NOTE ON A RELATION BETWEEN THE DISTRIBUTION FUNCTIONS AND CHARACTERISTIC FUNCTIONS

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1. When the limitting distribution of the random variable concerned is used in statistical inference, it is important to know how the distribution fits. On the other hand, the characteristic function is often used to prove that a sequence of distribution functions converges to some distribution function. In this case, therefore, it is desirable to know about the distance between distribution functions from that of characteristic functions. In this note we shall treat this problem. Let $F(x)$ and $G(x)$ be one-dimensional distribution functions and let $\varphi(t)$ and $\psi(t)$ their characteristic functions, respectively. Then we have the following:

THOREM *If $F(x)$ and $G(x)$ have second order moments,*

and
$$\int_{-T}^{T} \frac{|\varphi(t)-\psi(t)|}{|t|} dt=\varepsilon \qquad T \geqq 1, \tag{1}$$

then

$$|F(x)-G(x)| \leqq 2\sqrt{\frac{\log T+1}{T}}+\varepsilon \qquad x \notin K \tag{2}$$

where K is a set of measure less than $4\sqrt{\frac{\log T+1}{T}}$

It should be remarked that the intergral (1) is always convergent, and according as we take T larger or smaller, ε becomes larger or smaller. Therefore, in order to minimize the right hand side of (2), we must chose a suitable T. As for the set K, roughly speaking, it relates the discontinuity point of F and G.

2. LEMMA *Let $F(x)$ be a distribution function which has the second order moment, then for any $\delta > 0$ we have*

$$\int_{-\infty}^{\infty} (F(x+\delta)-F(x-\delta))dx=2\delta \tag{3}$$

PROOF: By the Tchebyschef's inequality we have

$$|F(x+\delta)-F(x-\delta)| \leqq \inf\left\{1, \frac{1}{(x\pm\delta-a)^2}\right\}$$

where a is a mean of $F(x)$. Therefore, the integral $\int_{-\infty}^{\infty}(F(x+\delta)-F(x-S))dx$ exists. For any real number p, we have

$$e^{-\frac{p^2x^2}{2}}\,|F(x+\delta)-F(x-\delta)| \leqq |F(x+\delta)-F(x-\delta)|=F(x+\delta)-F(x-\delta).$$

Consequently

$$\lim_{p\to 0}\int_{-\infty}^{\infty} e^{-\frac{p^2x^2}{2}}(F(x+\delta)-F(x-\delta)dx=\int_{-\infty}^{\infty}(F(x+\delta)-F(x-\delta))dx.$$

Put

$$B_p=\int_{-\infty}^{\infty} e^{-\frac{p^2x^2}{2}}(F(x+\delta)-F(x-\delta)dx.$$

Since $|F(x)| \leq 1$, by the integration by part we get

$$B_p=\int_{-\infty}^{\infty}\left(e^{-\frac{p^2(x-\delta)^2}{2}}-e^{-\frac{p^2(x+\delta)^2}{2}}\right)F(x)dx$$

$$=\frac{1}{p}\int_{-\infty}^{\infty}(\Phi(p(x-\delta))-\Phi(p(x+\delta))dF(x)$$

where

$$\Phi(x)=\int_{x}^{\infty} e^{-\frac{x^2}{2}}dx$$

Now

$$\Phi(x)=\Phi(0)+x\Phi'(0)+\frac{x^2}{2}\Phi''(\theta x),\ \ (0<\theta<1)$$

and

$$\Phi(0)=\sqrt{\frac{\pi}{2}},\quad \Phi'(0)=-1,\quad |\Phi''(\theta x)|=|\theta x\, e^{-\frac{\theta x^2}{2}}|\leq 1.$$

Therefore

$$|B_p-2\delta|=|B_p-\frac{1}{p}\int 2p\delta\, dF(x)| \leqq \frac{1}{p}\int_{-\infty}^{\infty}|p^2(x-\delta)^2-p^2(x+\delta)^2|\,dF(x)$$

$$=\frac{1}{p}\int_{-\infty}^{\infty} 2p^2(x^2+\delta^2)dF(x)=2p(\sigma+\delta^2),$$

where

$$\sigma=\int_{-\infty}^{\infty} x^2dF(x).$$

From this we have

$$\lim_{p\to 0} B_p=2\delta,$$

which proves the lemma.

COROLLARY *There exists a set K such that*

$$F(x+\delta)-F(x-\delta)<\sqrt{\delta}\qquad x\notin K,$$

and the measure of K is smaller than $2\sqrt{\delta}$.

Put

$$K=\{x: F(x+\delta)-F(x-\delta) \geqq \sqrt{\delta}\},$$

then we have

$$2\delta \geqq \int_K (F(x-\delta)-F(x-\delta))dx \geqq \sqrt{\delta}\, m(K),$$

for $F(x+\delta)-F(x-\delta)$ is non-negative where $m(K)$ denotes the Lebesque measure of K. Therefore

$$m(K) \leqq 2\sqrt{\delta}.$$

3. Now, we prove our theorem.

Put

$$\sqrt{\delta}=\sqrt{\frac{\log T+1}{T}}.$$

Then by the corollary in section 2, we can find sets K_F and K_G such that

$$m(K_F) \leqq 2\sqrt{\delta}, \quad m(K_G) \leqq 2\sqrt{\delta}$$

and

$$F(x+\delta)-F(x-\delta) < \sqrt{\delta} \qquad x \notin K_F$$
$$G(x+\delta)-G(x-\delta) < \sqrt{\delta} \qquad x \notin K_G.$$

Set $K=K_F \smile K_G$, and we have $m(K) \leqq 4\sqrt{\delta}$.

If for any $a \notin K$, is $|F(a)-G(a)| < 2\sqrt{\delta}=2\sqrt{\frac{\log T+1}{T}}$, we have nothing to prove. Therefore we may assume $|F(a)-G(a)|=M \geqq 2\sqrt{\delta}$. Moreover, we can assume $F(a) > G(a)$ without losing generality.

Then for $a+\delta \geqq x \geqq a$ it holds that

$$F(x) \geqq F(a)$$
$$G(x) \leqq G(a)+(G(a+\delta)-G(a)) \leqq G(a)+\sqrt{\delta}$$
$$F(x)-G(x) \geqq F(a)-G(a)-\sqrt{\delta} \geqq M-\frac{M}{2}=\frac{M}{2}.$$

Similarly for $a \geqq x \geqq a-\delta$

$$F(x) \geqq F(a)-\sqrt{\delta}$$
$$G(x) \leqq G(a)$$
$$F(x)-G(x) \geqq \frac{M}{2}.$$

As the second order moments of $F(x)$ and $G(x)$ exist, we have

$$\int_{-\infty}^{\infty} (F(x)-G(x))\, e^{ixt}\, dx=\frac{\varphi(t)-\psi(t)}{it}$$

Hence, for any $p>0$

$$\int_{-\infty}^{\infty}\frac{\varphi(t)-\psi(t)}{it}e^{-iat}e^{-\frac{p^2t^2}{2}}dt=\int_{-\infty}^{\infty}\left\{(F(x)-G(x))\int_{-\infty}^{\infty}e^{i(x-a)t-\frac{p^2t^2}{2}}dt\right\}dx$$

$$=\int_{-\infty}^{\infty}(F(x)-G(x))\,e^{-\frac{(x-a)^2}{2p^2}}\int_{-\infty}^{\infty}e^{-\frac{p^2\left(t-\frac{i(x-a)}{p^2}\right)^2}{2}}dt\,dx$$

$$=\frac{\sqrt{2\pi}}{p}\int_{-\infty}^{\infty}(F(x)-G(x))\,e^{-\frac{(x-a)^2}{2p^2}}dx$$

$$=\sqrt{2\pi}\int_{-\infty}^{\infty}\{F(a+px)-G(a+px)\}\,e^{-\frac{x^2}{2}}dx.$$

Now
$$F(a+px)-G(a+px)\geqq\frac{M}{2}\text{ for }|px|<\delta.$$

Therefore

$$\sqrt{2\pi}\int_{-\frac{\delta}{p}}^{\frac{\delta}{p}}\frac{M}{2}e^{-\frac{x^2}{2}}dx\leqq\sqrt{2\pi}\int_{-\frac{\delta}{p}}^{\frac{\delta}{p}}(F(a+px)-G(a+px))\,e^{-\frac{x^2}{2}}dx$$

$$\leqq\left|\int_{+\frac{\delta}{p}}^{\infty}(F(a+px)-G(a+px))\,e^{-\frac{x^2}{2}}dx\right|+\left|\int_{-\infty}^{-\frac{\delta}{p}}(F(a+px)-G(a+px))\,dx\right|$$

$$+\left|\int_{-T}^{T}\frac{\varphi(t)-\psi(t)}{it}\,e^{-iat}e^{-\frac{p^2t^2}{2}}\right|+\left|\int_{-T}^{\infty}\frac{\varphi(t)-\psi(t)}{it}\,e^{-iat}e^{-\frac{p^2t^2}{2}}dt\right|$$

$$+\left|\int_{-\infty}^{-T}\frac{\varphi(t)-\psi(t)}{it}\,e^{-iat}e^{-\frac{p^2t^2}{2}}dt\right|$$

$$\leqq 2\int_{\frac{\delta}{p}}^{\infty}e^{-\frac{x^2}{2}}dx+\int_{-T}^{T}\frac{|\varphi(t)-\psi(t)|}{|t|}dt+4\int_{T}^{\infty}\frac{e^{-\frac{p^2t^2}{2}}}{t}dt.$$

By the inequality $\int_A^{\infty}e^{-\frac{x^2}{2}}dx\leqq e^{-\frac{A^2}{2}}$, we get

$$M\left(\pi-e^{-\frac{\delta^2}{2p^2}}\right)\leqq\frac{M}{2}\cdot\sqrt{2\pi}\int_{-\frac{\delta}{p}}^{\frac{\delta}{p}}e^{-\frac{x^2}{2}}dx\leqq 2\,e^{-\frac{\delta^2}{2p^2}}+4\,e^{-\frac{p^2T^2}{2}}+\varepsilon\,.$$

Put
$$p^2=\frac{\log T+1}{T^2},$$

and we have

$$M\left(\pi-e^{-\frac{\log T+1}{2}}\right)\leqq 2\,e^{-\frac{\log T+1}{2}}+4\,e^{-\frac{\log T+1}{2}}+\varepsilon\,.$$

Consequently

$$M\left(\pi - e^{-\frac{1}{2}}\right) \leqq 6\frac{e^{-\frac{1}{2}}}{\sqrt{T}} + \varepsilon.$$

But $e^{-\frac{1}{2}} \leqq 0.61$ $\pi \geqq 3.14$ and $\log T \geqq 0$ $(T \geqq 1)$, therefore,

$$2.5M \leqq 3.66\frac{1}{\sqrt{T}} + \varepsilon$$

$$M \leqq 2\frac{1}{\sqrt{T}} + \varepsilon \leqq 2\sqrt{\frac{\log T + 1}{T}} + \varepsilon\,.$$

This proves the theorem.

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