On a Test in Paired Comparisons

By Hirojiro AOYAMA

(Received January 5, 1953)

§ 1. Introduction

In the ranking of occupations some people have a tendency to rank their own occupations higher than those of the others. But, is it always so? In the following we propose a simple test whether a person appreciates his occupation higher than the other.

Let H_0 be the null hypothesis that it happens at random that one appreciates his occupation higher or appreciate lower than the other. First we take n samples every one of which belongs to n different occupations, respectively.

When we have one pair of paired comparisons we put the value $+1$ for this pair if the two appreciate higher themselves each other, and -1 if the two appreciate lower themselves each other and 0 otherwise.

As we have assumed that these facts occur at random, the probabilities that these facts take place are $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$, respectively.

§ 2. Statistic S in test and its moments

We assume that there are n people belonging to n different occupations and they appreciate their occupations one another. Then we have $\binom{n}{2}$ pairs of comparisons. Let S be the algebraic sum of the values that are given for each pair. Then we have zero as the mean of S because of the symmetry of this distribution:

$$E(S) = 0 \tag{1}$$

As we assume the pairs occur independently, we have the variance of S as the sum of the variance $\frac{1}{2}$ of each random variable, that is,

$$\sigma^2(S) = \frac{k}{2}, \quad \text{where} \quad k = \binom{n}{2}. \tag{2}$$

For small n the probability can easily be calculated that S attains some value less than or equal to k. If $f(k, S = \alpha - \gamma)$, denotes the number of cases, where the S attains $\alpha - \gamma$ in k pairs, it holds

$$f(k, S = \alpha - \gamma) = \sum_{\alpha+\beta+\gamma=k} \frac{k!}{\alpha!\,\beta!\,\gamma!} 2^\beta \tag{3}$$

which is obtained by the generating function

$$g(x)=\left(x+2+\frac{1}{x}\right)^{k} \tag{4}$$

The value of $f(k, S)$ is also obtained by the recursion formula

$$f(k+1, S)=f(k, S+1)+2f(k, S)+f(k, S-1) \tag{5}$$

where $f(k, S)=0$ for $k<S$.

As to $f(k, S)$ we have the following table for small n.

Table of $P_r(|S| \geqq S_0)$

S_0 \ n	3	4	5	6	7	8	9
k	3	6	10	15	21	28	36
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.6875	0.7744	0.8238	0.8555	0.8776	0.8939	0.9063
2	0.2188	0.3877	0.5034	0.5847	0.6440	0.6889	0.7239
3	0.0313	0.1460	0.2632	0.3616	0.4408	0.5044	0.5560
4		0.0386	0.1153	0.2005	0.2800	0.3497	0.4096
5		0.0063	0.0414	0.0987	0.1641	0.2288	0.2888
6		$0.0^{3}49$	0.0118	0.0428	0.0884	0.1409	0.1945
7			0.0026	0.0161	0.0436	0.0814	0.1249
8			$0.0^{3}40$	0.0052	0.0195	0.0440	0.0764
9			$0.0^{4}40$	0.0014	0.0079	0.0222	0.0444
10			$0.0^{5}19$	$0.0^{3}32$	0.0029	0.0105	0.0245
11				$0.0^{4}59$	$0.0^{3}94$	0.0046	0.0128
12				$0.0^{5}84$	$0.0^{3}27$	0.0018	0.0063
13				$0.0^{6}87$	$0.0^{4}69$	$0.0^{3}69$	0.0029
14				$0.0^{7}58$	$0.0^{4}15$	$0.0^{3}23$	0.0013
15				$0.0^{8}19$	$0.0^{5}28$	$0.0^{4}73$	$0.0^{3}53$
16					$0.0^{6}44$	$0.0^{4}21$	$0.0^{3}20$
17					$0.0^{7}57$	$0.0^{5}54$	$0.0^{4}76$
18					$0.0^{8}56$	$0.0^{5}12$	$0.0^{4}26$
19					$0.0^{9}41$	$0.0^{6}26$	$0.0^{5}81$
20					$0.0^{10}20$	$0.0^{7}47$	$0.0^{5}24$
21					$0.0^{12}45$	$0.0^{8}75$	$0.0^{6}65$
22						$0.0^{8}10$	$0.0^{6}16$
23						$0.0^{9}12$	$0.0^{7}38$
24						$0.0^{10}11$	$0.0^{8}81$
25						$0.0^{12}81$	$0.0^{8}15$
26						$0.0^{13}44$	$0.0^{9}27$
27						$0.0^{14}16$	$0.0^{10}42$
28						$0.0^{16}28$	$0.0^{11}58$
29							$0.0^{12}70$
30							$0.0^{13}73$
31							$0.0^{14}64$
32							$0.0^{15}46$
33							$0.0^{16}26$
34							$0.0^{17}11$
35							$0.0^{19}31$
36							$0.0^{21}42$

For large n the variable

$$z = S\Big/\sqrt{\frac{k}{2}} \tag{6}$$

is asymptotically normally distributed with mean zero and variance 1 according to the central limit theorem. However, if we use the generating function (4), we can anew prove this fact as follows.

From equation (4) $g(x)$ satisfies the following recursion formula,

$$\begin{aligned} 4g^{(r+1)} + r^{[2]}g^{(r-1)} - r^{[3]}g^{(r-2)} + \dots + (-1)^r r^{[r]}g^{(1)} \\ = k\{2r^{[1]}g^{(r-1)} - 3r^{[2]}g^{(r-2)} + \dots + (-1)^{r+1}(r+1)r^{[r]}g\} \end{aligned} \tag{7}$$

where

$$g^{(i)} = \left[\frac{d^i g(x)}{dx^i}\right]_{x=1}, \qquad r^{[i]} = r(r-1)\dots(r-i+1).$$

Hence we have, for example,

$$g^{(1)} = 0, \qquad g^{(2)} = 2!\, k\cdot 4^{k-1}/1!\, 2^0$$

$$g^{(3)} = -3!\, k\cdot 4^{k-1}/0!\, 2^0, \qquad g^{(4)} = 4!\, k^2\cdot 4^{k-1}\left(1+\frac{7}{k}\right)\Big/2!\, 2^2$$

$$g^{(5)} = -5!\, k^2\cdot 4^{k-1}\left(1+\frac{3}{k}\right)\Big/1!\, 2^2, \quad g^{(6)} = 6!\, k^3\cdot 4^{k-1}\left(1+\frac{33}{k}+\frac{62}{k^2}\right)\Big/3!\, 2^4$$

....

and generally for $r \geqq 2$ by means of mathematical induction

$$\begin{aligned} g^{(2r)} &= \frac{(2r)!\, k^r 4^{k-1}}{r!\, 2^{2r-2}}\left(1+O\left(\frac{1}{k}\right)\right) \\ g^{(2r+1)} &= -\frac{(2r+1)!\, k^r 4^{k-1}}{(r-1)!\, 2^{2r-2}}\left(1+O\left(\frac{1}{k}\right)\right). \end{aligned} \tag{8}$$

Therefore, the $(2r-1)$th moment is zero and the $2r$th moment is given by the following formula

$$\mu_{2r} = \frac{1}{4^k}\sum_{j=1}^{2r}\left\{\binom{2r}{j}B^{(-j)}_{2r-j}g^{(j)}\right\} \tag{9}$$

where $B^{(l)}_m$ is the Bernoulli's number of order l and degree m.
From (8) and (9) we have

$$\mu_{2r} \doteqdot \frac{(2r)!\, k^r}{r!\, 2^{2r}}\left(1+O\left(\frac{1}{k}\right)\right) \tag{10}$$

and

$$\frac{\mu_{2r}}{\sigma^{2r}} = \frac{(2r)!}{2^r r!}\left(1+O\left(\frac{1}{k}\right)\right). \tag{11}$$

Thus we see that the variable S is asymptotically normally distributed with mean zero and variance $k/2$.

If we take m samples from each occupation, we have $m^2\binom{n}{2}$ comparisons. In this case, let S denote the total algebraic sum divided by m^2. Then $\sigma^2(S) = \frac{k}{2m}$ and it is also shown that S is normally distributed with mean 0 and variance $k/2m$.

§ 3. OC function of the test

Let H_1 be an alternative hypothesis that the probabilities for the values $+1$, -1 and 0 are p, q and r, respectively. To this hypothesis we have

$$P_r(k, S = \alpha - \gamma) = \sum \frac{k!}{\alpha!\,\beta!\,\gamma!} p^{\alpha} r^{\beta} q^{\gamma}, \tag{12}$$

whence

$$E(S) = k(p - q) \tag{13}$$

$$\sigma^2(S) = k((p + q) - (p - q)^2). \tag{14}$$

If $p = cq$, we have $r = 1 - q(c + 1)$ and the operating characteristic function (OC function) P for the l % level is

$$P(S_l, q, c) = \sum_{\alpha-\gamma=S_{l_1}}^{S_{l_2}} \frac{k!}{\alpha!\,(k - \alpha - \gamma)!\,\gamma!} c^{\alpha} q^{\alpha+\gamma} r^{\beta} \tag{15}$$

But, for large n it holds approximately

$$P(S_l, q, c) = \frac{1}{\sqrt{2\pi}} \int_{z_{l_1}}^{z_{l_2}} e^{-\frac{t^2}{2}} dt$$

where

$$z_{l_i} = \frac{S_{l_i} - kq(c - 1)}{\sqrt{k\{q(c + 1) - q^2(c - 1)^2\}}}, \quad i = 1, 2$$

and for 1 % level, for example, we can put approximately

$$S_{l_1} \fallingdotseq -3\sqrt{\frac{k}{2}}, \qquad S_{l_2} \fallingdotseq 3\sqrt{\frac{k}{2}}.$$

Thus the following figures are obtained. (The similar figure is obtainable for p, $q = cp$ and k)

If n, consequently k, tends to infinity, we have

$$z_{l_2} \to -\infty \quad \text{for} \quad c > 1$$

and

$$z_{l_1} \to +\infty \quad \text{for} \quad c < 1$$

so that anyway

$$P(S_l, q, c) \to 0$$

which proves the consistency of the test.

Institute of Statistical Mathematics

Fig. 1

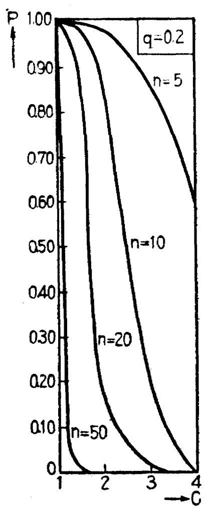

Fig. 2

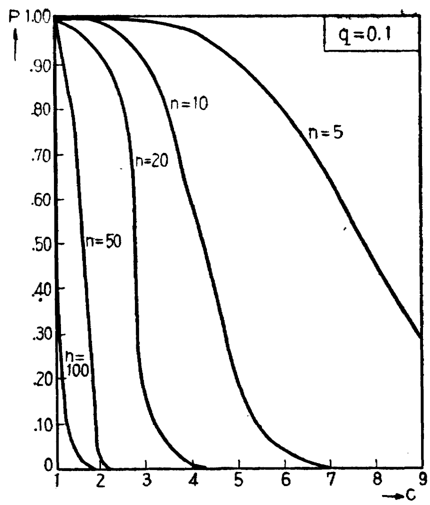

Fig. 3

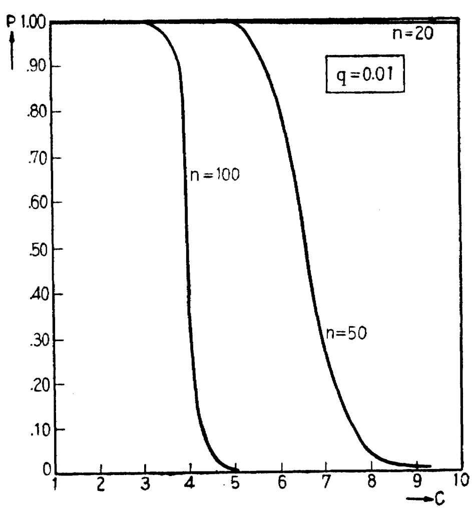

Fig. 4