+3\Delta_1^2\Delta_2+3\Delta_1\Delta_2^2+\Delta_1^3\Delta_2+\Delta_1\Delta_2^3)\Big\}$$

$$(47)\qquad +\frac{1}{4}\Big([rs|tu][vw|xy]+2[rs|vw][tu|xy]\Big)\Big\{m_1^2(4\Delta_1^2+4\Delta_1^3+\Delta_1^4)$$

$$+m_2^2(4\Delta_2^2+4\Delta_2^3+\Delta_2^4)+2m_1m_2(4\Delta_1\Delta_2+2\Delta_1^2\Delta_2+2\Delta_1\Delta_2^2+\Delta_1^2\Delta_2^2)\Big\}$$

$$+\frac{1}{4}\Big([rs][tu][vw|xy]+2[rs][vw][tu|xy]\Big)\Big\{m_1^3(2\Delta_1^3+\Delta_1^4)+m_2^3(2\Delta_2^3+\Delta_2^4)$$

$$+m_1^2m_2(\Delta_1^2\Delta_2^2+2\Delta_1^3\Delta_2+6\Delta_1^2\Delta_2)+m_1m_2^2(\Delta_1^2\Delta_2^2+2\Delta_1\Delta_2^3+6\Delta_1\Delta_2^2)\Big\}$$

$$+\frac{1}{16}[rs][tu][vw][xy]\Big\{m_1^4m_1^4+m_2^4\Delta_2^4+4m_1^3m_2\Delta_1^3\Delta_2+4m_1m_2^3\Delta_1\Delta_2^3$$

$$+6m_1^2m_2^2\Delta_1^2\Delta_2^2\Big\}B(\xi;\ \rho_1,\ \rho_2).$$

Substituting (45) into (27) and noting that

$$(48)\qquad \Delta_1B(\xi;\ \rho_1,\ \rho_2)=B(\xi;\ \rho_1+1,\ \rho_2)-B(\xi;\ \rho_1,\ \rho_2)=-\frac{1}{\rho_2-1}\beta(\xi;\ \rho_1+1,\ \rho_2-1),$$

$$(49)\qquad \Delta_2B(\xi;\ \rho_1,\ \rho_2)=B(\xi;\ \rho_1,\ \rho_2+1)-B(\xi;\ \rho_1,\ \rho_2)$$

$$=\frac{\rho_1}{\rho_2}\{B(\xi;\ \rho_1,\ \rho_2)-B(\xi;\ \rho_1+1,\ \rho_2)\}$$

$$=\frac{\rho_1}{\rho_2}\frac{1}{\rho_2-1}\beta(\xi;\ \rho_1+1,\ \rho_2-1),$$

we obtain

$$h_1(\lambda)=\frac{1}{4n}\sum_{rstu}\lambda_{ur}\lambda_{st}\Big[(\lambda^{ur}\lambda^{st}+\lambda^{us}\lambda^{rt})\Big\{m_1\Big(\frac{\rho_1+\rho_2}{\rho_1(\rho_1+1)}\frac{\xi^2}{1+\xi}+\frac{\xi}{\rho_1}\Big)$$

$$(50)\qquad -m_2\Big(\frac{\rho_1+\rho_2}{\rho_2(\rho_2+1)}\frac{\xi}{1+\xi}+\frac{\xi}{\rho_2}\Big)\Big\}$$

$$+\lambda^{rs}\lambda^{tu}\Big\{m_1^2\Big(\frac{\rho_1+\rho_2}{\rho_1(\rho_2+1)}\frac{\xi^2}{1+\xi}-\frac{\xi}{\rho_1}\Big)-m_2^2\Big(\frac{\rho_1+\rho_2}{\rho_2(\rho_2+1)}\frac{\xi}{1+\xi}-\frac{\xi}{\rho_2}\Big)$$

$$+2m_1m_2\Big(\frac{\rho_1+\rho_2}{\rho_1\rho_2}\frac{\xi}{1+\xi}-\frac{\xi}{\rho_1}\Big)\Big\}\Big]$$

$$=\frac{(m_1+m_2)}{2n}\Big[\Big\{\frac{p(p+m_1+1)}{m_1p+2}-1\Big\}\xi$$

$$-\Big\{\frac{p(p+m_1+1)}{m_1p+2}+\frac{p(p+m_2+1)}{m_2p+2}-2\Big\}\frac{\xi}{1+\xi}\Big].$$

Since $h_1(\lambda)$ is independent of λ, the derivatives like $h_1^{(rs)}(\lambda)$ are zero. Then substituting $h_1(\lambda)$, together with (45), (46) and (47) into (28), and putting $\xi = m_1/m_2\, F_\eta(m_1 p,\, m_2 p) \equiv m_1/m_2\, F$ (where $F_\eta(m_1 p,\, m_2 p)$ is the tabled value of F-distribution with $m_1 p$ and $m_2 p$ degrees of freedom for a particular significance level, η), we obtain, as the approximation of order n^{-2},

$$
\begin{aligned}
F(\eta) &= \frac{m_2}{m_1} h(l) \\
&= F + \frac{m_0}{2n}\{(pb_1\theta_1 - 1)F - (pb_1\theta_1 + pb_2\theta_2 - 2)\tau Q\} \\
&+ \frac{m_0}{n^2}\Bigg[\frac{m_0}{16}\left\{\frac{pm_2 m_0}{\tau(m_2 + m_1 F)} - \frac{pm_1 - 2}{F}\right\}\Big\{(pb_1\theta_1 - 1)F - (pb_2\theta_2 - 2)\tau Q\Big\}^2 \\
&- \frac{m_0}{8}\Big\{(pb_1\theta_1 - 1)F - (pb_1\theta_1 + pb_2\theta_2 - 2)\tau Q\Big\} \times \\
&\times \Big\{p(p+2)(p-1)m_1\theta_2 - 2(p+2)(p-1)\kappa_2 Q \\
&\qquad + (pm_1 + pm_2 + 2)(pb_1\theta_1 + pb_2\theta_2 - 2)Q^2\Big\} \\
&- \frac{1}{3}\Big\{[p\phi_1(c_1 + 3m_1 b_1) + p\theta_1(c_1 - 3m_2 b_1) - 3b_1 + 2(m_0 + m_1)]F \\
&\quad - [p\phi_1(c_1 + 3m_1 b_1) + p\phi_2(c_2 + 3m_2 b_2) + p\theta_1(c_1 - 3m_2 b_1) \\
&\qquad + p\theta_2(c_2 - 3m_1 b_2) - 3(b_1 + b_2 - 2m_0)]\tau Q \\
&\quad + [p\phi_2(c_2 + 3m_2 b_2] - p\phi_1(c_1 + 3m_1 b_1) + 3(b_1\kappa_1 - b_2\kappa_2)]\tau Q^2\Big\} \\
&+ \frac{1}{16}\Big\{(p\psi_1 a_{11} - p\phi_1 a_{12} + p\theta_1 a_{13} - a_{14})F \\
&\quad - (p\psi_1 a_{21} + p\psi_2 a_{22} - p\phi_1 a_{23} - p\phi_2 a_{24} - 4\omega_2 a_{25} + p\theta_1 a_{26} \\
&\qquad + p\theta_2 a_{27} + 2\kappa_2 a_{28} - 4a_{29})\tau Q \\
&\quad - (p\psi_1 a_{31} - 2p\psi_2 a_{32} - p\phi_1 a_{33} + p\phi_2 a_{34} + 8\omega_2 a_{35} - 2\omega_{12} a_{36} + 4\kappa_1 a_{37} - 2\kappa_2 a_{38})\tau Q^2 \\
&\quad - (p\psi_1 a_{41} + p\psi_2 a_{42} - 4\omega_1 a_{43} + 2\omega_{12} a_{44} - 4\omega_2 a_{45})\,\tau Q^3\Big\}\Bigg] \\
&+ 0\,(n^{-3}),
\end{aligned}
\tag{51}
$$

where $Q = m_1 F/(m_2 + m_1 F)$, $m_0 = m_1 + m_2$, $\tau = m_2/m_1$.

$$\theta_i = \frac{1}{pm_i + 2}, \qquad \kappa_i = \frac{pm_1 + pm_2 + 2}{pm_i + 2},$$

$$\phi_i = \frac{pm_1 + pm_2 + 2}{(pm_i + 2)(pm_i + 4)}, \qquad \omega_i = \frac{(pm_1 + pm_2 + 2)(pm_1 + pm_2 + 4)}{(pm_i + 2)(pm_i + 4)},$$

$$\omega_{12} = \frac{(pm_1 + pm_2 + 2)(pm_1 + pm_2 + 4)}{(pm_1 + 2)(pm_2 + 2)},$$

$$\psi_i=\frac{(pm_1+pm_2+2)(pm_1+pm_2+4)}{(pm_i+2)(pm_i+4)(pm_i+6)},$$

$$b_i=m_i+p+1, \qquad c_i=p^2+3p+4-2m_i^2,$$

$$a_{11}=a_{21}=a_{31}=a_{41}=b_1^2(pm_1+8)+4[b_1+(pm_1+2)],$$

$$a_{22}=a_{32}=a_{42}=b_2^2(pm_2+8)+4[b_2+(pm_2+2)],$$

$$a_{12}=a_{23}=a_{33}=b_1\{pm_1(3b_1-4b_2)+4[2m_2(pm_1+2)-m_1]\}$$
$$+12(m_1^2-4p^2)+20(2p+1)(p-1)$$

$$a_{24}=a_{34}=m_2b_2(3pm_2-p^2-p-4)+12(m_2^2-4p^2)+20(2p+1)(p-1),$$

$$a_{25}=a_{35}=a_{45}=b_2(pm_2+4)$$

$$a_{43}=b_1(pm_1+4),$$

$$a_{36}=a_{44}=pb_1b_2+4b_1+2m_2(pm_1+2)$$

$$a_{13}=a_{26}=b_1[4pm_0^2-pm_1b_1-2pm_2b_2-4(m_0+m_2)]+4m_2(pm_1+2)(3m_2-2b_2)$$
$$-4(p^2+p+1)(m_1^2-4)-4(2p+1)(p-1),$$

$$a_{27}=m_2b_2(3pm_2-p^2-p-4)-4(p^2+p+1)(m_2^2-4)-4(2p+1)(p-1),$$

$$a_{37}=pb_1(2m_0+m_2)-2(p+1)(pm_1+2)-2m_2(p+2)(p-1),$$

$$a_{28}=a_{38}=b_2[pb_1+2(pm_1+2)]-2m_1(p+2)(p-1)$$
$$+6p(m_2^2-4)-2(p-1)^2(p+4),$$

$$a_{14}=(m_0+m_2)[p(m_0+m_2)-2(p^2+p+4)]-m_2pm_0+(p+1)(p^2+p+20),$$

$$a_{29}=(m_0+m_2)(pm_0-p^2-p-4)+(pm_2^2+4p+4).$$

In the univariate case, the result becomes

$$F(\eta)=F_\eta(m_1,\ m_2),$$

which is to be expected.

The term of order n^{-2} in the above formula (51) is very complicated and would take a considerable time to compute. Consequently, we may often make errors in our calculation and hence it is unlikely to be of practical use. But by examining the numerical values of the term of order n^{-2} in several cases for usual significance levels, it seems to be sufficient for most practical use to use the terms of order up to n^{-1} for moderate values of n.

Finally, it must be noted that, in the theoretical point of view, the actual manner that the series (11) and (51) give good approximations of $T^2(\eta)$ and $F(\eta)$, respectively, is not known yet.

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