NOTE ON THE NEYMAN-PEARSON'S FUNDAMENTAL LEMMA

By YUKIO SUZUKI

(Received 25, March., 1955)

1. Introduction

In this paper we want to treat a generalization of the Neyman-Pearson's fundamental lemma.

At first we describe the fundamental lemma in a little generalized form which was obtained by George B. Dantzig and Wald [1].

Let R be a finite dimensional Euclidean space, B be a Borelfield of subsets of R. Let $f_0(x), f_1(x), \ldots, f_n(x)$ be probability density functions defined over B, that is,

$$f_i(x) \geqq 0$$

$$\int_R f_i(x)dx = 1 \qquad (i=0, 1, 2, \ldots, n)$$

For a given sequence $\{c_i\}$ $(i=1, 2, \ldots, n)$ with

$$0 \leqq c_i \leqq 1 \qquad (i=1, 2, \ldots, n)$$

we denote by $\mathfrak{S}$ the set of all Borel sets S such that

$$\int_S f_i(x)dx = c_i \qquad (i=1, 2, \ldots, n)$$

and by $\mathfrak{S}^0$ the set of all Borel sets S_0 such that S_0 belongs to $\mathfrak{S}$ and for any element S of $\mathfrak{S}$ we have

$$\int_{S_0} f_0(x)dx \geqq \int_S f_0(x)dx .$$

Then we get the following lemma.

LEMMA (*G. B. Dantzig and A. Wald*) *If $\mathfrak{S}$ is not empty, then $\mathfrak{S}^0$ is also not empty. It is a necessary and sufficient condition for any element S of $\mathfrak{S}$ to be an element of $\mathfrak{S}^0$ that a sequence $\{k_i\}$ $(i=1, 2, \ldots, n)$ exists such that*

$$f_0(x) \geqq \sum_1^n k_i f_i(x), \quad \textit{if } x \in S$$

$$f_0(x) \leqq \sum_1^n k_i f_i(x), \quad \textit{if } x \bar{\in} S$$

except perhaps on a set of measure zoro.

The problem we want to treat in this paper is to generalize the above lemma for the purpose of treating the same problem in case the number of probability density functions is countably infinite. But this problem is rather difficult in some points and we have to set some restrictions on the powers of tests to prove the necessary condition.

2. Sufficient condition

Let $\{f_i(x)\}$ $(i=0, 1, 2, \ldots)$ be a sequence of probability density functions which are defined on a finite dimensional Euclidean space R. For a given sequence $\{c_i\}$ $(i=1, 2, \ldots)$ such that

$$0 \leqq c_i \leqq 1, \qquad (i=1, 2, \ldots) \tag{2.1}$$

we denote by $\mathfrak{S}$ the set of all Borel sets S such that

$$\int_S f_i(x)dx = c_i \qquad (i=1, 2, \ldots).$$

and by $\mathfrak{S}^0$ the subset of $\mathfrak{S}$ such that for any element S_0 in $\mathfrak{S}_0$ and for any element S of $\mathfrak{S}$ the following inequality holds.

$$\int_{S_0} f_0(x)dx \geqq \int_S f_0(x)dx.$$

LEMMA 1. *It is a sufficient condition for any element S_0 of $\mathfrak{S}$ to be an element of $\mathfrak{S}^0$ that a sequence of real numbers $\{k_i\}$ $(i=1, 2, \ldots)$ exists such that*

$$\sum_{i=1}^{\infty} k_i f_i(x)$$

is convergent for all $x \in R$ except on a set of measure zero and for all n $\sum_{i=1}^{n} k_i f_i(x)$ is dominated by some integrable function $G(x)$, that is, $\sum_{i=1}^{n} k_i f_i(x) \leqq G(x)$ for all x except perhaps on a set of measure zero, and $\int_R G(x)dx < \infty$, and furthermore

$$\begin{aligned} &f_0(x) \geqq \sum_{i=1}^{\infty} k_i f_i(x) && \textit{if } x \in S_0 \\ &f_0(x) \leqq \sum_{i=1}^{\infty} k_i f_i(x) && \textit{if } x \bar{\in} S_0, \end{aligned} \tag{2.2}$$

except perhaps on a set of measure zero.

Proof. Suppose that the conditions are satisfied, then for any $S \in \mathfrak{S}$

$$\int_{S_0} f_0(x)dx - \int_S f_0(x)dx = \int_{S_0-S} f_0(x)dx - \int_{S-S_0} f_0(x)dx \geqq \int_{S_0-S} \sum_{i=1}^{\infty} k_i f_i(x)dx$$

$$-\int_{S-S_0} \sum_{i=1}^{\infty} k_i f_i(x)dx = \sum_{i=1}^{\infty} k_i \int_{S-S_0} f_i(x)dx - \sum_{i=1}^{\infty} k_i \int_{S-S_0} f_i(x)dx$$

$$= \sum_{i=1}^{\infty} k_i \Big(\int_{S_0-S} f_i(x)dx - \int_{S-S_0} f_i(x)dx \Big)$$

From the assumption that S and S_0 both belong to $\mathfrak{S}$

$$\int_{S_0} f_i(x)dx = \int_S f_i(x)dx \qquad (i=1, 2, \dots)$$

and so

$$\int_{S_0-S} f_i(x)dx = \int_{S-S_0} f_i(x)dx \qquad (i=1, 2, \dots).$$

Therefore we get the lemma.

3. Some concepts and lemma

Let $l^{(2)}$ consist of all sequences of real numbers $\{\xi_i\}$ $(i=0, 1, 2, \dots)$ such that $\sum_{i=0}^{\infty} \xi_i^2 < \infty$, that is, $l^{(2)}$ is a Hilbert space. As usual, for $\{\xi_i\} \in l^{(2)}$, we denote $\{\xi_i\}$ and $\sum_{i=0}^{\infty} \xi_i^2$, by ξ and $\|\xi\|^2$ respectively. Further, for given two elements ξ and η of $l^{(2)}$, we denote their inner product by (ξ, η), that is,

$$(\xi, \eta) = \sum_{i=0}^{\infty} \xi_i \eta_i, \quad \text{where } \eta = \{\eta_i\} \ (i=0, 1, 2, \dots).$$

Let

$$a = \{a_i\} \ (i=1, 2, \dots) \tag{3.1}$$

be a fixed element of $l^{(2)}$ such that $a_0=1$ and $a_i>0$ $(i=1, 2, \dots)$

Now, let us consider the sequence $\left\{a_i \int_S f_i(x)dx\right\}$ $(i=0, 1, 2, \dots)$ for any Borel set S and denote it by $V(S)$. Then for any $S \in B$, $V(S)$ belongs to $l^{(2)}$, because we get

$$\| V(S) \| = \sum_{i=0}^{\infty} \Big(a_i \int_S f_i(x)dx \Big)^2 \leqq \sum_{i=0}^{\infty} a_i^2 < \infty. \tag{3.2}$$

If S ranges over all measurable sets, we obtain a subset $\mathfrak{R}$ of $l^{(2)}$, that is, $\mathfrak{R} = \{V(S) : S \in B\}$

Let us denote the (i)-th term of $V(S)$ by $V_{i-1}(S)$ and the first term of $V(S)$ by $V_0(S)$. Consider the sequence $\{0, V_1(S), V_2(S), \dots\}$ for any measurable set S and denote it by $V^0(S)$. Letting S range over all measurable sets, we obtain a subset $\mathfrak{R}^0$ of $l^{(2)}$, that is, $\mathfrak{R}^0 = \{V^0(S); S \in B\}$. For any n, consider $\{V_0(S), V_1(S), \dots, V_n(S), 0, 0, \dots\}$ and denote it by $V^{(n)}(S)$. Let $\mathfrak{R}^{(n)}$ be the subset of $\mathfrak{R}$ consisting of all such $V^{(n)}(S)$ corresponding to all Borel sets S. Then by the Lyapounof's theorem $\mathfrak{R}^{(n)}$ is closed, bounded and convex. From these facts we get the following lemma.

LEMMA 2. *Let $\overline{\mathfrak{R}}$ and $\overline{\mathfrak{R}}^0$ be closures of $\mathfrak{R}$ and $\mathfrak{R}^0$, respectively, then $\overline{\mathfrak{R}}$ and $\overline{\mathfrak{R}}^0$ are bounded and convex.*

Proof: It is sufficient to prove the lemma in $\overline{\mathfrak{R}}$. Let ξ and η be any two elements of $\overline{\mathfrak{R}}$, and let α be any positive real number such that $0 \leqq \alpha \leqq 1$.

To prove the convexity of $\overline{\mathfrak{R}}$, it is sufficient to prove

$$\alpha\xi + (1-\alpha)\eta \in \overline{\mathfrak{R}}.$$

Since $\overline{\mathfrak{R}} \ni \xi, \eta$, there exist two sequences

$$\{V(S_{1i})\} \quad (i=1, 2, \dots) \quad \text{and} \quad \{V(S_{2i})\} \quad (i=1, 2, \dots)$$

such that

$$V(S_{1i}) \to \xi, \qquad V(S_{2i}) \to \eta \qquad (i \to \infty) \tag{3.3}$$

By Lyapounoff's lemma mentioned above, there exists

$$\{S_i^{(n)}\}, \qquad \begin{pmatrix} n=1, 2, \dots \\ i=1, 2, \dots \end{pmatrix}$$

such that

$$\alpha V^{(n)}(S_{1i}) + (1-\alpha) V^{(n)}(S_{2i}) = V^{(n)}(S_i^{(n)}). \qquad \begin{pmatrix} n=1, 2, \dots \\ i=1, 2, \dots \end{pmatrix}.$$

and hence by (3.2) and (3.3)

$$\begin{aligned} \lim_{n\to\infty} V(S_i^{(n)}) &= \lim_{n\to\infty} V^{(n)}(S_i^{(n)}) \\ &= \lim_{n\to\infty} \{\alpha V^{(n)}(S_{1i}) + (1-\alpha) V^{(n)}(S_{2i})\} \\ &= \alpha V(S_{1i}) + (1-\alpha) V(S_{2i}). \end{aligned}$$

Therefore we obtain,

$$\lim_{i\to\infty}(\lim_{n\to\infty} V(S_i^{(n)}))=\lim_{i\to\infty}\{\alpha V(S_{1i})+(1-\alpha)V(S_{2i})\}$$
$$=\alpha\xi+(1-\alpha)\eta .$$

and so we get

$$\lim_{n\to\infty} V(S_n^{(n)})=\alpha\xi+(1-\alpha)\eta .$$

This means that

$$\alpha\xi+(1-\alpha)\eta \in \overline{\mathfrak{R}} .$$

Thus, the convexity of $\overline{\mathfrak{R}}$ has been proved. The boundedness of $\overline{\mathfrak{R}}$ is the immediate result of (3.2).

From this lemma we can say $\mathfrak{R}$ is dense in the closed convex set $\overline{\mathfrak{R}}$. In the same way, we can verify that $\mathfrak{R}^0$ is dense in the closed set $\overline{\mathfrak{R}^0}$.

4. Convex set

In this section we want to prove a lemma concerning convex sets, in a rather general form.

Let L be a real normed space, that is, L is a vector space, and for any point x of L a real number $||x||$ is defined in such way that the following three conditions are satisfied,

(a) $\qquad ||x||\geqq 0$ and $||x||=0 \leftrightarrows x=0$.

(b) $\qquad ||x+y||\leqq ||x||+||y||$

(c) $\qquad ||\alpha x||\leqq |\alpha|\cdot||x||$ $\qquad$ where α is an arbitrary real number and $y \in L$.

If L is a normed space, we can define metric between arbitary two points x, y by $||x-y||$. If K is a subset of L and the following condition is satisfied: for any real number α such that $0\leqq\alpha\leqq 1$, and for arbitrary two points x and y of K, $\alpha x+(1-\alpha)y$ belongs to K, then K is called a convex set in L. Now, let K be convex and closed and consider the smallest of all subspaces of L containing K, say, L_K. Then we obtain the following lemma.

LEMMA 3. *If z is a boundary point of K when we regard L_K vs the whole space, then there exists a linear functional f_0 such that*

(4.1) $$\operatorname*{Sup}_{x\in K} f_0(x)\leqq f_0(z)$$

that is, K lies in the one handside of the hyperplane: $f_0(x)-f_0(z)=0$

Proof. Since we can translate K without loss of generality and K is closed, we may assume that

$$\operatorname*{Inf}_{x\in(L_K-K)} \|x\| = \alpha > 0$$

Now choose α' such that $0<\alpha'<\alpha$ and let K' be $K\frown\{x: \|x-x_K\| \geqq \alpha'$, for every boundary point x_K of $K\}$. Then K' is also a convex set. Let K'' be $\{x; \|x-x'\| \leqq \alpha'$ for some point $x' \in K'\}$. Obviously K'' is also a convex set. Furthermore, it is clear that

$$K''\frown L_K=K.$$

Obviously z is also a boundary point of K'' when we regard L as the whole space. Now, we define a function $p(x)$ on L as follows:

(4.2) $$p(x) = \inf_{\rho>0,\ \frac{x}{\rho}\in K''} \rho$$

Then, $p(x)$ is a sub-additive fuction, that is, for any two points x and y in L, and for any non-negative real number α

(4.3) $$p(x)+p(y) \geqq p(x+y)$$

(4.4) $$p(\alpha x)=\alpha p(x)$$

(4.4) is obvious. (4.3) is proved as follows. For an arbitrary positive number ε, and two arbitrary elements of L, say x and y,

$$\frac{x}{p(x)+\varepsilon} \in K'' \quad \text{and} \quad \frac{y}{p(y)+\varepsilon} \in K''$$

consist by virtue of the definition of $p(x)$. Since K'' is convex,

$$\frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}\cdot\frac{x}{p(x)+\varepsilon}+\frac{p(y)+\varepsilon}{p(x)+p(y)+2\varepsilon}\cdot\frac{y}{p(y)+\varepsilon} \in K'',$$

and so

$$\frac{x+y}{p(x)+p(y)+2\varepsilon} \in K''.$$

This means from the definition of $p(x+y)$,

$$p(x)+p(y)+2\varepsilon \geqq p(x+y)$$

Since ε may be arbitrarily small, this proves the relation (4.3).

Now, let L_1 be a subspace of L such that $L_1=\{x: x=\gamma z,\ -\infty<\gamma<\infty\}$. We define a function $f(x)$ on L_1 such that for $x=\gamma z$ we have $f(x)=\gamma$. Then $f(x)$ is a linear functional on L_1 and furthermore it is easy to verify

$$f_1(x) \leqq p(x) \qquad \text{on } L_1$$

Then by virtue of Hahn-Banach's extension theorem, we get a linear functional f on L such that

(4.5) $\qquad f(x) \leqq p(x) \qquad$ on L and $f(x)=f_1(x) \qquad$ on L_1

Therefore

$$\operatorname*{Sup}_{x\in K''} f(x) \leqq \operatorname*{Sup}_{x\in K''} p(x)=1=f_1(z)=f(z).$$

Since $K \subseteq K''$ we obtain

$$\operatorname*{Sup}_{x\in K} f(x) \leqq f(z)\,.$$

Thus the lemma was proved.

In the special case such that L is a Hilbert space $l^{(2)}$, linear functional $f(x)$ is expressed by Rietz' Theorem in the form of inner product, that is, $f(x)=(x, k)$, where $k \in l^{(2)}$, and so

$$f(x)-f(z)=(x-z,\ k)\,.$$

In the last section we proved that $\overline{\mathfrak{R}}$ or $\overline{\mathfrak{R}}^0$ is a closed and convex set and so by above lemma there exists a linear functional $f(x)$ such that

$\operatorname*{Sup}_{x\in\overline{\mathfrak{R}}} (f(x)-f(z)) \leqq 0,\quad$ where z is a boundary point of $\overline{\mathfrak{R}}$ regarding as the whole space the smallest subspace which contains $\overline{\mathfrak{R}}$.

§ 5. Necessary condition

In this section we want to prove the necessity of the condition (2.2) under some assumptions which would be rather weak.

Here, we use the same notations as used in previous sections without further explanations.

Assumption (I)

There exist an integrable function $G(x)$ on R and an element $\{a_i\}(i=0,\ 1,\ 2,\ \ldots)$ of $l^{(2)}$ such that $a_0=1$, $a_i > 0$ $(i=1,\ 2,\ \ldots)$ and for which $\sum_{i=1}^{\infty} a_i f_i(x) < \infty$ and $\sum_{i=1}^{n} a_i f_i(x) \leqq G(x)$ for all n. For any h such that $h \in \mathfrak{R}^0$, we denote $\{S\colon\ V^0(S)=h,$ where S is a measurable set$\}$ by $\mathfrak{S}_h$. Clearly, $\{a_i c_i\}(i=0,\ 1,\ 2,\ \ldots)$ is an element of $l^{(2)}$ and we denote it by c. Let $\mathfrak{S}_c^0$ be the set of all measurable sets S_0 such that

$$S_0 \in \mathfrak{S}_c$$

and

$$\int_{S_0} f_0(x)dx \geqq \int_{S} f(x)dx \ \text{ for any } S \in \mathfrak{S}_c,$$

Assumption (II)

$\mathfrak{S}_c^0$ is not empty.

We denote $\underset{S \in \mathfrak{S}_h}{\text{Sup}} \int_S f_0(x)dx$ by $a^*(\mathfrak{S}_z)$. Then we can state the third assumption as follows.

Assumption (III)

For an arbitrary positive number ε, there exists a positive number η such that

$$a^*(\mathfrak{S}_z) - \int_{S_0} f_0(x)px < \varepsilon$$

for any $h \in \mathfrak{R}^0$ such that $\| h - c \| < \eta$.

We can easily see that, if the conditions in Assumption (II) and (III) are statisfied, $V(S_0)$ is a boundary point of $\overline{\mathfrak{R}}$. Then we can obtain the following lemma.

LEMMA 4. *Let c be an inner point of $\overline{\mathfrak{R}^0}$, and S_0 be a measurable set which satisfies Assumptions (I), (II) and (III). Then there exists a sequence $\{k_i\}(i=1, 2, \ldots)$ of real numbers such that $\sum_{i=1}^{\infty} k_i^2 < \infty$,*

and
$$f_0(x) \geqq \sum_{i=1}^{\infty} k_i a_i f_i(x) \qquad \textit{if } x \in S_0$$
$$f_0(x) \leqq \sum_{i=1}^{\infty} k_i a_i f_i(x) \qquad \textit{other wise}$$

except perhaps on a set of measure zero.

Proof. First consider the special case, where

$$\underset{S \in \mathfrak{S}_c}{\text{Sup}}\ V_0(S) = \underset{S \in \mathfrak{S}_c}{\text{Inf}}\ V_0(S)$$

In this case, if there exists another inner point c' of $\overline{\mathfrak{R}^0}$ such that

$$\underset{S \in \mathfrak{S}_{c'}}{\text{Sup}}\ V_0(S) > \underset{S \in \mathfrak{S}_{c'}}{\text{Inf}}\ V_0(S),$$

We can find the third inner point c^* such that

$$c = \alpha c^* + \beta c', \quad \text{where} \quad \alpha, \beta \geqq 0 \text{ and } \alpha + \beta = 1,$$

and $\mathfrak{S}_{c^*} \neq 0$

Further there exist S' and S'' which belong to $\mathfrak{S}_{c'}$ and

$$V_0(S') > V_0(S'').$$

For any $S \in \mathfrak{S}_{c^*}$,

$$\alpha V(S) + \beta V(S') \in \overline{\mathfrak{R}}$$
$$\alpha V(S) + \beta V(S'') \in \overline{\mathfrak{R}}$$

from the convexity of $\overline{\mathfrak{R}}$

clearly $\alpha V(S)+\beta V(S')\neq\alpha V(S)+\beta V(S'')$

and $\alpha V_i(S)+\beta V_i(S)=\alpha V_i(S)+\beta V_i(S'')=c_i\,,\ (i=1,\ 2,\ \dots)$

This contradicts with the assumption: $\sup_{S\in\mathfrak{S}_c} V_0(S)=\inf_{S\in\mathfrak{S}_c} V_0(S)$, and so we get $\sup_{S\in\mathfrak{S}_{c'}} V_0(S)=\inf_{S\in\mathfrak{S}_{c'}} V_0(S)$. From this and the fact that $\mathfrak{R}^0$ lies densely in $\overline{\mathfrak{R}}^0$, there exists only one point in $\overline{\mathfrak{R}}$ corresponding to an arbitrary point c' in $\overline{\mathfrak{R}}^0$, that is, there exists a function defined on $\overline{\mathfrak{R}}^0$

$$f(g)=h_0 \text{ for any } g\in\overline{\mathfrak{R}}^0$$

such that $(h_0\,,\ g_1\,,\ g_2\,,\ \dots)\in\overline{\mathfrak{R}}\,.$

and for any $\alpha,\ \beta\geqq 0:\alpha+\beta=1$

$$f(\alpha g'+\beta g'')=\alpha f(g')+\beta f(g'') \quad \text{for any } g',\ g''\in\overline{\mathfrak{R}}^0$$

However, we can easily extend $f(x)$ defined on $\overline{\mathfrak{R}}^0$ to a linear functional on the subspace $l^{(2)}_0$ which we get by replacing the first elements of all elements of l^2 by zero. Now, let the extended linear functional be $F(x)$.

Then by the remark in § 4, there exists a point k of $l^{(2)}_0$ such that

$$F(x)=(k,\ x)=\sum_{i=1}^{\infty}k_ix_i\,.$$

Therefore, $V_0(S)=F(V^0(S))=\sum_{i=1}^{\infty}k_iV_i(S)$

where $V^0(S)=\{0,\ V_1(S),\ V_2(S),\ \dots\}$,

that is,

$$\int_S f_0(x)dx=\sum_{i=1}^{\infty}k_i\int_S a_if_i(x)dx$$

Then by Assumption (I) we have

$$\int_S f_0(z)dx=\int_S\sum_{i=1}^{k}k_ia_if_i(x)dx \qquad \text{for every Borel set } S.$$

Therefore we get

$$f_0(x)=\sum_{i=1}^{\infty}k_ia_if_i(x)$$

except on a set of measure zero. Thus, the proof has been carried out in the special case where $\sup_{S\in\mathfrak{S}_c} V_0(S)=\inf_{S\in\mathfrak{S}_c} V_0(S)$.

Secondly, we treat the case where $\operatorname{Sup}_{S\in\mathfrak{S}c} V_0(S') > \operatorname{Inf}_{S\in\mathfrak{S}c} V_0(S)$.

Because c is an inner point of $\overline{\mathfrak{R}}^0$, we can fiind a_0 such that

$$a^* = \operatorname{Sup}_{S\in\mathfrak{S}c} V_0(S) > a_0 > \operatorname{Inf}_{S\in\mathfrak{S}c} V_0(S) = a_* \quad \text{and}$$

$$\{a_0\,,\ a_1c_1\,,\ a_2c_2\,,\ \ldots\} \text{ is an inner point of } \overline{\mathfrak{R}}\,.$$

By virtue of lemma 3, there cxists a hyperplane II containing $(a^*,\ a_1c_1,\ a_2c_2\,,\ \ldots)$ such that II contains only boundary points of $\overline{\mathfrak{R}}$ as their common points and $\overline{\mathfrak{R}}$ lies entirely on one side of II.

Let the hyperplane be

$$k_0(x_0-a^*)-\sum_{i=1}^{\infty}k_i(x_i-a_ic_i)=0\,, \tag{5.1}$$

where

$$(k_0\,,\ k_1\,,\ k_2\,,\ \ldots) \in l^{(2)}$$

Because $(a_0\,,\ a_1c_1\,,\ a_2a_2\,,\ \ldots)$ is an inner point of $\overline{\mathfrak{R}}$

$$k_0(a_0-a^*) < 0 \tag{5.2}$$

Therefore we get $k_0 \neq 0$ and so we can assume $k_0=1$ without loss of generality. From (5.2), for any $g \in \overline{\mathfrak{R}}$, we get

$$(g_0-a^*)-\sum_{i=1}^{\infty}k_i(g_i-a_ic_i) \leqq 0\,.$$

This means

$$\int_S f_0(x)dx - \sum k_i \int a_if_i(x)dx \leqq a^* - \sum_{i=1}^{\infty} k_ia_ic_i\,,$$

that is, by virtue of Assumption (I)

$$\int_S (f_0(x)-\sum k_ia_if_i(x))dx \leqq \int_{S_0} (f_0(x)-\sum k_ia_if_i(x))dx$$

for any Borel set S. From the last inequality, we conclude

$$f_0(x)-\sum_{i=1}^{\infty}k_ia_if_i(x) \geqq 0 \qquad \text{if } x \in S_0$$

$$\text{〃} \qquad \leqq 0 \qquad \text{otherwise}.$$

q. e. d.

Finally we treat the case where c is a boundory point of $\overline{\mathfrak{R}}^0$. For this purpose, we shall introduce some difinitions and prove some lemmas.

Definition. Let ξ be a non zero element of $l^{(2)}$.

We call ξ maximal relative to c if

$$(\xi,\ g) \leqq (\xi,\ c) \quad \text{for all} \quad g \in \overline{\mathfrak{N}}^0$$

Definition. Let $\{\xi^i\}(i=1,\ 2,\ \ldots, r)$ be a finite sequence of elements of $l^{(2)}$. We call it a maximal set relative to c, if $\{\xi^i\}(i=1,\ 2,\ \ldots,\ r-1)$ is maximal relative to c and

$$(\xi^r,\ x) \leqq (\xi^r,\ c)$$

holds for all points $x \in \overline{\mathfrak{N}}^0$ for which

$$(\xi^i,\ g)=(\xi^i,\ c) \quad (i=1,\ 2,\ \ldots\ ,\ r-1).$$

Definition. We call an infinite sequence $\{\xi^i\}(i=1, 2, \ldots)$ of elements of $l^{(2)}$ maximal relative to c, if, for any positive integer r, $\{\xi^i\}(i,\ 1,\ 2,\ \ldots, r)$ is a maximal set relative to c.

Definition. We call $\{\xi^i\}(i=1,\ 2,\ \ldots,\ r)$ complete maximal relative to c, if $\{\xi^i\}(i=1,\ 2,\ \ldots\ ,\ r)$ is maximal relative to c and no element ξ^{r+1} of $l^{(2)}$ exists such that ξ^{r+1} is linearly independent of the sequence $\{\xi^i\}\ (i=1,\ 2,\ldots,\ r)$ and $\{\xi^i\}(i=1,\ 2,\ \ldots\ ,\ r,\ r+1)$ is maximal relative to c.

Difinition. We call $\{\xi^i\}(i=1,\ 2,\ \ldots)$ complete maximal relative to c if $\{\xi^i\}$ is maximal relative to c and there exists no element ξ of $l^{(2)}$ such that ξ is linearly independent of $\{\xi^i\}(i=1,\ 2,\ \ldots)$ and

$$(\xi,\ g) \leqq (\xi,\ c)$$

for any $g \in \overline{\mathfrak{N}}^0$ for which

$$(\xi^i,\ g)=(\xi^i,\ c) \qquad (i=1,\ 2,\ \ldots)$$

LEMMA 5. *If c is a boundary posnt of $\overline{\mathfrak{N}}^0$, then there exists a set $\{\xi^i\}(i=1,2,\ \ldots)$ of elements of $l^{(2)}$ that is a complete maximal set relative to c.*

Proof. Because c is a boundary point of $\overline{\mathfrak{N}}^0$, by virtue of lemma 3 there exists a hyperplaue Π cantaning c such that $\overline{\mathfrak{N}}^0$ lies entirely on one side of it.

Let the equation of Π be given by

$$(\xi,\ g-c)=0 \qquad \text{where} \qquad \xi \in l^{(2)}.$$

Since $\overline{\mathfrak{N}}^0$ lies entirely on one side of Π, either

$$(\xi,\ g-c) \geqq 0 \qquad \text{for all } g \in \overline{\mathfrak{N}}^0 \tag{5.3}$$

or

(5.4) $$(\xi,\ g-c)\leqq 0 \qquad \text{for all } g\in\overline{\mathfrak{R}}^0$$
holds.
Now put $\xi^1=-\xi$, if (5.3) holds, and $\xi^1=\xi$ otherwise. Clearly ξ^1 is maximal relative to c. If ξ^1 is not a complete maximal set relative to c, there exists ξ^2 such that ξ^2 is linearly independent of ξ^1 and $\{\xi^1,\ \xi^2\}$ is maximal relative to c, and so on.
Thus, the proof leads to completion.

LEMMA 6. *If* $\{\xi^i\}(i=1, 2, \dots)$ *is a maximal set of elements of* $l^{(2)}$ *relative to c, and if* $V(S_0)=c$, *then following two conditions are fulfilled for all* $x\in R^n$ *(perhops except on a set of measure zero.)*
a) If a point x in R^n *satisfies*

$$\sum_{j=1}^{\infty}\xi_j^i a_j f_j(x)=0 \qquad (i=1,\ 2,\ \dots,\ u-1)$$

and

$$\sum_{j=1}^{\infty}\xi_j^u a_j f_i(x)>0 \qquad \textit{for some integer } u\geqq 1,$$

then x belongs to S_0.
b) If a point x in R^n *satisfies*

$$\sum_{j=1}^{\infty}\xi_j^i a_j f_j(x)=0 \qquad (i=1,\ 2,\ \dots,\ u-1)$$

and

$$\sum_{j=1}^{\infty}\xi_j^u a_j f_j(x)<0 \qquad \textit{for some integer } u\geqq 1$$

then x does not belong to S_0.

Proof. Because $\{\xi^i\}(i=1,\ 2,\ \dots)$ is a maximal set relative to c, ξ^1 is maximal relative to c, that is,

$$(\xi^1,\ g-c)\leqq 0$$

hence
$$\sum_j \xi_j^1 g_j\leqq \sum_j \xi_j^1 a_j c_j \qquad \text{for all } g\in\mathfrak{R}^0$$

and so
$$\sum \xi_j^1\int_S a_j f_j(x)dx\leqq \sum \xi_j^1 a_j\int_{S_0} f_j(x)dx$$

$$\int_S\sum_{j=1}^{\infty}\xi_j^1 a_j f_j(x)dx\leqq \int_{S_0}\xi_j^1 a_j a_j f_j(x)dx \qquad \text{for all Borel set } S.$$

This means

$$\sum_{j=1}^{\infty}\xi_j^1 a_j f_j(x)\leqq 0 \qquad \text{if } x\,\overline{\in}\, S_0$$

$$\text{〃} \qquad \geqq 0 \qquad \text{if } x\in S_0$$

except on a set of measure zero. This implies that for all x (except perhaps on a set of measure zero) the following condition holds:

$$x \text{ belongs to } S_0\text{, if } \sum_{j=1}^{\infty} \xi_j^1 a_j f_j(x) > 0$$

$$x \text{ does not belong to } S_0\text{, if } \sum_{j=1}^{\infty} \xi_j^1 a_j f_j(x) < 0\,.$$

Thus, condition (a) and (b) of our lemma must be fulfilled for $u=1$. We shall now show that if (a) and (b) hold for $u=1, 2, \ldots, v$, then they must also hold for $u=v+1$. For this purpose, consider the set R_n' of all points x which has the following property:

$$\sum_{j=1}^{\infty} \xi_j^i a_j f_j(x) = 0 \qquad \text{for } i=1, 2, \ldots, v.$$

If R_n is replaced by R_n', then ξ^{v+1} is maximal relative to $c'=\{0, a_1 c_1', a_2 c_3', \ldots\}$ where

$$c_i' = \int_{S_0'} f_i(x) dx \qquad (S_0' = S \frown R_n')$$

Hence, for any $x \in R_n'$ (except perhops on a set of measure zero) the following condition holds:

$$x \in S_0'\text{, when } \sum \xi_j^{v+1} f_j'(x) > 0$$

$$x \in \overline{S}_0'\text{, when } \qquad < 0$$

This implies that (a) and (b) hold for $u=v+1$. This completes the proof of our lemma.

LEMMA 7. *Let* $\{\xi^i\}(i=1, 2, \ldots)$ *be a complete maximal set relative to* c *and let* T *be the set of all points* g *of* $\overline{\mathfrak{R}}^0$ *for which*

$$(\xi^i, g-c)=0 \qquad \text{for } i=1, 2. \ldots$$

Then T *is a bounded, closed, and convex set and* c *is an inner point of* T.

Proof. The boundedness and convexity are clear. The closedness follows from the continuity of inner product of $(\xi, g-c)$ in g.
Let c be a boundary point of T. Then there exists a hyperplane Π containing c such that the intersection of Π and T contains only boundary points of T and T lies enterely on one side of Π.
Let the equation of Π be given by $(\xi, g-c)=0$, where we can say ξ is independent of $\{\xi^i\}(i=1, 2. \ldots)$, because Π contains at most boundary points of T in their intersection. Since T lies on one side of Π, we have either

$$(\xi,\ g-c) \geqq 0 \qquad \text{for all } g \in T$$

or

$$(\xi,\ g-c) \leqq 0 \qquad \text{for all } g \in T.$$

Put $\xi'=\xi$ in the latter case, and $\xi'=-\xi$ in the former case.

Then $$(\xi',\ g-c) \leqq 0 \qquad \text{for all } g \in T.$$

This contradicts with the fact that $\{\xi^i\}(i=1,\ 2,\ \dots)$ is a complete maximal set. Thus c must be an interior point of T and our lemma has been proved.

LEMMA 8. *If is a boundary point of* $\overline{\mathfrak{R}}^0$ *and if* $\{\xi^i\}(i=1,\ 2,\ \dots)$ *is a complete mvximal set of elements of* $l^{(2)}$ *relative to c, then a necessary condition for a member S of* $\mathfrak{S}$ *to satisfy the Assumption (I), (II) and (III) is that there exists a sequence of real numbers* $\{k_i\}(i=1,\ 2,\ \dots)$ *such that* $\sum_{i=1}^{\infty} k_i^2 < \infty$ *and for all* $x \in R'$ *the inequalities*

$$f_0(x) \geqq \sum k_i f_i(x) \qquad \textit{if } x \in S$$
$$f_0(x) \leqq \sum k_i f_i(x) \qquad \textit{if } x \overline{\in} S$$

hold except perhaps on a set of measure zero, where R'_n *is the set of all points x for which*

$$\sum_{j=1}^{\infty} \xi^i_j a_j f_j(x) = 0 \qquad \text{for } i=1,\ 2,\ \dots$$

Proof. Let R^* be the set of points x which satisfy the following two conditions,

$$(5.5) \qquad \sum_j \xi^i_j a_j f_j(x) \neq 0 \qquad \text{for at least one value } i,$$

and

$$(5.6) \qquad \sum_j \xi^{i_0}_j a_j f_j(x) > 0 \qquad \text{where } i_0 \text{ is the smallest integer for which } \sum^{\infty} \xi^i_j a_j a_j f_j(x) \neq 0.$$

For any $S \in \mathfrak{S}$, let S^* be $S \frown (R_n - R'_n)$

It follows from lemma 6. that $R^* - (R^* \frown S^*)$ and $S^* - (R^* \frown S^*)$ are sets of measure zero. Thus

$$\int_{S^*} f_i(x)dx = \int_{R^*} f_i(x)dx \qquad (i=0,\ 1,\ 2, \dots)$$

for all $S \in \mathfrak{S}$. Now, let

$$f_i^*(x) = f_i(x) \qquad \text{if } x \in R'_n$$
$$f_i^*(x) = 0 \qquad x \overline{\in} R'_n \quad (i=1,\ 2,\ \dots)$$

and furthermore

$$c_i^* = c_i - \int_{R^*} f_i(x)dx \qquad (i=1,\ 2,\ \dots).$$

Let V^*, $V^{*(0)}$, $\mathfrak{R}^*$, $\mathfrak{R}^{*0}$, $\mathfrak{S}^*$ and $\mathfrak{S}_0^*$ have the same meaning with reference to the functions $f_0^*(x)$, $f_1^*(x)$, ... and the point $c^*=(0,\ a_1c_1^*,\ a_2c_2^*,\ \dots)$ as V, $V^{(0)}$, $\mathfrak{R}$, $\mathfrak{R}^0$, $\mathfrak{S}$ and $\mathfrak{S}_0$ have with reference to the functions $f_0(x)$, $f_1(x)$, ... and the point $c=(0,\ c_1,\ c_2,\ \dots)$.

It follows from lemma 6. that for any subset S of R_n for which $V^0(S)$ is a point of the set T defined in lemma 7, we have

$$\int_S f_i(x)dx = \int_S f_i^*(x)dx + \int_{R^*} f_i(x)dx \qquad (i=0,\ 1,\ 2,\ \dots)$$

Since the range of $V^{*0}(S)$ is equal to $\mathfrak{R}^{*0}$ even when S is restricted to subsets S for which $V^0(S) \in T$, $\mathfrak{R}^{*0}$ is obtained by a translation of the set T. The same translation brings the point c to c^*. It then follows from lemma 7 that c^* is an interior point of $\mathfrak{R}^{*0}$. An application of lemma 4 gives the following necessary condition for a member S_0 of $\mathfrak{S}^*$ to be a member of $\mathfrak{S}_0^*$, that is, it is necessary for S_0 to be a member of $\mathfrak{S}_0^*$ that there exists an element $\{k_i\}$ $(i=1,\ 2,\ \dots)$ of $l^{(2)}$ such that, for all x except perhaps on a set of measure zero,

$$f_0^*(x) \geqq \sum k_i a_i f_i^*(x) \qquad \text{if } x \in S$$
$$\text{〃} \quad \leqq \quad \text{〃} \qquad \text{other wise}$$

From the definition of $f^*(x)$ it holds that

$$f_0(x) \geqq \sum_{i=1}^{\infty} k_i a_i f_i(x) \qquad \text{if } x \in S \frown R'$$
$$\text{〃} \quad \leqq \quad \text{〃} \qquad \text{if } x \in (R-S) \frown R'$$

The lemma. follows from this and from the fact that every member of $\mathfrak{S}$ is a member of $\mathfrak{S}^*$ and that a member of S of $\mathfrak{S}^*$ is one of $\mathfrak{S}_0^*$ if and only if S is one of $\mathfrak{S}_0$.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] George B. Dantzig and A. Wald, On the fundamental lemma of Neyman and Pearson, *Ann. Math. Stat.* 1951.

[2] D. Blackwell, The range of certain vector integrals, *Proc. Amer. Math. Soc.*, Vol 2. 1951.