SOME THEOREMS ON THE SUM OF POSITIVE RANDOM VARIABLES

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1. Introduction

The purpose of this paper is to treat the limiting distribution and the fluctuation of the sum of identically distributed positive random variables without mean, which frequently appear in the theory of recurrence time.

Let $X_1, X_2, \cdots, X_n$ be positive random variables without mean, and $F(x)$ be their common distribution function, and let C_n be such that $1-F(C_n)\sim\frac{1}{n}$ (for large n). Then the sum

$$S_n=X_1+X_2+\cdots+X_n$$

has a stable limiting distribution, if and only if $\lim_{n\to\infty}\frac{C_{[n\lambda]}}{C_n}$ converges (Theorem 3). This statement is a special case of the results of Doeblin and Boboroff[1]. But, in case where $\lim\frac{C_{[n\lambda]}}{C_n}=\infty$ for $\lambda>1$, S_n shows a certain stability and we can consider a limiting distribution of S_n (Theorem 2). This case contains the distribution of the recurrence time for the return to the origin in a two-dimensional random walk. When $\lim\frac{C_{[n\lambda]}}{C_n}$ does not converge, we can also give the fluctuation of S_n comparing with n^s $(s>1)$ (Theorem 4).

The tool of the investigation is the theory of Laplace transformation which is known to be useful for the treatment of positive random variables (c.f. Kawata [5]).

2. Lemmas

In this section we shall state some fundamental properties of the Laplace transforms of distribution functions.

1) In the Boboroff's case, the definition of $\{C_n\}$ must be a little modified. (c.f. theorem 3).

LEMMA 1. *Let $F(x)$ be a distribution function of a positive random variables, and*

$$f(t)=\int_0^\infty e^{-tx}dF(x)$$

$$\varphi(x)=-\log f(t) ,$$

Then $\varphi(t)$ is a positive concave monotone increasing function and $\lim_{t\to 0}\varphi(t)=\varphi(0)=0$.

LEMMA 2. *Let $F(x)$, $\varphi(t)$ be the same as in lemma* 1. *Then we have $\varphi(t)\sim t\int_0^\infty (1-F(x))e^{-xt}dx$ for small t.*

PROOF. $f(t)=e^{-\varphi(t)}=1-\varphi(t)+{}_o(\varphi(t))$.

On the other hand, by partial integration we get

$$f(t)=1-t\int_0^\infty (1-F(x))e^{-xt}dx .$$

This proves the lemma.

LEMMA 3. *Let $\{F_n(x)\}$ be a sequence of distribution functions of positive random variables, $f_n(t)=\int_0^\infty e^{-xt}dF_n(x)$, and $\varphi_n(t)=-\log f_n(t)$. Then $\lim_{n\to\infty} F_n(x)=F(x)$ at the continuity point of x is equivalent to*

$$\lim_{n\to\infty} f_n(t)=f(t) \qquad 0<t<\infty ,$$

where $F(x)=\int_0^\infty e^{-xt}dF(x)$ and $\lim_{x\to\infty} F(x)$ can be smaller than 1.

This lemma is easily proved by the following two facts;

(1) $\{F_n(x)\}$ is a uniformly bounded, normal family,

(2) the inverse of the Laplace transform is uniquely determined.

3. Limiting Distribution of S_n

Let $X_1, X_2, \cdots, X_n \cdots$ be a sequence of positive random variables with the common distribution function $F(x)$. Then $S_n=X_1+\cdots+X_n$ has the distribution function $F^{(n)}(x)$ such that $F^{(n)}=F(x)*\cdots*F(x)$. Now we define the stability of S_n as follows.

DEFINITION. S_n is called stable if and only if there exists a positive increasing sequence of numbers $\{C_n\}$ such that

$$\lim_{n\to\infty} F\left(\frac{x}{C_n}\right) \to G(x)$$

where $0<G(x_0)<1$ for some $x_0>0$. (The case $\lim\limits_{x\to\infty} G(x)<1$ is admitted).

Since we are interested here mainly in the random variables without mean, the stability of the type, such as $\lim\limits_{n\to\infty} F^{(n)}\left(\frac{x-a_n}{C_n}\right)$ exists for some sequences $\{C_n\}$ and $\{a_n\}$, will be out of our consideration. On the other hand our definition is a little more general than the ordinary one; in the point that it admitts the case $\lim\limits_{x\to\infty} G(x)<1$.

THEOREM 1. *If S_n is stable and $\{C_n\}$ is such as in the above definition, then*

$$(1) \qquad \lim_{n\to\infty} \frac{C_{n\lambda}}{C_n} \quad \textit{exists}$$

or

$$(2) \qquad \lim_{n\to\infty} \frac{C_{n\lambda}}{C_n} = \infty \qquad \textit{for any } \lambda>1\,.$$

REMARK. We can define a fixed monotone function $C(x)$ for all positive value such that $C(n)=C_n$. In the above theorem $C_{n\lambda}$ means $C(n\lambda)$.

PROOF. By lemma 3, the condition that $F^{(n)}\left(\frac{x}{C_n}\right) \to G(x)$ and $0<G(x_0)<1$ for some $x_0>0$ is equivalent to $n\varphi\left(\frac{t}{C_n}\right) \to \mu(t)$ and $0<\mu(t)<\infty$ for $0<t<\infty$, where $\varphi(t)=-\log\left(\int_0^\infty e^{-xt}dF(x)\right)$ and $\mu(t)=-\log\left(\int_0^\infty e^{-xt}dG(x)\right)$. If $\underline{\lim}\, \frac{C_{n\lambda}}{C_n}=K<\overline{\lim}\, \frac{C_{n\lambda}}{C_n}=L$ for some $\lambda>1$ (L may be equal to ∞). For any K_1, L_1; $K<K_1<L_1<L$, we can take some subsequences $\{n_i\}$, $\{n_j\}$. Such that

$$(3) \qquad \frac{C_{n_i\lambda}}{C_{n_i}} < K_1 \qquad n_i > n_1$$

$$(4) \qquad \frac{C_{n_j\lambda}}{C_{n_j}} > L_1 \qquad n_j > n_2\,.$$

Therefore

$$\mu(t)=\lim_{n_i\to\infty} n_i\varphi\Big(\frac{t}{C_{n_i}}\Big)=\lim n_i\varphi\Big(\frac{C_{n_i\lambda}}{C_{n_i}}\cdot\frac{t}{C_{n_i\lambda}}\Big)$$
$$\leqq\frac{1}{\lambda}\,\overline{\lim}\, n_i\lambda\varphi\Big(K_1\frac{t}{C_{n_i\lambda}}\Big)=\frac{1}{\lambda}\mu(K_1 t)\ ,$$

i.e.,

(5) $$\mu(t)\leqq\frac{1}{\lambda}\mu(K_1 t)\ ,$$

and similary

(6) $$\mu(t)\geqq\frac{1}{\lambda}\mu(L_1 t) \qquad \text{for } t\neq 0.$$

Therefore we have from (5)

(7) $$\mu(t)<\lambda\mu(t)\leqq\mu(K_1 t)$$

and from (5) and (6)

(8) $$\mu(L_1 t)\leqq\mu(K_1 t) \qquad \text{where } K_1 t< L_1 t.$$

As $\mu(t)$ is monotone increasing, it holds that

$$\mu(s)=\mu(L_1 t)=\mu(K_1 t) \qquad \text{for } K_1 t\leqq s\leqq L_1 t.$$

Now, t is arbitrary, so we have

(9) $$\mu(t)\equiv C \qquad 0< s<\infty$$

which contradicts (7). q.e.d.

COROLLARY. *When S_n is stable and $\lim\frac{C_{n\lambda}}{C_n}=\infty$, then we have $\mu(t)\equiv$ constant.*

PROOF. In this case $L=\overline{\lim}\frac{C_{n\lambda}}{C_n}=\infty$. By (6) we then have $\mu(L_1 t)\leqq \lambda\mu(t)$ for arbitrary large L_1. Therefore $\mu(t)$ is bounded. From this together with the fact that $\mu(t)$ is monotone and convex, we have $\mu(t)\equiv$ constant.

THEOREM 2. *If S_n is stable and $\lim\frac{C_{n\lambda}}{C_n}=\infty$, then, taking $\overline{C}_n=C_{nk}$ for some positive constant k, we have*

(i) $P_r\{S_n<\overline{C}_{nx}\}\to e^{-1/x}$

or $P_r\{\overline{C}_{ny}<S_n<\overline{C}_{nx}\}\to e^{-1/x}-e^{-1/y}$ $\qquad (n\to\infty)$ *for* $x, y>0$.

(ii) $1-F(\overline{C}_n)\sim\frac{1}{n}$.

Conversely, if $1-F(\overline{C}_n)\sim\frac{1}{n}$ *and* $\lim\frac{\overline{C}_{n\lambda}}{\overline{C}_n}=\infty$ *for any* $\lambda>1$, *then* S_n *is stable with respect to* $\{C_n\}$.

First, we prove the following lemma.

LEMMA 4. When

$$1-F(C_n)<\frac{k}{n} \qquad \text{for } n>n_0, \tag{10}$$

and $\lim_{n\to\infty}\frac{C_{n\lambda}}{C_n}=\infty$ *for any* $\lambda>1$, *then we have*

$$\int_0^{x_0}(1-F(C_nx))dx<\frac{3x_0k}{n} \qquad \text{for large } n. \tag{11}$$

PROOF. We divide the integral into three parts:

$$I_n=\int_0^{x_0}(1-F(C_nx))\,dx=\int_0^{\frac{C\left(\frac{n}{2^{k+1}}\right)}{C_n}}+\sum_{K=1}^{k}\int_{\frac{C\left(\frac{2^{r+1}}{n}\right)}{C(n)}}^{\frac{C\left(\frac{2^r}{n}\right)}{C(n)}}+\int_{\frac{C\left(\frac{n}{2}\right)}{C(n)}}^{x_0}$$

From (10) we have

$$1-F\left(C\left(\frac{n}{2^{r+1}}\right)\right)<\frac{2^{r+1}}{n}k \text{ for } 0<r<k=\left[\frac{1}{2}\log_2 n\right] \quad \text{and } n>n_0^2. \tag{12}$$

On the other hand, $\lim_{n\to\infty}\frac{C_{n/2}}{C_n}=0$, therefore $\frac{C_{n/2}}{C_n}<\varepsilon$ for $n>n_1$. Thus we have $\frac{C\left(\frac{n}{2^{r+1}}\right)}{C\left(\frac{n}{2^r}\right)}<\varepsilon$ for $0<r<k$ and $n>n_1^2$, or

$$\frac{C\left(\frac{n}{2^{r+1}}\right)}{C(n)}<\varepsilon^r \qquad 0<r<k. \tag{13}$$

Especially

$$\frac{C\left(\frac{n}{2^{k+1}}\right)}{C(n)}<\varepsilon^{k+1}<n^{\frac{1}{2}\log_2\varepsilon}\ . \tag{14}$$

Therefore

$$I_n \leqq \frac{C\left(\frac{n}{2^{k+1}}\right)}{C(n)} + \sum_{r=1}^{k} \frac{K2^{r+1}}{n} \frac{C\left(\frac{n}{2^r}\right)}{C(n)} + \frac{2}{n} x_0 K$$

$$\leqq \frac{1}{n}\left\{n^{1+\frac{1}{2}\log_2 \varepsilon} + 2K \sum_{r=1}^{k} (2\varepsilon)^r + 2x_0 K\right\}$$

$$\leqq \frac{1}{n}\{n^{-p} + 4K\varepsilon + 2x_0K\} \qquad \text{for large } n,$$

where $p = -\left(1+\frac{1}{2}\log_2 \varepsilon\right) > 0$ (for sufficiently small ε). Consequently, for sufficiently large n we have

$$I_n < \frac{3x_0 k}{n} . \qquad \text{q.e.d.}$$

PROOF OF THEOREM. By the corollary of Theorem 1, we get $\mu(t) \equiv k$, therefore $\lim_{n\to\infty} n\varphi\left(\frac{t}{C_n}\right) = k$ for $0 < t < \infty$. Putting $\overline{C}_n = C_{nk}$, we have

$$\lim_{n\to\infty} n\varphi\left(\frac{t}{\overline{C}_n}\right) = \frac{1}{k} \lim nk\varphi\left(\frac{t}{C_{nk}}\right) = 1 \tag{15}$$

By the same argument, we have

$$\lim_{n\to\infty} n\varphi\left(\frac{t}{\overline{C}_{n\lambda}}\right) = \frac{1}{\lambda} . \tag{16}$$

If we define $F_\lambda(x)$ such that $F_\lambda(0)=0$, $F_\lambda(x) = e^{-1/\lambda}$ $x > 0$, then

$$\frac{1}{\lambda} = \int_0^\infty e^{-xt} dF_\lambda(x)$$

Therefore, by Lemma 3 we have

$$\lim_{n\to\infty} F^{(n)}\left(\frac{x}{\overline{C}_{n\lambda}}\right) = F_\lambda(x)$$

or

$$P_r\{S_n < \overline{C}_{n\lambda}\} \to e^{-1/\lambda}$$

which proves (i).

Next, we have $\varphi(t) \sim t\int_0^\infty (1-F(x))e^{-xt}dx$ by Lemma 2. On the other hand $n\varphi\left(\frac{t}{\overline{C}_n}\right) \to 1$ or $\varphi\left(\frac{t}{\overline{C}_n}\right) \sim \frac{1}{n}$ for $t > 0$. Thus we obtain

$$(17)\qquad \frac{1}{n}\sim\frac{t}{\overline{C}_n}\int_0^\infty(1-F(x))e^{-\frac{xt}{C_n}}dx\,.$$

From (17) for any $\varepsilon>0$ we have

$$\begin{aligned}1+\varepsilon&>\int_0^\infty n\left(1-F\left(\frac{\overline{C}_n x}{t}\right)\right)e^{-x}dx \qquad n>n_0\\ &\geqq\int_0^t n\left(1-F\left(\frac{\overline{C}_n x}{t}\right)\right)e^{-x}dx\geqq n(1-F(\overline{C}_n))\int_0^t e^{-x}dx\\ &\geqq n(1-F(\overline{C}_n))(1-e^{-t}) \qquad \text{for any large } t\ (n>n(t))\end{aligned}$$

Therefore

$$(18)\qquad 1+2\varepsilon\geqq n(1-F(\overline{C}_n)) \qquad n\geqq n_0\,.$$

From (18) it is evident that the conditions of Lemma 4 is satisfied, and we have

$$\int_0^{x_0}(1-F(\overline{C}_n))dx\leqq\frac{3Kx_0}{n}\,.$$

It follows that from (17)

$$\begin{aligned}(1-\varepsilon)&<t\int_0^\infty n(1-F(\overline{C}_n x))e^{-xt}dx\\ &\leqq t\int_0^1 n(1-F(\overline{C}_n x))dx+n(1-F(\overline{C}_n))\int_1^\infty te^{-xt}dx\\ &\leqq 3Kt+n(1-F(\overline{C}_n))e^{-t}\end{aligned}$$

for any small t and $n>n(t)$. Then we obtain (19) $1-2\varepsilon<n(1-F(\overline{C}_n))$ $n>n_1$. From (18) and (19) it follows that

$$1-F(\overline{C}_n)\sim\frac{1}{n}$$

which proves (ii).

Conversely, if

$$(19)\qquad 1-F(C_n)\sim\frac{1}{n}\quad\text{and}\quad\lim\frac{C_{n\lambda}}{C_n}=\infty\ \text{ for any } \lambda>1\,,$$

then the condition of Lemma 4 is satisfied (for $K=2$). That is,

$$(20)\qquad n\int_0^{\frac{C_{n(1-\varepsilon)}}{C_n}}(1-F(C_n x)dx<3K\frac{C_{n(1-\varepsilon)}}{C_n}\,.$$

(21) $$\frac{n}{C_n}\int_0^\infty (1-F(x))e^{-\frac{xt}{C_n}}dx = n\int_0^\infty (1-F(C_nx))e^{-xt}dx$$

$$= n\left\{\int_0^{\frac{C_{n(1-\varepsilon)}}{C_n}} + \int_{\frac{C_{n(1-\varepsilon)}}{C_n}}^{\frac{C_{n(1+\varepsilon)}}{C_n}} + \int_{\frac{C_{n(1+\varepsilon)}}{C_n}}^\infty\right\}.$$

From (20) we have

(22) $$n\int_0^{\frac{C_{n(1-\varepsilon)}}{C_n}} (1-F(C_nx))e^{-xt}dx \leqq n\int_0^{\frac{C_{n(1-\varepsilon)}}{C_n}} (1-F(C_nx))dx < 3K\frac{C_{n(1-\varepsilon)}}{C_n},$$

and

(23) $$n\int_{\frac{C_{n(1+\varepsilon)}}{C_n}}^\infty (1-F(C_nx))e^{-xt}dx \leqq \int_{\frac{C_{n(1-\varepsilon)}}{C_n}}^\infty e^{-xt}\,dx = \frac{1}{t}e^{-\frac{C_{n(1+\varepsilon)}}{C_n}}.$$

On the other hand, when $\frac{C_{n(1-\varepsilon)}}{C_n} \leqq x \leqq \frac{C_{n(1+\varepsilon)}}{C_n}$, we have from (19)

$$1-2\varepsilon \leqq n(1-F(C_nx)) \leqq 1+2\varepsilon \qquad \text{for sufficiently large } n.$$

Therefore

$$(1-2\varepsilon)\int_{\frac{C_{n(1-\varepsilon)}}{C_n}}^{\frac{C_{n(1+\varepsilon)}}{C_n}} e^{-xt} \leqq n\int_{\frac{C_{n(1-\varepsilon)}}{C_n}}^{\frac{C_{n(1+\varepsilon)}}{C_n}} (1-F(C_nx))e^{-xt}dx \leqq (1+2\varepsilon)\int_{\frac{C_{n(1-\varepsilon)}}{C_n}}^{\frac{C_{n(1+\varepsilon)}}{C_n}} e^{-xt}dx$$

or

(24) $$\frac{1}{t}(1-2\varepsilon)\left(e^{-\frac{C_{n(1-\varepsilon)}}{C_n}t} - e^{-\frac{C_{n(1+\varepsilon)}}{C_n}t}\right) \leqq n\int_{\frac{C_{n(1-\varepsilon)}}{C_n}}^{\frac{C_{n(1+\varepsilon)}}{C_n}}$$

$$\leqq \frac{1}{t}(1+2\varepsilon)\left(e^{-\frac{C_{n(1-\varepsilon)}}{C_n}} - e^{-\frac{C_{n(1+\varepsilon)}}{C_n}}\right).$$

Since $\frac{C_{n(1-\varepsilon)}}{C_n} \to 0$ and $\frac{C_{n(1+\varepsilon)}}{C_n} \to \infty$ $(n\to\infty)$, from (21), (22), (23) and (24) we have

$$\frac{1-2\varepsilon}{t} \leqq \frac{n}{C_n}\int_0^\infty (1-F(C_nx))\,e^{-\frac{xt}{C_n}}dx \leqq \frac{1+2\varepsilon}{t} \qquad \text{for } n > n(t).$$

Therefore we get

$$\lim_{n\to\infty}\frac{nt}{C_n}\int_0^\infty (1-F(x))e^{-\frac{xt}{C_n}}dx = 1,$$

or

$$\lim_{n\to\infty} n\varphi\left(\frac{t}{C_n}\right)=1 \qquad \text{for fixed } t\neq 0,$$

which means by Lemma 3

$$\lim_{n\to\infty} F^{(n)}\left(\frac{x}{C_n}\right)=F_1(x)\ .$$

This proves the converse. q.e.d.

When $\lim_{n\to\infty}\frac{C_{n\lambda}}{C_n}<\infty$, the problem is solved as a special case of Doeblin's theorem ($S_0>1$ in theorem 3) or Boboroff's theorem ($S_0=1$ in Theorem 3). Therefore we only state the results (c.f. Doeblin [1] and Kawata [5]).

THEOREM 3. *If* $\lim_{n\to\infty}\frac{C_{n\lambda}}{C_n}=K_\lambda$ *is finite and* S_n *is stable, then we have*

$$K_\lambda=\lambda^{s_0} \qquad s_0\geqq 1.$$

Further, under the condition that $\lim\frac{C_{n\lambda}}{C_n}=K_\lambda$ $K_\lambda=\lambda^{s_0}$ $(s_0\geqq 1)$

i) *for* $s_0>1$, S_n *is stable with respect to* $\{C_n\}$ *if and only if*

$$1-F(C_n)\sim\frac{k}{n} \qquad \text{for some positive } k.$$

ii) *for* $s_0=1$, S_n *is stable with respect to* $\{C_n\}$ *if and only if*

$$\frac{n}{C_n}\int_0^{C_n}(1-F(x))dx\to k \qquad (0<k<\infty)$$

$$n(1-F(C_n))\to 0 \qquad (n\to\infty).$$

In these cases, the limiting distribution of $\frac{S_n}{C_n}$ *is a stable law of index* $\frac{1}{s_0}$.

EXAMPLE. In the two-dimensional random walk, the moment generating function $P(s)$ of recurrence time for the return to the origin is given by the following relation

$$U(s)=\frac{1}{1-P(s)}$$

where

$$U(s)=\sum_{n=0}^{\infty}u_n s^n \text{ and } u_n\sim\frac{1}{2\pi n}\ . \qquad \text{(c.f. Feller [2] [3])}$$

Therefore,

$$U(s) \sim \frac{1}{2\pi} \log \frac{1}{(1-s)} \qquad \text{for } s \to 1,$$

and

$$1-P(e^{-t}) \sim 2\pi \frac{1}{\log \frac{1}{t}} \qquad \text{for } t \to 0,$$

or

$$\varphi(t) \sim 2\pi \frac{1}{\log \frac{1}{t}} \qquad \text{where } e^{-\varphi(t)} = P(e^{-t}) .$$

Therefore, we have

$$\lim_{n\to\infty} n\varphi\left(\frac{t}{e^{2\pi n}}\right) = 1 .$$

Thus, from Theorem 2 it follows that

$$P_r\{e^{2\pi ny} < S_n < e^{2\pi nx}\} = e^{-1/x} - e^{-1/y}$$

or $\dfrac{2\pi n}{\log S_n}$ is distributed according to an exponential distribution.
And

$$1-F(e^{2\pi n}) \sim \frac{1}{n} .$$

4. The Order of the Fluctuation of S_n

When S_n has no limiting distribution, we can also estimate the fluctuation order of S_n. For this purpose, we shall compare it with function $g(n) = n^s$ $(s \geqq 1)$.

THEOREM 4. *Let* $1-F(C_n) \sim \dfrac{1}{n}$,

and

$$s_1 = \inf \left\{ s \geqq 1, \ \overline{\lim} \frac{C_n}{n^s} = 0 \right\}$$

$$s_2 = \sup \left\{ s \geqq 1, \ \underline{\lim} \frac{C_n}{n^s} = \infty \right\} \quad (s_1 \geqq s_2, \ s_1, s_2 \textit{ may be infinity})$$

Then, we have

i) *for* $s > s_1$ *and any* $\varepsilon > 0$, *with probability one* $S_n < \varepsilon n^s$ *holds except at most finitely many* n.

ii) *for* $1 < s < s_2$ *and any* $K > 0$, *with probability one* $S_n > Kn^s$ *holds except at most finitely many* n.

iii) *if* $s_1 > s_2$, *then for* $s_1 > s > s_2$ *there exist some fixed subsequences* $\{n_i\}$ *and* $\{n_j\}$ *such that with probability one* $S_{n_i} > \varepsilon n_i^s$ *holds* (n_i *in* $\{n_i\}$) *except at most finitely many* n_i, *and with probability one* $S_{n_j} < Kn_j^s$ *holds* (n_j *in* $\{n_j\}$) *except at most finitely many* n_j.

PROOF. For $s > s_1$, we take s' such that $s > s' > s_1$. Then

$$\overline{\lim}\, \frac{C_n}{n^{s'}} = 0 \quad \text{and} \quad 1 - F(n^{s'}) \leqq 1 - F(C_n) \sim \frac{1}{n} \qquad \text{for large } n.$$

Therefore, we can choose a number x_0 such that

$$1 - F(x) \leqq \frac{2}{x^{\frac{1}{s'}}} \qquad \text{for } x > x_0, \tag{25}$$

from which it follows that

$$\int (1 - F(x))\, e^{-\frac{xt}{n^s}}\, dx < 2\int_{x_0}^{\infty} \frac{e^{-\frac{xt}{n^s}}}{x^{\frac{1}{s'}}} dx + \int_0^{x_0} < 2t^{\frac{1}{s'}-1}\, n^{s - \frac{s}{s'}} + x_0 \quad \text{for } n > n_0\,.$$

Therefore we have

$$e^{-n\varphi\left(\frac{t}{n^s}\right)} > e^{-t^{\frac{1}{s'}} n^{-\delta}} \qquad \text{where } \delta = \frac{s}{s'} - 1 > 0\,. \tag{26}$$

If we put $G^{(n)}(x)$ as a distribution function of $\dfrac{S_n}{n^s}$, then we have

$$G^{(n)}(x\varepsilon) + e^{-\varepsilon t} G^{(n)}(x\varepsilon) \geqq \int_0^{\infty} e^{-xt} dG^{(n)}(x) = e^{-n\varphi\left(\frac{t}{n^s}\right)} > e^{-3t^{\frac{1}{s'}} n^{-\delta}}\,.$$

Putting $t^{1/s'} = n^{\delta/2}$, we get for sufficiently large n.

$$G^{(n)}(x\varepsilon) \geqq e^{-3n^{-\frac{\delta}{2}}}\,, \tag{27}$$

or

$$P_r\{S_n > \varepsilon n^s\} < 3n^{-\frac{\delta}{2}} = 3n^{-\delta'} \qquad \text{for } n > n(\varepsilon, s). \tag{28}$$

But the event $S_n > \varepsilon n^s$ means $S_{2^r} > S_n > \varepsilon 2^{(r-1)s}$ for $2^{r-1} \leqq n < 2^r$. Therefore

$$P_r\{\bigcup_n [S_n > \varepsilon n^s]\} \leqq P_r\{\bigcup_r [S_{2^r} > \varepsilon 2^{(r-1)s}]\} \leqq r_0 + \sum_{r>r_0} 3\cdot\frac{2^s}{\varepsilon}\cdot 2^{-r\delta'} < \infty$$

where r_0 is such that $2^{r_0} > n(\varepsilon, s)$. That is,

$$P_r\{\overline{\lim}[S_n > \varepsilon n^s]\} = 0 \tag{29}$$

which proves (i).

On the other hand, for $1 < s < s_2$, applying the same method as above we have for $s < s'' < s_2$

$$1 - F(n^{s''}) > \frac{1}{n}, \quad \text{or} \quad 1 - F(x) > \frac{1}{2x^{\frac{1}{s''}}} \qquad x > x_0, \tag{30}$$

therefore,

$$\int_0^\infty (1-F(x))e^{-\frac{xt}{n^s}}dx > \frac{t^{\frac{1}{s''}-1} n^{s-\frac{s}{s''}}}{2} \qquad \text{for large } n,$$

and

$$e^{-n\varphi\left(\frac{t}{n^s}\right)} < e^{-\frac{t}{2}\frac{1}{s'}n^\delta} \qquad \delta > 0.$$

It follows from this that

$$G^{(n)}(xk)e^{-Kt} \leqq \int_0^\infty e^{-xt}dG^{(n)}(x) < e^{-3t\frac{1}{s''}n^\delta} \qquad \delta > 0. \tag{31}$$

Putting $t = n^{-\delta/2}$, we get for $n > n(K, \varepsilon)$

$$P_r\{S_n < Kn^s\} = G^{(n)}(Kx) < e^{-3n^{\delta'}} \qquad \left(\delta' > \frac{\delta}{2}\right). \tag{32}$$

and $\sum_n e^{-3n^{\delta'}}$ converges. Therefore we have by Borel-Cantelli's theorem

$$P_r\{\overline{\lim}[S_n < Kn^s]\} = 0,$$

which proves (ii).

For $s_1 > s > s_2$, we take s', s'' such that $s_1 > s'' > s > s' > s_2$. Then we determine the subsequences $\{n_i\}$ and $\{n_j\}$ such that

$$\lim_{n_i\to\infty} \frac{C_{n_i}}{n_i^{s'}} = 0$$
$$\lim_{n_j\to\infty} \frac{C_{n_j}}{n_j^{s''}} = \infty \tag{33}$$

Since for $\{n_i\}$ (25), (26) and (27) hold (if we substitute n_i for n), we can easily prove the first part of (iii).

Similary, for $\{n_j\}$ (30), (31), and (32) hold (if we substitute n_j for n), and the 2nd part of (iii) is easily verified.

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