# On the many-dimensional distribution functions*

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## 1. Introduction and summary

We shall discuss in 2 about a generalization to the many-dimensional case of the distance of two one-dimensional d. f. 's (distribution functions) introduced by P. Lévy [4]. The distance of two $p$-dimensional d. f. 's $F(x_1, \cdots, x_p)$ and $G(x_1, \cdots, x_p)$ is defined by the minimum value $\varepsilon$ such that

$$F(x_1-\varepsilon, \cdots, x_p-\varepsilon)-\varepsilon \leqq G(x_1, \cdots, x_p) \leqq F(x_1+\varepsilon, \cdots, x_p+\varepsilon)+\varepsilon$$

for all points $(x_1, \cdots, x_p)$. This metric is equivalent to the convergence of d. f. 's (THEOREM 1) and the metric space of $p$-dimensional d. f. 's is complete (THEOREM 2). By means of our metric we can give a simple proof to the continuity of the convolution of d. f. 's (COROLLARY to THEOREM 3).

Denote a $p$-dimensional d. f. $F(x_1, \cdots, x_p)$ by $F(\boldsymbol{x})$ where $\boldsymbol{x}=(x_1, \cdots, x_p)$. Let $\{F_n(\boldsymbol{x})\}$ be a sequence of $p$-dimensional d. f. 's and let $\{K_n\}$ be the corresponding sequence of classes. By the definition the convergence of $\{K_n\}$ means that there exist a sequence of positive numbers $\{a_n\}$ and a sequence of vectors $\{\boldsymbol{b}_n\}$ which give rise to the convergence of the sequence of d. f. 's $\{F_n(a_n\boldsymbol{x}+\boldsymbol{b}_n)\}$. In this case there may exist another sequences $\{a_n'\}$ and $\{\boldsymbol{b}_n'\}$ which give rise to the convergence of $\{F_n(a_n'\boldsymbol{x}+\boldsymbol{b}_n')\}$. In 3 we shall consider about the relations between sequences $\{a_n\}$, $\{\boldsymbol{b}_n\}$, $\{a_n'\}$, $\{\boldsymbol{b}_n'\}$ and the limit d. f. 's. This is a generalization of the known results in the one-dimensional case (K. Takano [5]) to the many-dimensional case. The main result is Theorem 6.

In 4 we shall discuss about a metrization of the class-convergences of many-dimensional d. f. 's. This is a generalization of the results in [6] to the many-dimensional case. We shall introduce the dispersions of the many-dimensional d. f. 's. If for a sequence of d. f. 's $\{F_n(\boldsymbol{x})\}$ there exist a sequence of positive numbers $\{a_n\}$ and a sequence of vectors $\{\boldsymbol{b}_n\}$ such that the d. f. $F_n(a_n\boldsymbol{x}+\boldsymbol{b}_n)$ tends to a proper d. f., we can substitute the dispersion of $F_n(\boldsymbol{x})$ for $a_n$ (Theorem 9 and 10).

Now, we shall give definitions of some terms used in this paper.

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Let $D_p$ be the set of all $p$-dimensional d. f.'s. In the theory of probability, the convergence in $D_p$ is defined as follows. Let $F_n = F_n(x_1, x_2, \cdots, x_p)$ be elements of $D_p$ for all $n = 0, 1, 2, \cdots$. If for any continuity point $(x_1, x_2, \cdots, x_p)$ of $F_0(x_1, x_2, \cdots, x_p)$ the sequence $\{F_n(x_1, \cdots, x_p);\ n=1, 2, \cdots\}$ converges to $F_0(x_1, \cdots, x_p)$, then the sequence of d. f.'s $\{F_n(x_1, \cdots, x_p)\}$ is said to converge to the d. f. $F_0(x_1, \cdots, x_p)$ and we write $\lim_{n\to\infty} F_n(x_1, \cdots, x_p) = F_0(x_1, \cdots, x_p)$ or briefly $\lim F_n = F_0$. Thus, the space $D_p$ is a convergence space.

We assume that any d. f. is normalized to be continuous to the right at every point, that is

$$F(x_1, x_2, \cdots, x_p) = F(x_1+0, x_2+0, \cdots, x_p+0)$$
$$= \lim_{h_1\downarrow 0,\, h_2\downarrow 0,\, \ldots,\, h_p\downarrow 0} F(x_1+h_1, x_2+h_2, \cdots, x_p+h_p)$$

at every point $(x_1, x_2, \cdots, x_p)$.

Let $F(x_1, \cdots, x_p)$ and $G(x_1, \cdots, x_p)$ be two elements of $D_p$. If there exist a positive number $a$ and a vector $(b_1, b_2, \cdots, b_p)$ such that

$$F(ax_1+b_1, ax_2+b_2, \cdots, ax_p+b_p) = G(x_1, x_2, \cdots, x_p)$$

for any point $(x_1, x_2, \cdots, x_p)$, then we write $F\sim G$. This relation $\sim$ satisfies the equivalence relations: $F\sim F$; if $F\sim G$ then $G\sim F$; if $F\sim G$ and $G\sim H$ then $F\sim H$. Therefore, the elements of $D_p$ are classified by letting $F$ and $G$ belong to the same class if and only if $F\sim G$. Classes of $p$-dimensional d. f.'s in this paper should be interpreted in this meaning.

Let $\{K_n;\ n = 0, 1, 2, \cdots\}$ be a sequence of classes of $p$-dimensional d. f.'s. If there exists a sequence of $p$-dimensional d. f.'s $\{F_n;\ n = 0, 1, 2, \cdots\}$ such that $F_n \in K_n\ (n = 0, 1, 2, \cdots)$ and $\lim F_n = F_0$, then $\{K_n;\ n = 1, 2, \cdots\}$ is said to converge to $K_0$, and we write $\lim K_n = K_0$.

By a unit d. f. we mean the d. f. $U(x_1, x_2, \cdots, x_p)$ of a distribution which has the whole probability 1 placed in a fixed point $(x_1, \cdots, x_p) = (a_1, \cdots, a_p)$:

$$U(x_1, x_2, \cdots, x_p) = \begin{cases} 0, \text{ if } x_r < a_r \text{ for some } r\ (1 \leqq r \leqq p), \\ 1, \text{ if } x_r \geqq a_r \text{ for all } r. \end{cases}$$

When a d. f. is not a unit d. f., it is called to be proper and otherwise improper. A proper d. f. may have the whole probability 1 placed in a hyperplane. All of unit d. f.'s form a class which is called improper. Other classes are called proper. The ch. f. (characteristic function, that is, Fourier-Stieltjes transform) of a. d. f. $F(x)$ is called proper or improper according as the d. f. $F(x)$ is proper or improper.

## 2. A metrization of the convergence space $D_p$ of p-dimensional d. f.'s

Let $F(x_1, x_2, \cdots, x_p)$ be a $p$-dimensional d. f. $(p = 1, 2, \cdots)$. It is convenient to write $x = (x_1, x_2, \cdots, x_p)$ and $F(x) = F(x_1, x_2, \cdots, x_p)$. Let $e = (1, 1, \cdots, 1)$. Then

$$F(x + \varepsilon e) = F(x_1 + \varepsilon, x_2 + \varepsilon, \cdots, x_p + \varepsilon),$$
$$F(x - \varepsilon e) = F(x_1 - \varepsilon, x_2 - \varepsilon, \cdots, x_p - \varepsilon).$$

DEFINITION. *We define a distance of two p-dimensional d.f.'s* $F(x)$ *and* $G(x)$, *as follows.*

(1) $d(F, G) = \min\{\varepsilon;\ F(x - \varepsilon e) - \varepsilon \leqq G(x) \leqq F(x + \varepsilon e) + \varepsilon \text{ for all } x\}$,
*where* $\{\varepsilon;\ C\}$ *denotes the set of all* $\varepsilon$ *satisfying the condition* $C$, *and* $\min\{\varepsilon;\ C\}$ *denotes the minimum value belonging to the set* $\{\varepsilon;\ C\}$.

In the particular case when $p = 1$, $\sqrt{2}\, d(F, G)$ is equal to the P. Lévy's distance of $F(x)$ and $G(x)$.

Clearly, $d(F, G)$ satisfies the following properties.

a) $0 \leqq d(F, G) \leqq 1$, $d(F, G) = 0$ if and only if $F = G$,

b) $d(F, G) = d(G, F)$,

c) $d(F, H) \leqq d(F, G) + d(G, H)$.

THEOREM 1. *Let* $\{F^n(x);\ n = 0, 1, 2, \cdots\}$ *be a sequence of p-dimensional d.f.'s. In order that we have*

(2) $\lim_{n \to \infty} F^n(x) = F^0(x)$

*in the convergence space* $D_p$, *it is necessary and sufficient that we have*

(3) $\lim_{n \to \infty} d(F^n, F^0) = 0$.

To prove THEOREM 1. we shall begin with the following lemmas.

LEMMA 1. *Let* $F(x_1, x_2, \cdots, x_p)$ *be a p-dimensional d.f. If* $a_1, a_2, \cdots, a_p$ *are continuity points of the marginal d.f.'s* $F_1(x_1) = F(x_1, \infty, \cdots, \infty)$, $F_2(x_2) = F(\infty, x_2\ \infty, \cdots, \infty)$, $\cdots$, $F_p(x_p) = F(\infty, \cdots, \infty, x_p)$, *respectively, then* $a = (a_1, a_2, \cdots, a_p)$ *is a continuity point of* $F(x) = F(x_1, x_2, \cdots, x_p)$. (H. Cramér [1], p. 79).

LEMMA 2. *If* $(a_1, \cdots, a_r)$ *and* $(a_{r+1}, \cdots, a_p)$ *are continuity points of* $F(x_1, \cdots, x_r, \infty, \cdots, \infty)$ *and* $F(\infty, \cdots, \infty, x_{r+1}, \cdots, x_p)$, *respectively, then* $(a_1, \cdots, a_p)$ *is a continuity point of* $F(x_1, \cdots, x_p)$.

The proof runs in the same way as LEMMA 1.

LEMMA 3. *Let* $F^n(x_1, \cdots, x_p)$ *be elements of* $D_p$ $(n = 0, 1, 2, \cdots)$.

*If in the convergence space* $D_p$

$$\lim_{n\to\infty} F^n(x_1, \cdots, x_p) = F^0(x_1, \cdots, x_p),$$

*then we have, for any* $r$ $(1 \leqq r < p)$, *in the convergence space* $D_r$

$$\lim_{n\to\infty} F^n(x_1, \cdots, x_r, \infty, \cdots, \infty) = F^0(x_1, \cdots, x_r, \infty, \cdots, \infty).$$

This is evident if we consider the sequence of the corresponding ch. f.'s, but, it is instructive to give a direct proof which I owe to Mr. O. Takenouchi.

PROOF. Let $(a_1, \cdots, a_r)$ be any continuity point of $F^0(x_1, \cdots, x_r, \infty, \cdots, \infty)$. If $(b_1, \cdots, b_s)$ is a continuity point of $F^0(\infty, \cdots, \infty, x_{r+1}, \cdots, x_p)$ $(r+s=p)$, from LEMMA 2, $(a_1, \cdots, a_r, b_1, \cdots, b_s)$ is a continuity point of $F^0(x_1, \cdots, x_p)$. Using simplified notations $\boldsymbol{a} = (a_1, \cdots, a_r)$, $\boldsymbol{b} = (b_1, \cdots, b_s)$ and $F^n(\boldsymbol{a}, \boldsymbol{b}) = F^n(a_1, \cdots, a_r, b_1, \cdots, b_s)$, we have

$$F^0(\boldsymbol{a}, \boldsymbol{b}) = \lim_{n\to\infty} F^n(\boldsymbol{a}, \boldsymbol{b}) \leqq \liminf_{n\to\infty} F^n(\boldsymbol{a}, \infty),$$

$$\therefore \quad F^0(\boldsymbol{a}, \boldsymbol{b}) \leqq \liminf F^n(\boldsymbol{a}, \infty).$$

Letting $(b_1, \cdots, b_s)$ tend to $(\infty, \cdots, \infty)$, we have

$$(4) \qquad F^0(\boldsymbol{a}, \infty) \leqq \liminf_{n\to\infty} F^n(\boldsymbol{a}, \infty).$$

Let $\boldsymbol{a}' = (a_1', \cdots, a_r')$ be a continuity point of $F^0(x_1, \cdots, x_r, \infty, \cdots, \infty)$ such that $a_1' > a_1, a_2' > a_2, \cdots, a_r' > a_r$, and let $\boldsymbol{b} = (b_1, \cdots, b_s)$ be a continuity point of $F^0(\infty, \cdots, \infty, x_{r+1}, \cdots, x_p)$. Then $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{a}', \boldsymbol{b})$ are both continuity points of $F^0(x_1, \cdots, x_p)$. It holds that

$$F^0(\boldsymbol{a}', \boldsymbol{b}) - F^0(\boldsymbol{a}, \boldsymbol{b}) = \lim_{n\to\infty} \{F^n(\boldsymbol{a}', \boldsymbol{b}) - F^n(\boldsymbol{a}, \boldsymbol{b})\}$$

$$\leqq \liminf_{n\to\infty} \{F^n(\boldsymbol{a}', \infty) - F^n(\boldsymbol{a}, \infty)\} \leqq \liminf \{1 - F^n(\boldsymbol{a}, \infty)\}$$

$$= 1 - \limsup F^n(\boldsymbol{a}, \infty),$$

$$\therefore \quad F^0(\boldsymbol{a}', \boldsymbol{b}) - F^0(\boldsymbol{a}, \boldsymbol{b}) \leqq 1 - \limsup F^n(\boldsymbol{a}, \infty).$$

Let $\boldsymbol{b} = (b_1, \cdots, b_s)$ tend to $(\infty, \cdots, \infty)$, and let $\boldsymbol{a}' = (a_1', \cdots, a_r')$ tend to $(\infty, \cdots, \infty)$, then we have

$$1 - F^0(\boldsymbol{a}, \infty) \leqq 1 - \limsup F^n(\boldsymbol{a}, \infty)$$

from which it follows

$$(5) \qquad F^0(\boldsymbol{a}, \infty) \geqq \limsup F^n(\boldsymbol{a}, \infty).$$

(4) and (5) completes the proof.

PROOF OF THEOREM 1. To prove the sufficiency, assume (3). Write $d_n = d(F^n, F^0)$. Then

$$F^0(x-d_n e)-d_n \leqq F^n(x) \leqq F^0(x+d_n e)+d_n, \quad (\text{for all } x).$$

If we let $n \to \infty$, then $d_n \to 0$ by the hypothesis, and we have

$$F^0(x-0) \leqq \liminf F^n(x) \leqq \limsup F^n(x) \leqq F^0(x+0).$$

Therefore, for any continuity point $x$ of $F^0(x)$, we have

$$\lim F^n(x) = F^0(x).$$

To prove the necessity assume (2). Denote the marginal d. f. of $F^0(x)$ corresponding to the $r$-th variate by $F_r^0(x)$ $(r = 1, 2, \cdots, p)$. For any given positive number $\varepsilon$, we can choose $M$ such that

$1-F^0(M, M, \cdots, M) < \varepsilon,$

$F_r^0(-M) < \varepsilon, \quad (r = 1, 2, \cdots, p),$

$\pm M$ are both common continuity points of all $F_1^0(x)$, $F_2^0(x)$, $\cdots$, $F_p(x)$.

For every $r(= 1, 2, \cdots, p)$, we can devide the interval $(-M, M)$ by

$$-M = m_r^0 < m_r^1 < \cdots < m_r^{k_r} = M$$

where every $m_r^i$ is continuity point of $F_r^0(x)$ and

$$|m_r^i - m_r^{i-1}| < \varepsilon, \qquad i = 1, 2, \cdots, k_r.$$

Let

$$S_r = \{m_r^0, m_r^1, \cdots, m_r^{k_r}, \infty\}$$

be the set composed of $m_r^0, m_r^1, \cdots, m_r^{k_r}$ and $\infty$, and let

$$S = S_1 \times S_2 \times \cdots \times S_p$$

be the direct product of $S_1, S_2, \cdots$, and $S_p$. Then $S$ is a finite set and from LEMMA 1 each element of $S$ is a continuity point of $F^0(x)$. Here, for example, that a point $(a_1, \cdots, a_r, \infty, \cdots, \infty)$ is a continuity point of $p$-dimensional d. f. $F(x, \cdots, x_p)$ means that the point $(a_1, \cdots, a_r)$ is a continuity point of $r$-dimensional d. f. $F(x_1, \cdots, x_r, \infty, \cdots, \infty)$. From the hypothesis (2) and LEMMA 3, there exists $N$ such that for all $n > N$ and for all $x \in S$ we have

$$|F^n(x) - F^0(x)| < \varepsilon. \tag{6}$$

We prove that

$$F^n(x-\varepsilon e)-2\varepsilon \leqq F^0(x) \leqq F^n(x+\varepsilon e)+2\varepsilon \tag{7}$$

for all $n > N$ and for all points $x = (x_1, \cdots, x_p)$. If (7) is proved we have $d(F^n, F^0) \leqq 2\varepsilon$, which completes the proof, since $\varepsilon$ is arbitrary positive number.

Now assume that $n > N$. We treat the following four cases determined by the argument of $x = (x_1, x_2, \cdots, x_p)$.

Case (a) $x_r \leqq -M$ for some $r$ $(1 \leqq r \leqq p)$. We have

$$(8) \quad \begin{cases} F^0(x) \leqq F_r^0(-M) < \varepsilon, \\ F^n(x) \leqq F_r^n(-M) < F_r^0(-M) + \varepsilon < 2\varepsilon, \end{cases}$$

using that $(\infty, \cdots, \infty, -M, \infty, \cdots, \infty) \in S$. (8) implies (7).

Case (b) $x_r > M$ for all $r$ $(1 \leqq r \leqq p)$. We have

$$(9) \quad \begin{cases} F^0(x) \geqq F^0(M, M, \cdots, M) > 1-\varepsilon, \\ F^n(x) \geqq F^n(M, M, \cdots, M) > F^0(M, M, \cdots, M) - \varepsilon > 1 - 2\varepsilon, \end{cases}$$

using that $(M, M, \cdots, M) \in S$. (9) implies (7).

Case (c) $|x_r| \leqq M$ for all $r$ $(1 \leqq r \leqq p)$. Choose $x'_r$, $x''_r$ $(r = 1, 2, \cdots, p)$ such that

$$(10) \qquad x_r - \varepsilon \leqq x'_r \leqq x_r \leqq x''_r \leqq x_r + \varepsilon, \quad x'_r \in S_r, \quad x''_r \in S_r,$$

and set

$$x' = (x'_1, x'_2, \cdots, x'_p), \quad x'' = (x''_1, x''_2, \cdots, x''_p),$$

then $x'$ and $x''$ belong to $S$ and we have (6) for $x = x'$ and $x''$. We have

$$\begin{aligned} F^n(x-\varepsilon e) - \varepsilon &\leqq F^n(x') - \varepsilon \leqq F^0(x') \leqq F^0(x) \leqq F^0(x'') \leqq F^n(x'') + \varepsilon \\ &\leqq F^n(x+\varepsilon e) + \varepsilon, \end{aligned}$$

which implies (7).

Case (d) $x_1, x_2, \cdots, x_p$ are divided into two sets, one of which is characterized by $|x_i| \leqq M$ and another by $x_j > M$. In this case, we may assume that

$$\begin{aligned} &|x_i| \leqq M, \text{ for } i = 1, 2, \cdots, s, \\ &x_j > M, \text{ for } j = s+1, \cdots, p, \end{aligned}$$

where $1 \leqq s < p$, without loss of generality. For each $r = 1, 2, \cdots, s$, choose $x'_r$ and $x''_r$ such that (10) holds. Let

$$\begin{aligned} x' &= (x'_1, \cdots, x'_s, \infty, \cdots, \infty), \\ x'' &= (x''_1, \cdots, x''_s, \infty, \cdots, \infty), \end{aligned}$$

then $x'$ and $x''$ belong to $S$, and we have (6) for $x = x'$ and $x''$. Therefore, it holds that

$$(11) \quad F^n(x-\varepsilon e) - \varepsilon \leqq F^n(x') - \varepsilon \leqq F^0(x') \leqq F^0(x_1, \cdots, x_s, \infty, \cdots, \infty).$$

And we have

$$\begin{aligned} &F^0(x_1, \cdots, x_s, \infty, \cdots, \infty) - F^0(x_1, \cdots, x_s, x_{s+1}, \cdots, x_p) \\ (12) \quad &\leqq F^0(\infty, \cdots, \infty, \infty, \cdots, \infty) - F^0(\infty, \cdots, \infty, x_{s+1}, \cdots, x_p) \\ &\leqq 1 - F^0(M, \cdots, M) < \varepsilon. \end{aligned}$$

From (11) and (12) it follows

$$(13) \qquad F^m(x-\varepsilon e)-2\varepsilon \leqq F^0(x).$$

Next we have

$$(14) \qquad F^m(x+\varepsilon e)+\varepsilon \geqq F^m(x_1+\varepsilon, \cdots, x_s+\varepsilon, M+\varepsilon, \cdots, M+\varepsilon)+\varepsilon \geqq F^0(x_1, \cdots, x_s, M, \cdots, M)$$

applying a result of case (*c*) for $(x_1, \cdots, x_s, M, \cdots, M)$.
And

$$F^0(x_1, \cdots, x_s, x_{s+1}, \cdots, x_p)-F^0(x_1, \cdots, x_s, M, \cdots, M)$$
$$(15) \qquad \leqq F^0(\infty, \cdots, \infty, x_{s+1}, \cdots, x_p)-F^0(\infty, \cdots, \infty, M, \cdots, M)$$
$$\leqq 1-F^0(M, \cdots, M) < \varepsilon.$$

From (14) and (15) it follows

$$(16) \qquad F^n(x+\varepsilon e)+2\varepsilon \geqq F^0(x).$$

(13) and (16) implies (7). q. e. d.

THEOREM 2. *The metric space $D_p$ with the distance* (1) *is complete.*

PROOF. Let $\{F^n;\ n=1, 2, \cdots\}$ be a sequence of $p$-dimensional d. f. 's. Under the assumption that

$$(17) \qquad \lim_{n,\, m\to\infty} d(F^n, F^m)=0,$$

we shall prove that there exists a $p$-dimensional d. f. $F^0$ such that

$$(18) \qquad \lim_{n\to\infty} d(F^n, F^0)=0.$$

By making use of the diagonal method we can choose a subsequence $\{F^{n_i}\}$ such that for any rational point $r=(r_1, r_2, \cdots, r_p)$ we have the limit

$$(19) \qquad \lim_{i\to\infty} F^{n_i}(r)=F(r).$$

$F(r)$ is defined only for rational points $r$ and it holds that $0 \leqq F(r) \leqq 1$ and $F(r_1', r_2', \cdots, r_p') \geqq F(r_1, r_2, \cdots, r_p)$ if $r_i' \geqq r_i$ for all $i$. Using this $F(r)$ define a function $F^0(x)$ as follows

$$(20) \qquad F^0(x)=\inf_{r>x} F(r),$$

where $(r_1, \cdots, r_p) > (x_1, \cdots, x_p)$ means that $r_i > x_i$ for all $i$. For any fixed point $x$ and for any positive number $\varepsilon$ there exists a rational point $s$ such that $s>x$ and $F(s)< F^0(x)+\varepsilon$. And it holds that $F^0(x) \leqq F(r) < F^0(x)+\varepsilon$ for any rational point $r$ such that $x < r < s$. Thus we have

$$(21) \qquad F^0(x)=\lim_{r\downarrow x} F(r),$$

and

(22) $F^0(x)$ is continuous to the right.

Let $x = (x_1, \cdots, x_p)$ and $y = (y_1, \cdots, y_p)$ be two arbitrary points except that $x_1 \leqq y_1, x_2 \leqq y_2, \cdots, x_p \leqq y_p$. For any function $G(x_1, x_2, \cdots, x_p)$ of $p$ variables set

$$(23) \quad \begin{aligned} P(x, y;\ G) = {} & G(y_1, y_2, \cdots, y_p) \\ & -G(x_1, y_2, \cdots, y_p) - \cdots - G(y_1, \cdots, y_{p-1}, x_p) \\ & +G(x_1, x_2, y_3, \cdots, y_p) + \cdots + G(y_1, \cdots, y_{p-2}, x_{p-1}, x_p) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & +(-1)^p G(x_1, x_2, \cdots, x_p)\,. \end{aligned}$$

For any fixed $x$ and $y \geqq x$, choose $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)$ and $\delta = (\delta_1, \delta_2, \cdots, \delta_p)$ such that $x+\varepsilon$ and $y+\delta$ are rational points and $0 < \varepsilon < \delta$, then it holds that

$$P(x+\varepsilon, y+\delta;\ F^{n_i}) \geqq 0\,.$$

Letting $n_i \to \infty$ we have $P(x+\varepsilon, y+\delta;\ F) \geqq 0$. Letting $\delta \downarrow 0$ we have

$$(24) \qquad P(x, y;\ F^0) \geqq 0\,.$$

From our assumption (17) for any positive number $\varepsilon$ there exists $N$ such that

$$(25) \qquad F^m(r-\varepsilon e)-\varepsilon \leqq F^{n_i}(r) \leqq F^m(r+\varepsilon e)+\varepsilon$$

for all $r$ if $m$ and $n_i$ both $> N$. Assuming that $r$ is rational, let $n_i$ tend to $\infty$, then we have

$$F^m(r-\varepsilon e)-\varepsilon \leqq F(r) \leqq F^m(r+\varepsilon e)+\varepsilon$$

Let $x$ be any point and let $r \downarrow x$, then we have

$$(26) \qquad F^m(x-\varepsilon e)-\varepsilon \leqq F^0(x) \leqq F^m(x+\varepsilon e)+\varepsilon\,, \quad (m > N(\varepsilon))\,,$$

since $F^m(x)$ are continuous to the right. From (26) we have

$$(27) \quad \begin{aligned} & F^0(-\infty, x_2, \cdots, x_p) = 0\,,\ F^0(x_1, -\infty, x_3, \cdots, x_p) = 0, \cdots \\ & \cdots, F^0(x_1, x_2, \cdots, x_{p-1}, -\infty) = 0 \quad \text{and} \\ & F^0(\infty, \cdots, \infty) = 1\,. \end{aligned}$$

From (22), (24) and (27) $F^0(x)$ is proved to be a $p$-dimensional d.f. From (26) we have $d(F^m, F^0) \leqq \varepsilon$ if $m > N(\varepsilon)$. Therefore (18) holds.

THEOREM 3. *For any $p$-dimensional d.f. $F$, $G$ and $H$ it holds that*

$$d(F*H, G*H) \leqq d(F, G)$$

*where $F*H$ denotes the convolution of $F$ and $H$.*

PROOF. Write $\varepsilon = d(F, G)$, then

$$F(x-\varepsilon e)-\varepsilon \leqq G(x) \leqq F(x+\varepsilon e)+\varepsilon\,, \quad \text{for all } x.$$

From this it is derived that

$$\int \cdots \int [F(x-\varepsilon e-y)-\varepsilon]dH(y) \leqq \int \cdots \int G(x-y)dH(y)$$
$$\leqq \int \cdots \int [F(x+\varepsilon e-y)+\varepsilon]dH(y),$$

that is,

$$(F*H)(x-\varepsilon e)-\varepsilon \leqq (G*H)(x) \leqq (F*H)(x+\varepsilon e)+\varepsilon.$$

Therefore

$$d(F*H, G*H) \leqq \varepsilon = d(F, G).$$

COROLLARY. *Let $F^n$ and $G^n$ be p-dimensional d.f.'s for all $n = 0$, 1, 2, $\cdots\cdot$. Assume that*

$$\lim F^n = F^0 \quad \text{and} \quad \lim G^n = G^0$$

*in the convergence space $D_p$, then it holds that*

$$\lim (F^n * G^n) = F^0 * G^0.$$

This is evident from the following inequalities.

$$d(F^n * G^n, F^0 * G^0) \leqq d(F^n * G^n, F^0 * G^n) + d(F^0 * G^n, F^0 * G^0)$$
$$\leqq d(F^n, F^0) + d(G^n, G^0).$$

We can introduce another metric in the convergence space $D_p$ of $p$ dimensional d.f.'s, using the fact that the convergence of a sequence of d.f.'s is equivalent to the uniform convergence in any finite interval of the sequence of the corresponding characteristic functions. Let $F_1(x_1, \cdots\cdot, x_p)$ and $F_2(x_1, \cdots\cdot, x_p)$ be two $p$-dimensional d.f.'s and let $\varphi_1(t_1, \cdots\cdot, t_p)$ and $\varphi_2(t_1, \cdots\cdot, t_p)$ be the corresponding characteristic functions. Define a distance of two d.f.'s $F_1$ and $F_2$ as follows:

$$(28) \qquad \rho(F_1, F_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{\|t\| \leq n} |\varphi_1(t_1, \cdots\cdot, t_p) - \varphi_2(t_1, \cdots\cdot, t_p)|$$

where $\|\boldsymbol{t}\| = (t_1^2 + t_2^2 + \cdots\cdot + t_p^2)^{\frac{1}{2}}$. Then this metric is equivalent to the convergence in $D_p$, and the metric space $D_p$ with the distance (28) is complete.

## 3. Class-convergence of p-dimensional d. f.'s

We begin with the following lemmas.

LEMMA 4. *Let $\{f_n(t);\ n = 1, 2, \cdots\cdot\}$ be a sequence of one-dimensional ch.f.'s and let both $\varphi(t)$ and $\psi(t)$ be proper one-dimensinal ch.f.'s. If there exist sequences of positive numbers $\{a_n\}$, $\{a_n'\}$ and sequences of real numbers $\{b_n\}$, $\{b_n'\}$ such that*

$$\lim e^{-itb_n t/a_n} f_n(t/a_n) = \varphi(t)\,,$$
$$\lim e^{-itb'_n t/a'_n} f_n(t/a'_n) = \psi(t)\,, \qquad (-\infty < t < \infty)$$

*then there exist the limits*

$$\lim \frac{a'_n}{a_n} = A > 0\,, \qquad \lim \frac{b'_n - b_n}{a_n} = B$$

*and it holds that*

$$\psi(t) = e^{-iBt/A}\varphi(t/A)\,, \qquad (-\infty < t < \infty)\,.$$

LEMMA 5. *In the above* LEMMA 4 *if* $\varphi(t)$ *is improper and* $\psi(t)$ *is proper, then we have*

$$\lim \frac{a'_n}{a_n} = 0\,, \qquad \lim \frac{b'_n - b_n}{a_n} = B\,, \quad \text{and} \quad \varphi(t) = e^{iBt}.$$

LEMMA 4 and 5 are restatements of Theorem 4 and Corollary 2 to it in K. TAKANO [5] respectively.

LEMMA 6. *Let both* $\varphi(\boldsymbol{t})$ *and* $\psi(\boldsymbol{t})$ *be ch. f.'s of proper p-dimensional d.f.'s. Then there exist linearly independent p vectors* $\boldsymbol{t}_1, \boldsymbol{t}_2, \cdots, \boldsymbol{t}_p$ *such that both* $\varphi(\boldsymbol{t}_j t)$ *and* $\psi(\boldsymbol{t}_j t)$ *are proper one-dimensional ch. f.'s as functions of* $t(-\infty < t < \infty)$ $(j = 1, 2, \cdots, p)$.

PROOF. Let $\{\boldsymbol{s}_{kl};\ k = 1, 2, \cdots, 2p,\ l = 1, 2, \cdots, p\}$ be a set of $p$-dimensional vectors such that any $p$ of them are linearly independent(1). Then $\boldsymbol{s}_{k1}, \boldsymbol{s}_{k2}, \cdots, \boldsymbol{s}_{kp}$ are linearly independent for any $k$. If LEMMA 6 were not true, there should exist $l = l(k)$ for any $k$, such that it holds that either

$$\varphi(\boldsymbol{s}_{kl} t) \tag{29}$$

or

$$\psi(\boldsymbol{s}_{kl} t) \tag{30}$$

is improper. Since $k$ runs from 1 to $2p$, either (29) or (30) holds for at least $p$ $k$-values. Assume that

$$\varphi(\boldsymbol{s}_{k_j l_j} t) \text{ are improper, } (j = 1, 2, \cdots, p,\ k_1 < k_2 < \cdots < k_p).$$

Write $\boldsymbol{t}_j = \boldsymbol{s}_{k_j l_j}$. Then

$$\boldsymbol{t}_1, \boldsymbol{t}_2, \cdots, \boldsymbol{t}_p \text{ are linearly independent} \tag{31}$$

and

$$\varphi(\boldsymbol{t}_j t) \text{ are improper,} \qquad (j = 1, 2, \cdots, p)\,. \tag{32}$$

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(1) For example, let $\{s_{kl};\ k = 1, 2, \ldots, 2p,\ l = 1, 2, \ldots, p\}$ be a set of real numbers such that any two of them are different, and write $\boldsymbol{s}_{kl} = (1, s_{kl}, s_{kl}^2 \ldots, s_{kl}^{p-1})$.

Let $\boldsymbol{X} = (X_1, X_2, \cdots, X_p)$ be a random vector with the ch. f. $\varphi(\boldsymbol{t})$. Then from (32) we have

$$(33) \qquad \boldsymbol{t}_j\boldsymbol{X} = \text{const}, \qquad (j = 1, 2, \cdots, p),$$

with probability 1, where $\boldsymbol{t}_j\boldsymbol{X}$ denotes the inner product of two vectors $\boldsymbol{t}_j$ and $\boldsymbol{X}$. From (31) and (33) $\boldsymbol{X}$ must be a certain vector with probability 1. This contradict that $\varphi(\boldsymbol{t})$ is the ch. f. of a proper d. f.

First we have to notice the special role of the improper class in the convergence space of classes of d. f.'s. In the following, whenever nothing is explicitly mentioned we shall denote a $p$-dimensional d. f., and a $p$-dimensional vector merely by a d. f. and a vector, respectively.

THEOREM 4. *Any arbitrarily given sequence of classes $\{K_n;\ n = 1, 2, 3, \cdots\}$ converges to the improper class, i. e., by choosing an adequate d. f. $F_n$ from each $K_n$ we can make the sequence $\{F_n\}$ converge to a unit d. f.*

This was first proved by A. I. Khintchine [2] in case $p = 1$.

PROOF. Let $F_n$ be a d. f. belonging to $K_n$ for any $n$. For each $n$, we can choose a positive number $a_n$ such that

$$F_n(-a_n, \infty, \cdots, \infty) < \frac{1}{n}, \quad F_n(\infty, -a_n, \infty, \cdots, \infty) < \frac{1}{n}, \quad \cdots,$$

$$F_n(\infty, \cdots, \infty, -a_n) < \frac{1}{n}, \quad F_n(a_n, a_n, \cdots, a_n) > 1 - \frac{1}{n}.$$

Then the sequence of d. f.'s

$$F_n(na_nx_1, na_nx_2, \cdots, na_nx_p), \quad (n = 1, 2, \cdots),$$

converges to the unit d. f. $U(x_1, \cdots, x_p)$ of a distribution which has the whole probability 1 at the origin $(x_1, \cdots, x_p) = (0, \cdots, 0)$.

THEOREM 5. *Let $\{F_n(\boldsymbol{x});\ n = 1, 2, \cdots\}$ be a sequence of d. f.'s, let $\{a_n\}$ and $\{a_n'\}$ be sequences of positive numbers and let $\{\boldsymbol{b}_n\}$ and $\{\boldsymbol{b}_n'\}$ be sequences of vectors. Assume that the sequence $\{F_n(a_n\boldsymbol{x} + \boldsymbol{b}_n)\}$ converges to a d. f. $F_0(\boldsymbol{x})$ in $D_p$, and that there exist the limits*

$$\lim \frac{a_n'}{a_n} = A \qquad \text{and} \qquad \lim \frac{\boldsymbol{b}_n' - \boldsymbol{b}_n}{a_n} = \boldsymbol{B}.$$

*If $A > 0$, the sequence $\{F_n(a_n'\boldsymbol{x} + \boldsymbol{b}_n')\}$ converges to the d. f. $F_0(A\boldsymbol{x} + \boldsymbol{B})$, and if $A = 0$ we have*

$$F_0(\boldsymbol{B} - 0) \leqq \liminf F_n(a_n'\boldsymbol{x} + \boldsymbol{b}_n') \leqq \limsup F_n(a_n'\boldsymbol{x} + \boldsymbol{b}_n') \leqq F_0(\boldsymbol{B} + 0).$$

PROOF. We can assume that $a_n = 1$ and $\boldsymbol{b}_n = 0$ $(n = 1, 2, \cdots)$ without loss of generality. Let $\boldsymbol{x}$ be a fixed vector. Then from the hypothesis $a'_n\boldsymbol{x}+\boldsymbol{b}'_n$ tends to $A\boldsymbol{x}+\boldsymbol{B}$ as $n \to \infty$. Hence for any positive number $\varepsilon$ there exists a number $N$ such that

$$A\boldsymbol{x}+\boldsymbol{B}-\varepsilon\boldsymbol{e} < a'_n\boldsymbol{x}+\boldsymbol{b}'_n < A\boldsymbol{x}+\boldsymbol{B}+\varepsilon\boldsymbol{e}, \quad \text{for } n > N,$$

where $\boldsymbol{e} = (1, 1, \cdots, 1)$ and $(a_1, \cdots, a_p) < (b_1, \cdots, b_p)$ means that $a_j < b_j$ for all $j$. We have

$$F_n(A\boldsymbol{x}+\boldsymbol{B}-\varepsilon\boldsymbol{e}) \leqq F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n) \leqq F_n(A\boldsymbol{x}+\boldsymbol{B}+\varepsilon\boldsymbol{e}).$$

If $A\boldsymbol{x}+\boldsymbol{B}\pm\varepsilon\boldsymbol{e}$ are both continuity points of $F_0(\boldsymbol{x})$, it holds that

$$F_0(A\boldsymbol{x}+\boldsymbol{B}-\varepsilon\boldsymbol{e}) \leqq \liminf F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n) \leqq \limsup F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n)$$
$$\leqq F_0(A\boldsymbol{x}+\boldsymbol{B}+\varepsilon\boldsymbol{e}).$$

As $\varepsilon$ may be arbitrarily small, we have

$$F_0(A\boldsymbol{x}+\boldsymbol{B}-0) \leqq \liminf F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n) \leqq \limsup F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n)$$
$$\leqq F_0(A\boldsymbol{x}+\boldsymbol{B}+0)$$

from which THEOREM 5 follows.

THEOREM 6. *Let $\{F_n(\boldsymbol{x});\ n = 1, 2, \cdots\}$ be a sequence of d. f.'s. Assume that there exist sequences of positive numbers $\{a_n\}$, $\{a'_n\}$ and sequences of vectors $\{\boldsymbol{b}_n\}$, $\{\boldsymbol{b}'_n\}$; and that there exist proper d.f.'s $\Phi(\boldsymbol{x})$, $\Psi(\boldsymbol{x})$ such that*

$$\lim F_n(a_n\boldsymbol{x}+\boldsymbol{b}_n) = \Phi(\boldsymbol{x}),$$
$$\lim F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n) = \Psi(\boldsymbol{x})$$

*in $D_p$. Then there exist the limits*

$$\lim \frac{a'_n}{a_n} = A > 0, \qquad \lim \frac{\boldsymbol{b}'_n-\boldsymbol{b}_n}{a_n} = \boldsymbol{B}$$

*and we have*

$$\Psi(\boldsymbol{x}) = \Phi(A\boldsymbol{x}+\boldsymbol{B}).$$

*Accordingly, if a sequence of classes $\{K_n;\ n = 1, 2, \cdots\}$ converges to a proper class $K_0$, the limit proper class $K_0$ is uniquely determined by the sequence $\{K_n\}$.*

PROOF. Denote the ch. f.'s of $F_n(\boldsymbol{x})$, $\Phi(\boldsymbol{x})$ and $\Psi(\boldsymbol{x})$ by $f_n(\boldsymbol{t})$, $\varphi(\boldsymbol{t})$ and $\psi(\boldsymbol{t})$, respectively. Then the ch. f.'s of $F_n(a_n\boldsymbol{x}+\boldsymbol{b}_n)$ and $F_n(a'_n\boldsymbol{x}+\boldsymbol{b}'_n)$ are,

$$e^{-i\boldsymbol{b}_n\boldsymbol{t}/a_n}f_n(\boldsymbol{t}/a_n) \qquad \text{and} \qquad e^{-i\boldsymbol{b}'_n\boldsymbol{t}/a'_n}f_n(\boldsymbol{t}/a'_n)$$

respectively. From the hypothesis it holds that

$$\lim e^{-ib_n t/a_n} f_n(t/a_n) = \varphi(t),$$
$$\lim e^{-ib'_n t/a'_n} f_n(t/a'_n) = \psi(t).$$

Substituting $tt$ for $t$ and considering $t$ as a fixed vector, we have

$$(34)\qquad \begin{aligned} &\lim e^{-ib_n tt/a_n} f_n(tt/a_n) = \varphi(tt), \\ &\lim e^{-ib'_n tt/a'_n} f_n(tt/a'_n) = \psi(tt). \end{aligned} \qquad (-\infty < t < \infty).$$

If the constant vector $t$ satisfies that both $\varphi(tt)$ and $\psi(tt)$ are proper ch. f. 's as functions of $t$, from Lemma 4, it is seen that there exist the limits

$$(35)\qquad \lim \frac{a'_n}{a_n} = A > 0,$$

$$(36)\qquad \lim \frac{b'_n t - b_n t}{a_n} = B(t).$$

On the other hand, from LEMMA 6, there exist linearly independent $p$ vectors $t_1, t_2, \cdots, t_p$ such that both $\varphi(t_j t)$ and $\psi(t_j t)$ are proper ch. f. 's as functions of $t$ for all $j = 1, 2, \cdots, p$. Therefore, for each of these $p$ vectors $t_1, t_2, \cdots, t_p$ there exist the limits

$$\lim (b'_n - b_n)t_j/a_n = B(t_j).$$

As $t_1, t_2, \cdots, t_p$ are linearly independent, there exists the limit

$$(37)\qquad \lim (b'_n - b_n)/a_n = B.$$

From (35) and (37) we have

$$\lim F_n(a'_n x + b'_n) = \Phi(Ax + B)$$

according to THEOREM 5. Therefore we have

$$\Psi(x) = \Phi(Ax + B).$$

COROLLARY 1. *Assume that*

$$\lim F_n(a_n x + b_n) = \Phi(x), \qquad (a_n > 0),$$
$$\lim F_n(a'_n x + b'_n) = \Phi(x), \qquad (a'_n > 0),$$

*in* $D_p$, *and that* $\Phi(x)$ *is proper. Then we have*

$$\lim \frac{a'_n}{a_n} = 1, \qquad \lim \frac{b'_n - b_n}{a_n} = 0.$$

PROOF. From the above theorem we have

$$\Phi(x) = \Phi(Ax + B)$$

with $A = \lim a'_n/a_n$ and $\boldsymbol{B} = \lim(\boldsymbol{b}'_n - \boldsymbol{b}_n)/a_n$. Denote the marginal d. f. of $\Phi(x)$ corresponding to the $r$-th variate by $\Phi_r(x)$ ($r = 1, 2, \cdots, p$), and put $\boldsymbol{B} = (B_1, B_2, \cdots, B_p)$. Then

$$\Phi_r(x) = \Phi_r(Ax + B_r), \quad (r = 1, 2, \cdots, p).$$

As $\Phi(x)$ is proper, $\Phi_r(x)$ is proper, for at least one $r$, hence we have $A = 1$ and $B_r = 0$. (cf. Corollary 1 to Theorem 4 in K. Takano [5]). From $\Phi_k(x) = \Phi_k(x + B_k)$ we have $B_k = 0$ ($k = 1, 2, \cdots, p$).

COROLLARY 2. *Assume that*

$$\lim F_n(a_n x + \boldsymbol{b}_n) = \Phi(x), \qquad (a_n > 0),$$
$$\lim F_n(a'_n x + \boldsymbol{b}'_n) = \Psi(x), \qquad (a'_n > 0),$$

*in $D_p$, and assume that $\Phi(x)$ is improper and $\Psi(x)$ is proper. Then there exist limits*

$$\lim \frac{a'_n}{a_n} = 0, \qquad \lim \frac{\boldsymbol{b}'_n - \boldsymbol{b}_n}{a_n} = \boldsymbol{B}$$

*and $\Phi(x)$ is the d. f. of a distribution which has the whole probability* 1 *at the point $x = \boldsymbol{B}$.*

This is shown by applying LEMMA 5 to (34).

By the statement that a sequence of random vectors $\boldsymbol{X}_n = (X_{n1}, X_{n2}, \cdots, X_{np})$ converges to a certain vector $\boldsymbol{c} = (c_1, c_2, \cdots, c_p)$ in probability, we mean that $P_r\{\sum_k |X_{nk} - c_k| < \varepsilon\}$ tends to 1 as $n \to \infty$, for any $\varepsilon > 0$. In order that $\boldsymbol{X}_n$ tends to $\boldsymbol{c}$ in probability, it is necessary and sufficient that the d. f. of $\boldsymbol{X}_n$ tends to the unit d. f. of the distribution which has the whole probability 1 at the fixed point $x = \boldsymbol{c}$.

COROLLARY 3. *Let $\{\boldsymbol{X}_n\}$ be a sequence of random vectors, let $\{a_n\}$ be a sequence of positive numbers, and let $\{\boldsymbol{b}_n\}$ be a sequence of vectors. Assume that the d. f. of $\boldsymbol{X}_n$ tends to a proper d. f. Then, in order that the $(\boldsymbol{X}_n - \boldsymbol{b}_n)/a_n$ tends to a certain vector $\boldsymbol{B}$ in probability, it is necessary and sufficient that*

$$\lim a_n = \infty \qquad \textit{and} \qquad \lim \boldsymbol{b}_n/a_n = -\boldsymbol{B}.$$

The necessity is the restatement of COROLLARY 2. The sufficiency can be shown from THEOREM 1.

## 4. A metrization of class-convergences of p-dimensional d. f.'s

Like in the one-dimensional case, we define also in the $p$-dimentional case a distance between two classes by the distance (1)* between representative d. f.'s adequately chosen from the two classes, respectively. As the representative one it seems natural to choose a d.f., whose disperson is equal to a fixed constant. Denote the dispersion (without definition) of the probability distribution determined by a d.f. $F(\boldsymbol{x})$ by $D(F) = D(F(\boldsymbol{x}))$. $D(F)$ should satisfy the following conditions:

a *For any d.f.* $F(\boldsymbol{x})$, *a real number* $D(F)$ *is defined and* $D(F) \geqq 0$.

b $D(F) = 0$ *if and only if* $F(\boldsymbol{x})$ *is a unit d.f.*

c $D(F(a\boldsymbol{x}+\boldsymbol{b})) = D(F(\boldsymbol{x}))/a$. $(a > 0)$.

d *If* $\lim F_n = F_0$ *then* $\lim D(F_n) = D(F_0)$.

However, there exists no $D(F)$ which satisfies these conditions.(2) So, we shall substitute the following $b'$ for $b$.

b′ $D(F) = 0$ *if and only if the class which contains* $F$ *belongs to a neighbourhood of the improper class.*

We shall show that there exists the dispersions $D(F)$ which satisfies a, b′, c and d. Let $F(\boldsymbol{x}) = F(x_1, x_2, \cdots, x_p)$ be any $p$-dimensional d.f. and write its marginal d.f.'s as follows:

$$F_1(x) = F(x, \infty, \cdots, \infty), \quad F_2(x) = F(\infty, x, \infty, \cdots, \infty), \cdots$$

$$F_p(x) = F(\infty, \cdots, \infty, x).$$

We shall call the convolution of the marginal d.f.'s

$$F^*(x) = F_1(x) * F_2(x) * \cdots * F_p(x)$$

the trace d.f. (or the trace briefly) of the d.f. $F(\boldsymbol{x})$. Then, we have:

In order that *a* $p$-dimensional d.f. $F(\boldsymbol{x})$ be proper, it is necessary and sufficient that its trace $F^*(x)$ be proper.

Denoting the trace of a d.f. $F(\boldsymbol{x})$ by $F^*(x)$, the trace of a d.f. $F(a\boldsymbol{x}+\boldsymbol{b})$ is $F^*(ax+b_1+\cdots+b_p)$ where $\boldsymbol{b} = (b_1, \cdots, b_p)$.

Let $F_n(\boldsymbol{x})$ be $p$-dimensional d.f.'s and let $F_n^*(x)$ be the corresponding trace d.f.'s $(n = 0, 1, 2, \cdots)$. If the sequence $\{F_n(\boldsymbol{x})\}$ converges to $F_0(\boldsymbol{x})$ in the convergence space $D_p$, then the sequence $\{F_n^*(x)\}$ converges to $F_0^*(x)$ in the convergence space $D_1$.

By the m.c.f. (mean concentration function) and the dispersion function of a $p$-dimensional d.f. $F(\boldsymbol{x})$, we mean the m.c.f. and the dispersion

(2) If there exists $D(F)$ which satisfies the set of conditions *a, b, c* and *d*, the convergence space of all proper classes is shown to be metrizable, which is impossible.

function, respectively, which were introduced by K. Kunisawa [3], of the trace $F^*(x)$ of the d.f. $F(x)$, and we shall denote them by $\psi_{F(x)}(l)$ and $D_{F(x)}(\gamma)$:

$$\psi_{F(x)}(l) = \psi_{F^*(x)}(l) = \int_{-\infty}^{\infty} \frac{l^2}{l^2+x^2} d\tilde{F}^*(x), \quad (0 < l < \infty),$$

$$D_{F(x)}(\gamma) = \min\{l;\ \psi_{F(x)}(l) \geqq \gamma\}, \quad (0 < \gamma < 1),$$

where

$$\tilde{F}^*(x) = \int_{-\infty}^{\infty} F^*(x-y)d[1-F^*(-y)].$$

It follows that

(38) $\psi_{F(ax+b)}(l) = \psi_{F(x)}(al)$, $D_{F(ax+b)}(\gamma) = \dfrac{1}{a} D_{F(x)}(\gamma)$, $(a>0)$.

and

(39) in case a sequence of d.f.'s $\{F_n(x)\}$ converges to a d.f. $F_0(x)$ in the convergence space $D_p$, it holds that

$$\lim \psi_{F_n}(l) = \psi_{F_0}(l), \qquad (0 < l < \infty),$$

$$\lim D_{F_n}(\gamma) = D_{F_0}(\gamma), \qquad (0 < \gamma < 1).$$

From (38) we have

$$\psi_{F(ax+b)}(+0) = \psi_{F(x)}(+0), \qquad (a > 0).$$

Thus, $\psi_F(+0)$ is invariant as $F(x)$ runs in the same class $K$. We denote this value by $\psi_K(0)$. It holds that

(40) $0 \leqq \psi_K(0) \leqq 1$, $\psi_K(0) = 1$ if and only if $K$ is improper, and

(41) if $\psi_K(0) < \gamma < 1$ and $F \in K$ then $D_F(\gamma) > 0$.

For a fixed $\gamma (0 < \gamma < 1)$ we shall call $D_{F(x)}(\gamma)$ the dispersion of $F(x)$ for parameter $\gamma$. Put $D(F) = D_{F(x)}(\gamma)$. The dispersion $D(F)$ satisfies the conditions $a$, $b'$, $c$ and $d$, where by the neighbourhood of the improper class we mean the set $\{K;\ \gamma \leqq \psi_K(0) \leqq 1\}$.

Let $\Omega$ be the set of all proper classes of $p$-dimensional d.f.'s, and for any fixed $\gamma(0 < \gamma < 1)$ let $\Omega_\gamma = \{K;\ \psi_K(0) < \gamma\}$. Then $\Omega_\gamma$ is monotone increasing with $\gamma$ and $\Omega = \bigcup_\gamma \Omega_\gamma$.

On fixing $\gamma$, we shall define a metric in the set $\Omega_\gamma$. Let $K_1$ and $K_2$ belong to $\Omega_\gamma$. From (38) and (41), we can choose $F_1(x)$ and $F_2(x)$ such that $F_1 \in K_1$, $F_2 \in K_2$, $D_{F_1}(\gamma) = 1$ and $D_{F_2}(\gamma) = 1$. Any d.f. $F(x)$ such that

$F \in K_1$, and $D_F(\gamma) = 1$, is equal to $F_1(\boldsymbol{x}+\boldsymbol{c})$ for some vector $\boldsymbol{c}$. Define the distance of two classes $K_1$ and $K_2$ as follows:

$$(42) \qquad d_T(K_1, K_2) = \inf_{c_1, c_2} d(F_1(\boldsymbol{x}+\boldsymbol{c}_1), F_2(\boldsymbol{x}+\boldsymbol{c}_2)) = \inf_{\boldsymbol{c}} d(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}+\boldsymbol{c})),$$

where $d(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}))$ denotes the distance defined by (1).

Let $g(\boldsymbol{c}) = d(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}+\boldsymbol{c})$, then $g(\boldsymbol{c})$ is continuous:

$$(43) \qquad |g(\boldsymbol{c})-g(\boldsymbol{c}')| \leqq d(F_2(\boldsymbol{x}+\boldsymbol{c}), F_2(\boldsymbol{x}+\boldsymbol{c}')) \leqq ||\boldsymbol{c}-\boldsymbol{c}'||,$$

where $||(c_1, \cdots, c_p)|| = \max(|c_1|, \cdots, |c_p|)$. Denote the marginal d. f. 's corresponding to the $k$-th variate of $F_1(\boldsymbol{x})$ and $F_2(\boldsymbol{x})$ by $F_{1k}(x)$ and $F_{2k}(x)$, respectively, $(k = 1, 2, \cdots, p)$. Then, it follows from the definition that

$$d(F_{1k}(x), F_{2k}(x+c_k)) \leqq d(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}+\boldsymbol{c})), \ (k = 1, 2, \cdots, p),$$

where $\boldsymbol{c} = (c_1, c_2, \cdots, c_p)$. Since, for any $k$,

$$\lim_{|c|\to\infty} d(F_{1k}(x), \quad F_{2k}(x+c)) = 1,$$

we have

$$(44) \qquad \lim_{||\boldsymbol{c}||\to\infty} g(\boldsymbol{c}) = 1 = \sup_{\boldsymbol{c}} g(\boldsymbol{c}).$$

From (43) and (44) we have

$$(45) \qquad d_T(K_1, K_2) = \min_{\boldsymbol{c}} d(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}+\boldsymbol{c})) = \min_{\boldsymbol{c}} d(F_1(\boldsymbol{x}+\boldsymbol{c}), F_2(\boldsymbol{x})).$$

We have the following theorems which are easily proved by the same methods as THEOREM 1 and 2 in my previous paper [6].

THEOREM 7. *$d_T(K_1, K_2)$ satisfies the axioms of distance in $\Omega_T$:*

$$(46) \quad d_T(K_1, K_2) \geq 0, \quad d_T(K_1, K_2) = 0 \quad \textit{if and only it } K_1 = K_2,$$

$$(47) \qquad d_T(K_1, K_2) = d_T(K_2, K_1),$$

$$(48) \qquad d_T(K_1, K_3) \leq d_T(K_1, K_2) + d_T(K_2, K_3).$$

THEOREM 8. *Let $K_n \in \Omega_T$ $(n = 0, 1, 2, \cdots)$. In order that the sequence of classes $\{K_n\}$ converges to the class $K_0$, it is necessary and sufficient that $\{K_n\}$ converges to $K_0$ in the sense of the distance (45).*

COROLLARY. *If a sequence of classes of d. f. 's converges to a proper class, then the limiting proper class is uniquely determined by the sequence.*

We shall denote the dispersion of the d.f. of a $p$-dimensional random vector $\boldsymbol{X}$ by $D_{\boldsymbol{X}}(\gamma)$.

THEOREM 9. *Let* $\{\boldsymbol{S}_n;\ n=1, 2, \cdots\cdot\}$ *be a sequence of $p$-dimensional random variables and let $G(\boldsymbol{x})$ be a $p$-dimensional d.f. Assume that* $D_G(\gamma)>0$, *and* $D_{\boldsymbol{S}_n}(\gamma)>0$, $(n=1, 2, \cdots\cdot)$, *for some* $\gamma$, $(0<\gamma<1)$. *Fixing such a* $\gamma$, *put* $D=D_G(\gamma)$ *and* $D_n=D_{\boldsymbol{S}_n}(\gamma)$. *Then, in order that there exists a sequence of positive numbers* $\{a_n\}$ *such that the sequence of the d.f.'s. of* $\boldsymbol{S}_n/a_n$ *converges to* $G(\boldsymbol{x})$, *it is necessary and sufficient that the sequence of the d.f.'s of* $\dfrac{\boldsymbol{S}_n}{D_n/D}$ *converges to* $G(\boldsymbol{x})$.

This is a conclusion from (39) and THFOREM 5. (cf. K. Takano (6), THEOREM 5).

THEOREM 10. *Assume, moreover, that the median of any marginal d.f. of $G(\boldsymbol{x})$ is uniquely determined and is equal to 0. Let* $\boldsymbol{m}_n=(m_{n1}, m_{n2}, \cdots\cdot, m_{np})$ *be a vector such that* $m_{ni}$ *is a median of the i-th component of* $\boldsymbol{S}_n$ $(n=1, 2, \cdots\cdot)$. *Then, in order that there exist a sequence of positive numbers* $\{a_n\}$ *and a sequence of vectors* $\{\boldsymbol{b}_n\}$ *such that the sequence of the d.f.'s of* $(\boldsymbol{S}_n-\boldsymbol{b}_n)/a_n$ *converges to* $G(\boldsymbol{x})$, *it is necessary and sufficient that the sequence of the d.f.'s of* $\dfrac{\boldsymbol{S}_n-\boldsymbol{m}_n}{D_n/D}$ *converges to* $G(\boldsymbol{x})$.

(cf. K. Takano [6], THEOREM 6)

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