Note on Wiener's Prediction Theory

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1. Let $\{X(t, \omega);\ -\infty < t < \infty,\ \omega \in \Omega\}$ be a measurable strictly stationary ergodic stochastic process, defined on a probability space Ω, with $\mathrm{E}\{|X(0)|^2\} < \infty$, and assume that $X(t)$ is continuous in mean. Then the covariance function

$$(1) \qquad R(u) = \mathrm{E}\{X(t+u)\overline{X(t)}\}$$

does not depend on t. Following Doob [1], we shall not set the condition that $\mathrm{E}\{X(t)\}$ is independent of t. Therefore, $R(u)$ is not in general a true covariance. It is well known that $R(u)$ is continuous in u, and can be expressed in the form

$$(2) \qquad R(u) = \int_{-\infty}^{\infty} e^{iux}\, dF(x),$$

where F is monotone non-decreasing, bounded and is called the spectral distribution function of the process. From our hypothesis it is proved that

$$(3) \qquad R(u) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t+u, \omega)\, \overline{X(t, \omega)}\, dt$$

for all $u(-\infty < u < \infty)$ and for almost all ω. (Wiener [1], 10.)

Let $f(t)$ be a sample function of a process which satisfies the above hypotheses, and suppose that the spectral distribution function F is known and that $f(t)$ satisfies the condition (3), that is,

$$(3') \qquad R(u) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+u)\, \overline{f(t)}\, dt, \qquad -\infty < u < \infty.$$

Under these circumstances, Wiener [2] studied the problem to predict the future value of $f, f(t+\alpha)$, at a period α units of time later than t, knowing the past and present values $\{f(t-\tau);\ 0 \leqq \tau < \infty\}$ up to the time t. Assuming $f(t)$ to be bounded, he investigated the predicting operator with the expression

$$(4) \qquad \int_0^{\infty} f(t-\tau)\, dK(\tau),$$

where $K(\tau)$ is of finite total variation. Wiener considered the predicting operator (4) to be optimum, if the mean square error

$$(5) \qquad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| f(t+\alpha) - \int_0^{\infty} f(t-\tau)\, dK(\tau) \right|^2 dt$$

is minimum, from the point of view of the theory of least squares.

However, even if $f(t)$ is supposed to be bounded and continuous, it is not clear whether the limit (5) exists, or whether the limit (5), even if it exists, attains the minimum value, when $K(\tau)$ runs over all functions of finite total variation. In this paper we want to give a way of removing these vague points. For this purpose we formulate the problem as follows.

Put

$$(6) \qquad \delta = \inf_{n,\tau_j,a_j} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t+\alpha) - \sum_{j=1}^{n} a_j f(t-\tau_j)|^2 dt \right\}^{1/2}$$

where inf means the greatest lower bound of the set of the indicated limits, when n runs over all positive integers, $\tau_1, \tau_2, \cdots, \tau_n$ assume all non-negative real numbers, and $a_1, a_2, \cdots, a_n$ assume all complex numbers. For a given $\varepsilon>0$, we shall aim at finding n, τ_j and a_j $(j=1, 2, \cdots, n)$ which satisfy

$$(7) \qquad \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t+\alpha) - \sum_{j=1}^{n} a_j f(t-\tau_j)|^2 dt \right\}^{1/2} \leq \delta + \varepsilon,$$

$$\tau_j \geq 0 .$$

2. Let $L(f)$ be the linear space generated by the set of shifted $f(t)$'s, $\{f(t+\tau)\}$, where τ is a parameter which runs over all real numbers. Then any element of $L(f)$ can be written in the form

$$p(t) = \sum_{j=1}^{n} a_j f(t+\tau_j),$$

where n is a positive integer, $\tau_1, \cdots, \tau_n$ are real numbers, and $a_1, \cdots, a_n$ are complex numbers. Now, we introduce an inner product in the linear space $L(f)$ as follows:

$$(p, q) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(t)\, \overline{q(t)}\, dt, \qquad p, q \in L(f).$$

The existence of the limit follows from (3'). Then, define the norm of any element p of $L(f)$ by

$$\|p\| = (p, p)^{1/2},$$

and identify p and $q\,(\in L(f))$, if and only if $\|p-q\|=0$. The space $L(f)$, thus, becomes a linear normed space.

Let $L_2(F)$ be the L_2-space with respect to F-measure. In $L_2(F)$, the inner product and the norm are defined as follows:

$$(g, h)_F = \int_{-\infty}^{\infty} g(x)\, \overline{h(x)}\, dF(x), \qquad g, h \in L_2(F),$$

$$\|g\|_F = \{(g, g)_F\}^{1/2},$$

and two elements g and h $(g, h \in L_2(F))$ are identified, if and only if $\|g-h\|_F=0$. Further, let $L(F)$ be the linear subspace of $L_2(F)$ generated by the set $\{e^{i\tau x}\}$, where τ is a parameter which runs over all real numbers. From (3′) and (2), we have

$$(f(t+\tau), f(t+\tau'))=R(\tau-\tau')=(e^{i\tau x}, e^{i\tau' x})_F,$$

and in general

$$\left(\sum_{j=1}^{n} a_j f(t+\tau_j),\ \sum_{k=1}^{m} b_k f(t+\sigma_k)\right)=\left(\sum_{j=1}^{n} a_j e^{i\tau_j x},\ \sum_{k=1}^{m} b_k e^{i\sigma_k x}\right)_F.$$

This means that the two linear normed spaces $L(f)$ and $L(F)$ are isomorphic by the correspondense

$$\sum_{j=1}^{n} a_j f(t+\tau_j) \longleftrightarrow \sum_{j=1}^{n} a_j e^{i\tau_j x}. \tag{8}$$

Let $L(f;0)$ $(L(F;0))$ be the linear subspace generated by the set $\{f(t-\tau);\ 0\leqq\tau<\infty\}$ $(\{e^{-i\tau x};\ 0\leqq\tau<\infty\})$, and denote the closure of $L(F;0)$ by $L_2(F;0)$.

Our problem stated in **1** is to find p which satisfies

$$\begin{aligned}&\|f(t+\alpha)-p(t)\| \leqq \delta+\varepsilon,\\ &p\in L(f;0),\end{aligned} \tag{9}$$

where δ is defined by (6) and ε is a given positive number.

From (6) we have

$$\begin{aligned}\delta &= \inf_{p\in L(f;0)} \|f(t+\alpha)-p(t)\| = \inf_{g\in L(F;0)} \|e^{i\alpha x}-g(x)\|_F\\ &= \min_{g\in L_2(F;0)} \|e^{i\alpha x}-g(x)\|_F = \|e^{i\alpha x}-h(x)\|_F \qquad \text{(say).}\end{aligned} \tag{10}$$

According to the theory of Hilbert space, the element h which minimizes $\|e^{i\alpha x}-h(x)\|_F$ under the condition that

$$h\in L_2(F;0), \tag{11}$$

is given by the projection of $e^{i\alpha x}$ to the closed linear subspace $L_2(F;0)$. For $h(x)$ to be such a element, it is necessary and sufficient that $e^{i\alpha x}-h(x)$ is orthogonal to $e^{-i\tau x}$ for all $\tau\geqq 0$:

$$\int_{-\infty}^{\infty} (e^{i\alpha x}-h(x))\, e^{i\tau x}\, dF(x)=0, \qquad \tau\geqq 0,$$

i.e.,

$$\int_{-\infty}^{\infty} e^{i\tau x} h(x)\, dF(x)=\int_{-\infty}^{\infty} e^{i\tau x} e^{i\alpha x}\, dF(x), \qquad \tau\geqq 0. \tag{12}$$

This equation has been derived in T. Kawata [1]. The function $h(x)$ which satisfies (11) and (12) exists and is uniquely determined except for the equivalence with respect to F-measure.

Since $L(F; 0)$ is dense in $L_2(F; 0)$, for any given $\varepsilon > 0$, we may take g such that

$$\| g-h \|_F \leq \varepsilon, \qquad g \in L(F; 0).$$

Let us denote by p the element of $L(f; 0)$ corresponding to g by the correspondence (8). Then, we have

$$\| f(t+\alpha) - p(t) \| = \| e^{i\alpha x} - g(x) \|_F \leq \| e^{i\alpha x} - h(x) \|_F + \| h(x) - g(x) \|_F \leq \delta + \varepsilon .$$

Thus, p satisfies (9). If

$$g(x) = \sum_{j=1}^{n} a_j e^{-i\tau_j x},$$

then

$$p(t) = \sum_{j=1}^{n} a_j f(t-\tau_j)$$

and $n, \tau_j, a_j\ (j=1, 2, \cdots, n)$ satisfy (7).

3. We now wish to solve equation (12) under condition (11), assuming, following Wiener [2], that the spectral distribution function $F(x)$ is absolutely continuous and that

$$\int_{-\infty}^{\infty} \frac{|\log F'(x)|}{1+x^2}\, dx < \infty. \tag{13}$$

According to the well-known Paley-Wiener theorem (Paley and Wiener [1], Theorem XII, p. 16), there exists a function $\Psi(x)$ of $L_2(-\infty, \infty)$ such that

$$|\Psi(x)|^2 = F'(x) \tag{14}$$

and the Fourier transform in $L_2(-\infty, \infty)$ of $\Psi(x)$,

$$\psi(y) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \Psi(x)\, e^{ixy} dx, \tag{15}$$

vanishes for negative values of y, i.e.,

$$\psi(y) = 0, \qquad y < 0, \tag{16}$$

except at a set of points of zero measure. In this case, (12) is rewritten as

$$\int_{-\infty}^{\infty} e^{i\tau x}(x)\, h(x)\, \Psi(x)\, \overline{\Psi(x)}\, dx = \int_{-\infty}^{\infty} e^{i(\tau+\alpha)x}\, \Psi(x)\, \overline{\Psi(x)}\, dx, \qquad \tau \geq 0. \tag{17}$$

Let us denote the Fourier transform in $L_2(-\infty, \infty)$ of $h(x)\Psi(x)$ by

$$\xi(y) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} h(x)\Psi(x)\, e^{ixy}\, dx. \tag{18}$$

Then we have

$$h(x)\Psi(x) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \xi(y)\, e^{-ixy}\, dy,$$

and therefore, noticing that $\Psi(x) \neq 0$ almost everywhere because of (13),

$$h(x)=\frac{1}{\Psi(x)} \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \xi(y)\, e^{-ixy}\, dy. \tag{19}$$

Thus, $h(x)$ can be expressed by $\xi(y)$. Concerning this $\xi(y)$, we can prove that

$$\xi(y) = 0, \qquad y < 0, \tag{20}$$

except at a set of points of zero measure. From (11) it follows that there exists a sequence $\{h_n\}$ of elements belonging to $L(F; 0)$, such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} | h(x) - h_n(x) |^2\, dF(x) = 0,$$

which implies that

$$h(x)\Psi(x) = \underset{n\to\infty}{\text{l.i.m.}}\, h_n(x)\Psi(x). \tag{21}$$

Now, $h_n(x)$ can be written as

$$h_n(x) = \sum_{m=1}^{N_n} a_{nm}\, e^{-i\tau_{nm}x}$$

with

$$\tau_{nm} \geq 0. \tag{22}$$

If we make the Fourier transform of both sides of (21), we obtain

$$\xi(y) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{m=1}^{N_n} a_{nm}\, \psi(y - \tau_{nm}).$$

From (16) and (22), we have (20). On the other hand, from (17) and the Parseval's theorem it follows that

$$\int_0^{\infty} \xi(y+\tau)\, \overline{\psi(y)}\, dy = \int_0^{\infty} \psi(y+\alpha+\tau)\, \overline{\psi(y)}\, dy, \qquad \tau \geq 0. \tag{23}$$

Now let us define $\xi_0(y)$ and $h_0(x)$ as follows,

$$\begin{aligned} \xi_0(y) &= \psi(y+\alpha), & y > 0, \\ &= 0, & y < 0, \end{aligned}$$

$$h_0(x) = \frac{1}{\Psi(x)} \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \xi_0(y)\, e^{-ixy}\, dy.$$

Then we have

$$h_0(x) = \frac{1}{\Psi(x)} \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_0^{A} \psi(y+\alpha)\, e^{-ixy}\, dy, \tag{24}$$

(cf. Wiener [1], p. 63, (2.0393)), and

$$\int_{-\infty}^{\infty} | h_0(x) |^2\, dF(x) = \int_{-\infty}^{\infty} | h_0(x)\, \Psi(x) |^2\, dx = \int_{-\infty}^{\infty} | \xi_0(y) |^2\, dy < \infty,$$

that is,

$$h_0(x) \in L_2(F).$$

It is easily proved that $h_0(x)$ satisfies (12) (See T. Kawata [1], Theorem 3). In order to show that $h_0(x)$ is equivalent to the $h(x)$ in 2, we, further, need to prove that $h_0(x)$ belongs to $L_2(F; 0)$, which, however, is yet to be proved.

Finally, we want to make an additional remark.

Suppose that

$$\int_{-\infty}^{\infty} x^{2p} dF(x) < \infty,$$

for a positive integer p. Then we have

$$a_0 + a_1 x + a_2 x^2 \cdots + a_p x^p \in L_2(F; 0),$$

for arbitrary complex numbers $a_0, a_1, \cdots, a_p$.

To prove this, we wish to show that

$$(ix)^k \in L_2(F; 0), \qquad k = 1, 2, \cdots, p.$$

This, however, follows from

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\{n(1-e^{-ix/n})\}^k - (ix)^k|^2 dF(x) = 0,$$

and this holds, since we have

$$\lim_{n\to\infty} n(1-e^{-ix/n}) = ix,$$

$$|\{n(1-e^{-ix/n})\}^k - (ix)^k| \leq |n(1-e^{-ix/n})|^k + |ix|^k \leq |x|^k + |x|^k = 2|x|^k,$$

and

$$\int_{-\infty}^{\infty} 4|x|^{2k} dF(x) < \infty.$$

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