Absolute Moments in 3-dimensional Normal Distribution.

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In the previous paper the author derived the formulæ for the absolute moments in the bivariate normal distribution with means zero. It is the purpose of this paper to extend the foregoing results to the trivariate case.

First we shall prove the following theorem.

Theorem. Let $x_1, x_2, \ldots, x_r$ be distributed according to an r-dimensional distribution, and denote their characteristic function by $\varphi(t_1, t_2, \ldots, t_r)$. For a fixed set of non-negative integers $n_1, n_2, \ldots, n_r$ $\left(\sum_{j=1}^{r} n_j = n\right)$, and for any set of non-negative integers $m_1, m_2, \ldots, m_r$ such that $m_1 \leqq n_1$, $m_2 \leqq n_2$, $\ldots$, $m_r \leqq n_r$, let the absolute moments $E(|x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}|)$ be finite.

If $n_1, n_2, \ldots, n_p$ $(0 \leqq p \leqq r)$ are odd and $n_{p+1}, \ldots, n_r$ are even, then

$$(1)\qquad E(|x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}|) = \frac{1}{i^{p+n}\pi^p} \int \cdots \int \frac{dt_1 \ldots dt_p}{t_1 \ldots t_p} \left[\frac{\partial^n \varphi(t_1, t_2, \ldots, t_r)}{\partial t_1^{n_1} \partial t_2^{n_2} \ldots \partial t_r^{n_r}} \right]_{t_{p+1} = \cdots = t_r = 0}$$

Here the integral with respect to t's is to be understood in the sense of Cauchy's principal value, i.e.,

$$(2)\qquad \lim_{\substack{\varepsilon_1, \ldots, \varepsilon_p \to 0 \\ C_1, \ldots, C_p \to \infty}} \left(\int_{-C_1}^{-\varepsilon_1} + \int_{\varepsilon_1}^{C_1} \right) dt_1 \ldots \ldots \left(\int_{-C_p}^{-\varepsilon_p} + \int_{\varepsilon_p}^{C_p} \right) dt_p$$

Proof. Denote by $F(x_1, x_2, \ldots, x_r)$ the distribution function of $x_1, x_2, \ldots, x_r$. Then, using the function

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

we can express as follows.

$$(3)\qquad E(|x_1^{n_1} x_2^{n_2} \cdots x^{n_r}|) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r} \operatorname{sgn}(x_1) \ldots \operatorname{sgn}(x_p)\, dF(x_1, \ldots, x_r).$$

Substituting in the right hand side of (3) the wellknown integral expression,

$$\operatorname{sgn}(x) = \frac{2}{\pi} \lim_{\substack{\varepsilon \to 0 \\ C \to \infty}} \int_{\varepsilon}^{C} \frac{\sin tx}{t} dt = \frac{2}{\pi} \lim_{\substack{\varepsilon \to 0 \\ C \to \infty}} \int_{\varepsilon}^{C} \frac{e^{itx} - e^{-itx}}{2it} dt$$

$$= \frac{1}{i\pi} \lim_{\substack{\varepsilon \to 0 \\ C \to \infty}} \left(\int_{-C}^{-\varepsilon} + \int_{\varepsilon}^{C} \right) \frac{e^{itx}}{t} dt,$$

and interchanging the order of limit-signs with integral-sign with respect to F, which is permitted by the dominated convergence from the assumption that $E(|x_1^{n_1} \dots x_r^{n_r}|)$ is finite, we have

(4) $E(|x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}|)$

$$= \frac{1}{i^n} \left(\frac{1}{i\pi} \right)^p \int \cdots \int \frac{dt_1 \dots dt_p}{t_1 \dots t_p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (ix_1)^{n_1} (ix_2)^{n_2} \dots (ix_r)^{n_r}$$

$$\exp\left(i \sum_{j=1}^{p} t_j x_j \right) dF(x_1, \dots, x_r),$$

the integrals with respect to t's being also to be understood in the sense of (2). In (4) the integral with respect to F is just

$$\left[\frac{\partial^n \varphi(t_1, \dots, t_r)}{\partial t_1^{n_1} \dots \partial t_r^{n_r}} \right]_{t_{p+1} = \dots = t_r = 0}$$

since by the assumption the partial derivations are possible. *q. e. d.*

Applying the above theorem, the author has derived the formulæ for the absolute moments up to 12th degree in the 3-dimensional normal distribution with the means zero. We shall illustrate the method in some detail by taking an example.

Let x_1, x_2, x_3 be distributed according to the 3-dimensional normal distribution, with the means zero, with the standard deviations $\sigma_1, \sigma_2, \sigma_3$ respectively, and with the correlation matrix

$$\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$$

the determinant of which being denoted by R. Moreover, in the results given below, we shall denote by $\rho_{12\cdot3}, \rho_{13\cdot2}, \rho_{23\cdot1}$ the partial correlation coefficients, *i. e.*,

$$\rho_{12\cdot3} = \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{1-\rho_{13}^2}\sqrt{1-\rho_{23}^2}} \quad \text{etc.}$$

We shall calculate $E(|x_1^2 x_2 x_3|)$ for an example. The characteristic function of x_1, x_2, x_3 is, as is known,

$$\varphi(t_1, t_2, t_3) = \exp\left(-\frac{1}{2} \sum_{j,k=1}^{3} \sigma_j \sigma_k \rho_{jk} t_j t_k \right).$$

Differentiating two times with respect to t_1, one time with respect to t_2 and t_3, and putting $t_1 = 0$, we have

$$(5)\quad \left[\frac{\partial^4\varphi(t_1,t_2,t_3)}{\partial t_1^2\partial t_2\partial t_3}\right]_{1=0} = (\rho_{23}+2\rho_{12}\rho_{13}-T_2T_3-2\rho_{12}T_1T_3-2\rho_{13}T_1T_2 - \rho_{23}T_1^2+T_1^2T_2T_3)\sigma_1^2\sigma_2\sigma_3 \times \exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\},$$

where

$$T_1=\rho_{12}\sigma_2t_2+\rho_{13}\sigma_3t_3,$$
$$T_2=\sigma_2t_2+\rho_{23}\sigma_3t_3,$$
$$T_3=\rho_{23}\sigma_2t_2+\sigma_3t_3.$$

To calculate $E(|x_1^2x_2x_3|)$, it is necessary to find the value of the integral of the type

$$(6)\quad [\alpha,\beta,\gamma]=\iint\frac{T_1^\alpha T_2^\beta T_3^\gamma}{t_2t_3} \times\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\}dt_2dt_3,$$

for which $\alpha\leqq 2$, $\beta\leqq 1$, $\gamma\leqq 1$, such that $\alpha+\beta+\gamma$ is an even number. We start from the well-known integral

$$\iint\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\}dt_2dt_3=\frac{2\pi}{\sigma_2\sigma_3\sqrt{1-\rho_{23}^2}}.$$

Integrating both sides with respect to ρ_{23}, and taking into account the case $\rho_{23}=0$ to determine the integral constant there, we have

$$(7)\quad \iint\frac{1}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\}dt_2dt_3=-2\pi\sin^{-1}\rho_{23},$$

i. e., $[0,0,0]=-2\pi\sin^{-1}\rho_{23}$

Differentiating both sides of (7) with respect to σ_2,σ_3 and ρ_{23}, and multiplying by $-\sigma_2$, $-\sigma_3$, and -1 respectively, we have

$$(8)\quad \iint\frac{\sigma_2t_2T_2}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\}dt_2dt_3=0,$$

$$(9)\quad \iint\frac{\sigma_3t_3T_3}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\}dt_2dt_3=0,$$

$$(10)\quad \iint\frac{\sigma_2\sigma_3t_2t_3}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2+2\rho_{23}\sigma_2\sigma_3t_2t_3+\sigma_3^2t_3^2)\right\}dt_2dt_3=\frac{2\pi}{\sqrt{1-\rho_{23}^2}}.$$

From (8), (9), (10) and from the equalities

$$T_1^2=\rho_{12}^2\sigma_2t_2T_2+\rho_{13}^2\sigma_3t_3T_3+(2\rho_{12}\rho_{13}-\rho_{12}^2\rho_{23}-\rho_{13}^2\rho_{23})\sigma_2\sigma_3t_2t_3,$$
$$T_1T_2=\rho_{12}\sigma_2t_2T_2+\rho_{13}\rho_{23}\sigma_3t_3T_3+\rho_{13}(1-\rho_{23}^2)\sigma_2\sigma_3t_2t_3,$$

$$T_1T_3 = \rho_{12}\rho_{23}\sigma_2t_2T_2 + \rho_{13}\sigma_3t_3T_3 + \rho_{12}(1-\rho_{23}^2)\sigma_2\sigma_3t_2t_3,$$

$$(11)\qquad T_2T_3 = \rho_{23}\sigma_2t_2T_2 + \rho_{23}\sigma_3t_3T_3 + (1-\rho_{23}^2)\sigma_2\sigma_2t_2t_3,$$

we obtain

$$(12)\qquad [2,0,0] = \frac{2\pi}{\sqrt{1-\rho_{23}^2}}(2\rho_{12}\rho_{13} - \rho_{12}^2\rho_{23} - \rho_{13}^2\rho_{23}),$$

$$[1,1,0] = 2\pi\rho_{13}\sqrt{1-\rho_{23}^2},$$

$$[1,0,1] = 2\pi\rho_{12}\sqrt{1-\rho_{23}^2},$$

$$[0,1,1] = 2\pi\sqrt{1-\rho_{23}^2}.$$

If we substitute in the left hand side of (12) the right hand side of (6) putting $\alpha = 2$, $\beta = \gamma = 0$, differentiate with respect to σ_2 and multiply by $-\sigma_2$, we obtain

$$\iint \frac{\sigma_2t_2T_1^2T_2}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2 + 2\rho_{23}\sigma_2\sigma_3t_2t_3 + \sigma_3^2t_3^2)\right\}dt_2dt_3$$

$$-2\rho_{12}\iint \frac{\sigma_2t_2T_1}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2 + 2\rho_{23}\sigma_2\sigma_3t_2t_3 + \sigma_3^2t_3^2)\right\}dt_2dt_3 = 0.$$

And again using (8) and (10) with

$$\sigma_2t_2T_1 = \rho_{12}\sigma_2t_2T_2 + (\rho_{13} - \rho_{12}\rho_{23})\sigma_2\sigma_3t_2t_3$$

in the second integral, we have

$$(13)\qquad \iint \frac{\sigma_2t_2T_1^2T_2}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2 + 2\rho_{23}\sigma_2\sigma_3t_2t_3 + \sigma_3^2t_3^2)\right\}dt_2dt_3$$

$$= \frac{2\pi}{\sqrt{1-\rho_{23}^2}}(2\rho_{12}\rho_{13} - 2\rho_{12}^2\rho_{23}).$$

Similarly

$$(14)\qquad \iint \frac{\sigma_3t_3T_1^2T_3}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2 + 2\rho_{23}\sigma_2\sigma_3t_2t_3 + \sigma_3^2t_3^2)\right\}dt_2dt_3$$

$$= \frac{2\pi}{\sqrt{1-\rho_{23}^2}}(2\rho_{12}\rho_{13} - 2\rho_{13}^2\rho_{23}).$$

On the other hand, from the differentiation of (12) with respect to ρ_{23}, we obtain

$$(15)\qquad \iint \frac{\sigma_2\sigma_3t_2t_3T_1^2}{t_2t_3}\exp\left\{-\frac{1}{2}(\sigma_2^2t_2^2 + 2\rho_{23}\sigma_2\sigma_3t_2t_3 + \sigma_3^2t_3^2)\right\}dt_2dt_3$$

$$= \frac{2\pi}{\sqrt{1-\rho_{23}^2}}\,\frac{\rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{23}^2}.$$

In view of (11), if we multiply both sides of (13), (14) and (15) by ρ_{23}, ρ_{23} and $1-\rho_{23}^2$, and sum them up, we have

$$[2,1,1] = 2\pi\sqrt{1-\rho_{23}^2}\left(2\rho_{12}^2 + 2\rho_{13}^2 + \frac{-\rho_{12}^2 - \rho_{13}^2 + 2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{23}^2}\right).$$

We have thus obtained $[\alpha, \beta, \gamma]$ for all the necessary cases.

Combining these with the coefficients given in (5) and multiplying by $-1/\pi$ we have the final result

$$E(|x_1^2x_2x_3|) = \frac{2}{\pi}\left[\sqrt{1-\rho_{23}^2}(1+\rho_{12}^2+\rho_{13}^2) + (\rho_{23}+2\rho_{12}\rho_{13})\sin^{-1}\rho_{23}\right]\sigma_1^2\sigma_2\sigma_3.$$

In the following we give the results $E(|x_1^{n_1}x_2^{n_2}x_3^{n_3}|)$ using the notation (n_1, n_2, n_3), for all the cases $n_1 \geqq n_2 \geqq n_3 \geqq 1$ and $n_1 + n_2 + n_3 \leqq 12$. For other cases of order not higher than 12, we may obtain the similar formulæ by a permutation of the indices. For the cases $n_3 = 0$, see also the author's previous paper.

$$(1,1,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left[\sqrt{R} + (\rho_{12}+\rho_{13}\rho_{23})\sin^{-1}\rho_{12\cdot3}\right.$$
$$\left. + (\rho_{13}+\rho_{12}\rho_{23})\sin^{-1}\rho_{13\cdot2} + (\rho_{23}+\rho_{12}\rho_{13})\sin^{-1}\rho_{23\cdot1}\right]\sigma_1\sigma_2\sigma_3$$

$$(2,1,1) = \frac{2}{\pi}\left[\sqrt{1-\rho_{23}^2}(1+\rho_{12}^2+\rho_{13}^2) + (\rho_{23}+2\rho_{12}\rho_{13})\sin^{-1}\rho_{23}\right]\sigma_1^2\sigma_2\sigma_3$$

$$(3,1,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left[\sqrt{R}(2+\rho_{12}^2+\rho_{13}^2) + (3\rho_{12}+3\rho_{13}\rho_{23}+3\rho_{12}\rho_{13}^2\right.$$
$$-\rho_{13}^3\rho_{23})\sin^{-1}\rho_{12\cdot3} + (3\rho_{13}+3\rho_{12}\rho_{23}+3\rho_{12}^2\rho_{13}-\rho_{12}^3\rho_{23})\sin^{-1}\rho_{13\cdot2}$$
$$\left.+(2\rho_{23}+6\rho_{12}\rho_{13})\sin^{-1}\rho_{23\cdot1}\right]\sigma_1^3\sigma_2\sigma_3$$

$$(2,2,1) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1+2\rho_{12}^2+\rho_{13}^2+\rho_{23}^2+4\rho_{12}\rho_{13}\rho_{23}-\rho_{13}^2\rho_{23}^2)\sigma_1^2\sigma_2^2\sigma_3$$

$$(4,1,1) = \frac{2}{\pi}\left[\sqrt{1-\rho_{23}^2}\left(2+8\rho_{12}^2+8\rho_{13}^2+\rho_{23}^2-4\rho_{12}\rho_{13}\rho_{23}-2\rho_{12}^4\right.\right.$$
$$\left.\left.+4\rho_{12}^2\rho_{13}^2-2\rho_{13}^4+\frac{R^2}{1-\rho_{23}^2}\right)+(3\rho_{23}+12\rho_{12}\rho_{13})\sin^{-1}\rho_{23}\right]\sigma_1^4\sigma_2\sigma_3$$

$$(3,2,1) = \frac{2}{\pi}\left[\sqrt{1-\rho_{13}^2}(2+6\rho_{12}^2+\rho_{13}^2+2\rho_{23}^2+6\rho_{12}\rho_{13}\rho_{23}\right.$$
$$\left.-2\rho_{13}^2\rho_{23}^2)+(3\rho_{13}+6\rho_{12}\rho_{23}+6\rho_{12}^2\rho_{13})\sin^{-1}\rho_{13}\right]\sigma_1^3\sigma_2^2\sigma_3$$

$$(2,2,2) = (1+2\rho_{12}^2+2\rho_{13}^2+2\rho_{23}^2+8\rho_{12}\rho_{13}\rho_{23})\sigma_1^2\sigma_2^2\sigma_3^2$$

$$(5,1,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left[\sqrt{R}\left(7+11\rho_{12}^2+11\rho_{13}^2+\rho_{23}^2-4\rho_{12}\rho_{13}\rho_{23}-3\rho_{12}^4\right.\right.$$

$$+ 4\rho_{12}^2\rho_{13}^2 - 3\rho_{13}^4 + \frac{R^2}{1-\rho_{23}^2}\Big) + (15\rho_{12} + 15\rho_{13}\rho_{23}$$
$$+ 30\rho_{12}\rho_{13}^2 - 10\rho_{13}^3\rho_{23} - 5\rho_{12}\rho_{13}^4 + 3\rho_{13}^5\rho_{23}) \sin^{-1}\rho_{12\cdot 3}$$
$$+ (15\rho_{13} + 15\rho_{12}\rho_{23} + 30\rho_{12}^2\rho_{13} - 10\rho_{12}^3\rho_{23} - 5\rho_{12}^4\rho_{13}$$
$$+ 3\rho_{12}^5\rho_{23}) \sin^{-1}\rho_{13\cdot 2} + (8\rho_{23} + 40\rho_{12}\rho_{13}) \sin^{-1}\rho_{23\cdot 1}\Big]\sigma_1^5\sigma_2\sigma_3$$

$$(4,2,1) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(3 + 12\rho_{12}^2 + 6\rho_{13}^2 + 3\rho_{23}^2 + 24\rho_{12}\rho_{13}\rho_{23} + 12\rho_{12}^2\rho_{13}^2$$
$$- \rho_{13}^4 - 6\rho_{13}^2\rho_{23}^2 - 8\rho_{12}\rho_{13}^3\rho_{23} + 3\rho_{13}^4\rho_{23}^2)\,\sigma_1^4\sigma_2^2\sigma_3$$

$$(3,3,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\Big[\sqrt{R}\,(4 + 11\rho_{12}^2 + 2\rho_{13}^2 + 2\rho_{23}^2 + 8\rho_{12}\rho_{13}\rho_{23} - 3\rho_{13}^2\rho_{23}^2)$$
$$+ (9\rho_{12} + 9\rho_{13}\rho_{23} + 6\rho_{12}^3 + 9\rho_{12}\rho_{13}^2 + 9\rho_{12}\rho_{23}^2 + 18\rho_{12}^2\rho_{13}\rho_{23}$$
$$- 3\rho_{13}^3\rho_{23} - 3\rho_{13}\rho_{23}^3 - 9\rho_{12}\rho_{13}^2\rho_{23}^2 + 3\rho_{13}^3\rho_{23}^3) \sin^{-1}\rho_{12\cdot 3}$$
$$+ (6\rho_{13} + 18\rho_{12}\rho_{23} + 18\rho_{12}^2\rho_{13} + 6\rho_{12}^3\rho_{23}) \sin^{-1}\rho_{13\cdot 2}$$
$$+ (6\rho_{23} + 18\rho_{12}\rho_{13} + 18\rho_{12}^2\rho_{23} + 6\rho_{12}^3\rho_{13}) \sin^{-1}\rho_{23\cdot 1}\Big]\sigma_1^3\sigma_2^3\sigma_3$$

$$(3,2,2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(2 + 6\rho_{12}^2 + 6\rho_{13}^2 + 4\rho_{23}^2 + 24\rho_{12}\rho_{13}\rho_{23} + 6\rho_{12}^2\rho_{13}^2)\,\sigma_1^3\sigma_2^2\sigma_3^2$$

$$(6,1,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{23}^2}\Big\{2 + 76\rho_{12}^2 + 76\rho_{13}^2 + 13\rho_{23}^2 - 50\rho_{12}\rho_{13}\rho_{23} - 38\rho_{12}^4$$
$$+ 44\rho_{12}^2\rho_{13}^2 - 4\rho_{12}^2\rho_{23}^2 - 38\rho_{13}^4 - 4\rho_{13}^2\rho_{23}^2 + 8\rho_{12}^3\rho_{13}\rho_{23} + 8\rho_{12}\rho_{13}^3\rho_{23}$$
$$- 4\rho_{12}\rho_{13}\rho_{23}^3 + 8\rho_{12}^6 + 16\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 8\rho_{13}^6 - 16\rho_{12}^3\rho_{13}^3\rho_{23}$$
$$+ \frac{R^2}{1-\rho_{23}^2}(12 - 4\rho_{12}^2 - 4\rho_{13}^2 - 4\rho_{12}\rho_{13}\rho_{23}) + \frac{R^3}{(1-\rho_{23}^2)^2}\Big\}$$
$$+ (15\rho_{23} + 90\rho_{12}\rho_{13}) \sin^{-1}\rho_{23}\Big]\sigma_1^6\sigma_2\sigma_3$$

$$(5,2,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{13}^2}\,(8 + 40\rho_{12}^2 + 9\rho_{13}^2 + 8\rho_{23}^2 + 50\rho_{12}\rho_{13}\rho_{23} + 20\rho_{12}^2\rho_{13}^2$$
$$- 2\rho_{13}^4 - 16\rho_{13}^2\rho_{23}^2 - 20\rho_{12}\rho_{13}^3\rho_{23} + 8\rho_{13}^4\rho_{23}^2) + (15\rho_{13} + 30\rho_{12}\rho_{23}$$
$$+ 60\rho_{12}^2\rho_{13}) \sin^{-1}\rho_{13}\Big]\sigma_1^5\sigma_2^2\sigma_3$$

$$(4,3,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{23}^2}\,(6 + 36\rho_{12}^2 + 12\rho_{13}^2 + 3\rho_{23}^2 + 36\rho_{12}\rho_{13}\rho_{23} + 6\rho_{12}^4$$
$$+ 36\rho_{12}^2\rho_{13}^2 - 2\rho_{13}^4 - 12\rho_{13}^2\rho_{23}^2 - 24\rho_{12}\rho_{13}^3\rho_{23} + 8\rho_{13}^4\rho_{23}^2)$$
$$+ (9\rho_{23} + 36\rho_{12}\rho_{13} + 36\rho_{12}^2\rho_{23} + 24\rho_{12}^3\rho_{13}) \sin^{-1}\rho_{23}\Big]\sigma_1^4\sigma_2^3\sigma_3$$

$$(4,2,2) = (3 + 12\rho_{12}^2 + 12\rho_{13}^2 + 6\rho_{23}^2 + 48\rho_{12}\rho_{13}\rho_{23} + 24\rho_{12}^2\rho_{13}^2)\,\sigma_1^4\sigma_2^2\sigma_3^2$$

$$(3,3,2) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{12}^2}\,(4 + 11\rho_{12}^2 + 12\rho_{13}^2 + 12\rho_{23}^2 + 54\rho_{12}\rho_{13}\rho_{23}$$
$$+ 6\rho_{12}^2\rho_{13}^2 + 6\rho_{12}^2\rho_{23}^2) + (9\rho_{12} + 18\rho_{13}\rho_{23} + 6\rho_{12}^3 + 18\rho_{12}\rho_{13}^2$$
$$+ 18\rho_{12}\rho_{23}^2 + 36\rho_{12}^2\rho_{13}\rho_{23})\sin^{-1}\rho_{12}\Big]\sigma_1^3\sigma_2^3\sigma_3^2$$

$$(7,1,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\Big[\sqrt{R}\,\Big\{32 + 126\rho_{12}^2 + 126\rho_{13}^2 + 16\rho_{23}^2 - 64\rho_{12}\rho_{13}\rho_{23}$$
$$- 68\rho_{12}^4 + 60\rho_{12}^2\rho_{13}^2 - 5\rho_{12}^2\rho_{23}^2 - 68\rho_{13}^4 - 5\rho_{13}^2\rho_{23}^2 + 12\rho_{12}^3\rho_{13}\rho_{23}$$
$$+ 12\rho_{12}\rho_{13}^3\rho_{23} - 4\rho_{12}\rho_{13}\rho_{23}^3 + 15\rho_{12}^6 - 4\rho_{12}^4\rho_{13}^2 - 4\rho_{12}^2\rho_{13}^4$$
$$+ 16\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 15\rho_{13}^6 - 16\rho_{12}^3\rho_{13}^3\rho_{23} + \frac{R^2}{1-\rho_{23}^2}(14 - 5\rho_{12}^2 - 5\rho_{13}^2$$
$$- 4\rho_{12}\rho_{13}\rho_{23}) + 2\frac{R^3}{(1-\rho_{23}^2)^2}\Big\} + (105\rho_{12} + 105\rho_{13}\rho_{23} + 315\rho_{12}\rho_{13}^2$$
$$- 105\rho_{13}^3\rho_{23} - 105\rho_{12}\rho_{13}^4 + 63\rho_{13}^5\rho_{23} + 21\rho_{12}\rho_{13}^6 - 15\rho_{13}^7\rho_{23})\sin^{-1}\rho_{12\cdot3}$$
$$+ (105\rho_{13} + 105\rho_{12}\rho_{23} + 315\rho_{12}^2\rho_{13} - 105\rho_{12}^3\rho_{23} - 105\rho_{12}^4\rho_{13}$$
$$+ 63\rho_{12}^5\rho_{23} + 21\rho_{12}^6\rho_{13} - 15\rho_{12}^7\rho_{23})\sin^{-1}\rho_{13\cdot2}$$
$$+ (48\rho_{23} + 336\rho_{12}\rho_{13})\sin^{-1}\rho_{23\cdot1}\Big]\sigma_1^7\sigma_2\sigma_3$$

$$(6,2,1) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(15 + 90\rho_{12}^2 + 45\rho_{13}^2 + 15\rho_{23}^2 + 180\rho_{12}\rho_{13}\rho_{23} + 180\rho_{12}^2\rho_{13}^2$$
$$- 15\rho_{13}^4 - 45\rho_{13}^2\rho_{23}^2 - 120\rho_{12}\rho_{13}^3\rho_{23} - 30\rho_{12}^2\rho_{13}^4 + 3\rho_{13}^6 + 45\rho_{13}^4\rho_{23}^2$$
$$+ 36\rho_{12}\rho_{13}^5\rho_{23} - 15\rho_{13}^6\rho_{23}^2)\,\sigma_1^6\sigma_2^2\sigma_3$$

$$(5,3,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\Big[\sqrt{R}\,(16 + 83\rho_{12}^2 + 18\rho_{13}^2 + 8\rho_{23}^2 + 68\rho_{12}\rho_{13}\rho_{23} + 6\rho_{12}^4$$
$$+ 47\rho_{12}^2\rho_{13}^2 - 4\rho_{13}^4 - 25\rho_{13}^2\rho_{23}^2 - 40\rho_{12}\rho_{13}^3\rho_{23} + 15\rho_{13}^4\rho_{23}^2)$$
$$+ (45\rho_{12} + 45\rho_{13}\rho_{23} + 60\rho_{12}^3 + 90\rho_{12}\rho_{13}^2 + 45\rho_{12}\rho_{23}^2 + 180\rho_{12}^2\rho_{13}\rho_{23}$$
$$- 30\rho_{13}^3\rho_{23} - 15\rho_{13}\rho_{23}^3 + 60\rho_{12}^3\rho_{13}^2 - 15\rho_{12}\rho_{13}^4 - 90\rho_{12}\rho_{13}^2\rho_{23}^2$$
$$- 60\rho_{12}^2\rho_{13}^3\rho_{23} + 9\rho_{13}^5\rho_{23} + 30\rho_{13}^3\rho_{23}^3 + 45\rho_{12}\rho_{13}^4\rho_{23}^2$$
$$- 15\rho_{13}^5\rho_{23}^3)\sin^{-1}\rho_{12\cdot3} + (30\rho_{13} + 90\rho_{12}\rho_{23} + 180\rho_{12}^2\rho_{13} + 60\rho_{12}^3\rho_{23}$$
$$+ 30\rho_{12}^4\rho_{13} - 6\rho_{12}^5\rho_{23})\sin^{-1}\rho_{13\cdot2} + (24\rho_{23} + 120\rho_{12}\rho_{13} + 120\rho_{12}^2\rho_{23}$$
$$+ 120\rho_{12}^3\rho_{13})\sin^{-1}\rho_{23\cdot1}\Big]\sigma_1^5\sigma_2^3\sigma_3$$

$$(5,2,2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(8 + 40\rho_{12}^2 + 40\rho_{13}^2 + 16\rho_{23}^2 + 160\rho_{12}\rho_{13}\rho_{23}$$
$$+ 120\rho_{12}^2\rho_{13}^2)\,\sigma_1^5\sigma_2^2\sigma_3^2$$

$$(4,4,1)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(9+72\rho_{12}^2+18\rho_{13}^2+18\rho_{23}^2+144\rho_{12}\rho_{13}\rho_{23}+24\rho_{12}^4$$
$$+72\rho_{12}^2\rho_{13}^2+72\rho_{12}^2\rho_{23}^2-3\rho_{13}^4-36\rho_{13}^2\rho_{23}^2-3\rho_{23}^4+96\rho_{12}^3\rho_{13}\rho_{23}$$
$$-48\rho_{12}\rho_{13}^3\rho_{23}-48\rho_{12}\rho_{13}\rho_{23}^3-72\rho_{12}^2\rho_{13}^2\rho_{23}^2+18\rho_{13}^4\rho_{23}^2+18\rho_{13}^2\rho_{23}^4$$
$$+48\rho_{12}\rho_{13}^3\rho_{23}^3-15\rho_{13}^4\rho_{23}^4)\,\sigma_1^4\sigma_2^4\sigma_3$$

$$(4,3,2)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(6+36\rho_{12}^2+24\rho_{13}^2+18\rho_{23}^2+144\rho_{12}\rho_{13}\rho_{23}+6\rho_{12}^4$$
$$+72\rho_{12}^2\rho_{13}^2+36\rho_{12}^2\rho_{23}^2+48\rho_{12}^3\rho_{13}\rho_{23}-6\rho_{12}^4\rho_{23}^2)\,\sigma_1^4\sigma_2^3\sigma_3^2$$

$$(3,3,3)=\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\Big[\sqrt{R}\,(8+22\rho_{12}^2+22\rho_{13}^2+22\rho_{23}^2+100\rho_{12}\rho_{13}\rho_{23}+6\rho_{12}^2\rho_{13}^2$$
$$+6\rho_{12}^2\rho_{23}^2+6\rho_{13}^2\rho_{23}^2)+(18\rho_{12}+54\rho_{13}\rho_{23}+12\rho_{12}^3+54\rho_{12}\rho_{13}^2$$
$$+54\rho_{12}\rho_{23}^2+108\rho_{12}^2\rho_{13}\rho_{23}+18\rho_{13}^3\rho_{23}+18\rho_{13}\rho_{23}^3+54\rho_{12}\rho_{13}^2\rho_{23}^2$$
$$-6\rho_{13}^3\rho_{23}^3)\sin^{-1}\rho_{12\cdot 3}+(18\rho_{13}+54\rho_{12}\rho_{23}+54\rho_{12}^2\rho_{13}+12\rho_{13}^3$$
$$+54\rho_{13}\rho_{23}^2+18\rho_{12}^3\rho_{23}+108\rho_{12}\rho_{13}^2\rho_{23}+18\rho_{12}\rho_{23}^3+54\rho_{12}^2\rho_{13}\rho_{23}^2$$
$$-6\rho_{12}^3\rho_{23}^3)\sin^{-1}\rho_{13\cdot 2}+(18\rho_{23}+54\rho_{12}\rho_{13}+54\rho_{12}^2\rho_{23}+54\rho_{13}^2\rho_{23}$$
$$+12\rho_{23}^3+18\rho_{12}^3\rho_{13}+18\rho_{12}\rho_{13}^3+108\rho_{12}\rho_{13}\rho_{23}^2+54\rho_{12}^2\rho_{13}^2\rho_{23}$$
$$-6\rho_{12}^3\rho_{13}^3)\sin^{-1}\rho_{23\cdot 1}\Big]\,\sigma_1^3\sigma_2^3\sigma_3^3$$

$$(8,1,1)=\frac{2}{\pi}\Big[\sqrt{1-\rho_{23}^2}\,\Big\{-58+868\rho_{12}^2+868\rho_{13}^2+168\rho_{23}^2-592\rho_{12}\rho_{13}\rho_{23}$$
$$-650\rho_{12}^4+396\rho_{12}^2\rho_{13}^2-100\rho_{12}^2\rho_{23}^2-650\rho_{13}^4-100\rho_{13}^2\rho_{23}^2$$
$$+168\rho_{12}^3\rho_{13}\rho_{23}+168\rho_{12}\rho_{13}^3\rho_{23}-104\rho_{12}\rho_{13}\rho_{23}^3+272\rho_{12}^6+112\rho_{12}^4\rho_{13}^2$$
$$+24\rho_{12}^4\rho_{23}^2+112\rho_{12}^2\rho_{13}^4+456\rho_{12}^2\rho_{13}^2\rho_{23}^2+272\rho_{13}^6+24\rho_{13}^4\rho_{23}^2$$
$$-32\rho_{12}^5\rho_{13}\rho_{23}-448\rho_{12}^3\rho_{13}^3\rho_{23}+32\rho_{12}^3\rho_{13}\rho_{23}^3-32\rho_{12}\rho_{13}^5\rho_{23}$$
$$+32\rho_{12}\rho_{13}^3\rho_{23}^3-48\rho_{12}^8-32\rho_{12}^6\rho_{13}^2-96\rho_{12}^4\rho_{13}^4-128\rho_{12}^4\rho_{13}^2\rho_{23}^2$$
$$-32\rho_{12}^2\rho_{13}^6-128\rho_{12}^2\rho_{13}^4\rho_{23}^2-48\rho_{13}^8+128\rho_{12}^5\rho_{13}^3\rho_{23}+128\rho_{12}^3\rho_{13}^5\rho_{23}$$
$$+\frac{R^2}{1-\rho_{23}^2}(144-94\rho_{12}^2-94\rho_{13}^2-100\rho_{12}\rho_{13}\rho_{23}+24\rho_{12}^4+32\rho_{12}^2\rho_{13}^2$$
$$+24\rho_{13}^4+32\rho_{12}^3\rho_{13}\rho_{23}+32\rho_{12}\rho_{13}^3\rho_{23})+\frac{R^3}{(1-\rho_{23}^2)^2}(16-6\rho_{12}^2$$
$$-6\rho_{13}^2-4\rho_{12}\rho_{13}\rho_{23})+3\frac{R^4}{(1-\rho_{23}^2)^3}\Big\}+(105\rho_{23}$$
$$+840\rho_{12}\rho_{13})\sin^{-1}\rho_{23}\Big]\,\sigma_1^8\sigma_2\sigma_3$$

$$(7,2,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{13}^2}\,(48 + 336\rho_{12}^2 + 87\rho_{13}^2 + 48\rho_{23}^2 + 462\rho_{12}\rho_{13}\rho_{23}$$
$$+ 378\rho_{12}^2\rho_{13}^2 - 38\rho_{13}^4 - 144\rho_{13}^2\rho_{23}^2 - 364\rho_{12}\rho_{13}^3\rho_{23} - 84\rho_{12}^2\rho_{13}^4$$
$$+ 8\rho_{13}^6 + 144\rho_{13}^4\rho_{23}^2 + 112\rho_{12}\rho_{13}^5\rho_{13} - 48\rho_{13}^6\rho_{23}^2) + (105\rho_{13}$$
$$+ 210\rho_{12}\rho_{23} + 630\rho_{12}^2\rho_{13})\sin^{-1}\rho_{13}\Big]\sigma_1^7\sigma_2^2\sigma_3$$

$$(6,3,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{23}^2}\Big(36 + 252\rho_{12}^2 + 72\rho_{13}^2 + 3\rho_{23}^2 + 306\rho_{12}\rho_{13}\rho_{23}$$
$$+ 108\rho_{12}^4 + 576\rho_{12}^2\rho_{13}^2 + 18\rho_{12}^2\rho_{23}^2 - 12\rho_{13}^4 - 72\rho_{13}^2\rho_{23}^2 + 6\rho_{23}^4$$
$$- 72\rho_{12}^3\rho_{13}\rho_{23} - 432\rho_{12}\rho_{13}^3\rho_{23} - 36\rho_{12}\rho_{13}\rho_{23}^3 - 12\rho_{12}^6 + 72\rho_{12}^4\rho_{13}^2$$
$$- 108\rho_{12}^2\rho_{13}^4 + 72\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 120\rho_{13}^4\rho_{23}^2 - 48\rho_{12}^3\rho_{13}^3\rho_{23}$$
$$+ 144\rho_{12}\rho_{13}^5\rho_{23} - 48\rho_{13}^6\rho_{23}^2 - 6\frac{R^3}{1-\rho_{23}^2}\Big) + (45\rho_{23} + 270\rho_{12}\rho_{13}$$
$$+ 270\rho_{12}^2\rho_{23} + 360\rho_{12}^3\rho_{13})\sin^{-1}\rho_{23}\Big]\sigma_1^6\sigma_2^3\sigma_3$$

$$(6,2,2) = (15 + 90\rho_{12}^2 + 90\rho_{13}^2 + 30\rho_{23}^2 + 360\rho_{12}\rho_{13}\rho_{23} + 360\rho_{12}^2\rho_{13}^2)\,\sigma_1^6\sigma_2^2\sigma_3^2$$

$$(5,4,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{13}^2}\,(24 + 240\rho_{12}^2 + 27\rho_{13}^2 + 48\rho_{23}^2 + 300\rho_{12}\rho_{13}\rho_{23}$$
$$+ 120\rho_{12}^4 + 120\rho_{12}^2\rho_{13}^2 + 240\rho_{12}^2\rho_{23}^2 - 6\rho_{13}^4 - 96\rho_{13}^2\rho_{23}^2 - 8\rho_{23}^4$$
$$+ 240\rho_{12}^3\rho_{13}\rho_{23} - 120\rho_{12}\rho_{13}^3\rho_{23} - 160\rho_{12}\rho_{13}\rho_{23}^3 - 240\rho_{12}^2\rho_{13}^2\rho_{23}^2$$
$$+ 48\rho_{13}^4\rho_{23}^2 + 56\rho_{13}^2\rho_{23}^4 + 160\rho_{12}\rho_{13}^3\rho_{23}^3 - 48\rho_{13}^4\rho_{23}^4) + (45\rho_{13}$$
$$+ 180\rho_{12}\rho_{23} + 360\rho_{12}^2\rho_{13} + 240\rho_{12}^3\rho_{23} + 120\rho_{12}^4\rho_{13})\sin^{-1}\rho_{13}\Big]\sigma_1^5\sigma_2^4\sigma_3$$

$$(5,3,2) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{12}^2}\,(16 + 83\rho_{12}^2 + 80\rho_{13}^2 + 48\rho_{23}^2 + 390\rho_{12}\rho_{13}\rho_{23} + 6\rho_{12}^4$$
$$+ 220\rho_{12}^2\rho_{13}^2 + 54\rho_{12}^2\rho_{23}^2 + 60\rho_{12}^3\rho_{13}\rho_{23} - 12\rho_{12}^4\rho_{23}^2) + (45\rho_{12}$$
$$+ 90\rho_{13}\rho_{23} + 60\rho_{12}^3 + 180\rho_{12}\rho_{13}^2 + 90\rho_{12}\rho_{23}^2 + 360\rho_{12}^2\rho_{13}\rho_{23}$$
$$+ 120\rho_{12}^3\rho_{13}^2)\sin^{-1}\rho_{12}\Big]\sigma_1^5\sigma_2^3\sigma_3^2$$

$$(4,4,2) = (9 + 72\rho_{12}^2 + 36\rho_{13}^2 + 36\rho_{23}^2 + 288\rho_{12}\rho_{13}\rho_{23} + 24\rho_{12}^4 + 144\rho_{12}^2\rho_{13}^2$$
$$+ 144\rho_{12}^2\rho_{23}^2 + 192\rho_{12}^3\rho_{13}\rho_{23})\,\sigma_1^4\sigma_2^4\sigma_3^2$$

$$(4,3,3) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{23}^2}\,(12 + 72\rho_{12}^2 + 72\rho_{13}^2 + 33\rho_{23}^2 + 324\rho_{12}\rho_{13}\rho_{23}$$
$$+ 12\rho_{12}^4 + 216\rho_{12}^2\rho_{13}^2 + 36\rho_{12}^2\rho_{23}^2 + 12\rho_{13}^4 + 36\rho_{13}^2\rho_{23}^2 + 72\rho_{12}^3\rho_{13}\rho_{23}$$
$$+ 72\rho_{12}\rho_{13}^3\rho_{23} - 12\rho_{12}^4\rho_{23}^2 - 12\rho_{13}^4\rho_{23}^2) + (27\rho_{23} + 108\rho_{12}\rho_{13}$$
$$+ 108\rho_{12}^2\rho_{23} + 108\rho_{13}^2\rho_{23} + 18\rho_{23}^3 + 72\rho_{12}^3\rho\ +{}_{13}72\rho_{12}\rho_{13}^3$$

$$+ 216\rho_{12}\rho_{13}\rho_{23}^3 + 216\rho_{12}^2\rho_{13}^2\rho_{23}) \sin^{-1}\rho_{23}\Big]\sigma_1^4\sigma_2^3\sigma_3^3$$

$$(9,1,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\Big[\sqrt{R}\Big\{151 + 1638\rho_{12}^2 + 1638\rho_{13}^2 + 233\rho_{23}^2 - 912\rho_{12}\rho_{13}\rho_{23}$$
$$- 1314\rho_{12}^4 + 733\rho_{12}^2\rho_{13}^2 - 145\rho_{12}^2\rho_{23}^2 - 1314\rho_{13}^4 - 145\rho_{13}^2\rho_{23}^2$$
$$+ 312\rho_{12}^3\rho_{13}\rho_{23} + 312\rho_{12}\rho_{13}^3\rho_{23} - 124\rho_{12}\rho_{13}\rho_{23}^3 + 575\rho_{12}^6 + 16\rho_{12}^4\rho_{13}^2$$
$$+ 35\rho_{12}^4\rho_{23}^2 + 16\rho_{12}^2\rho_{13}^4 + 551\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 575\rho_{13}^6 + 35\rho_{13}^4\rho_{23}^2$$
$$- 60\rho_{12}^5\rho_{13}\rho_{23} - 556\rho_{12}^3\rho_{13}^3\rho_{23} + 40\rho_{12}^3\rho_{13}\rho_{23}^3 - 60\rho_{12}\rho_{13}^5\rho_{23}$$
$$+ 40\rho_{12}\rho_{13}^3\rho_{23}^3 - 105\rho_{12}^8 - 20\rho_{12}^6\rho_{13}^2 - 100\rho_{12}^4\rho_{13}^4 - 160\rho_{12}^4\rho_{13}^2\rho_{23}^2$$
$$- 20\rho_{12}^2\rho_{13}^6 - 160\rho_{12}^2\rho_{13}^4\rho_{23}^2 - 105\rho_{13}^8 + 160\rho_{12}^5\rho_{13}^3\rho_{23} + 160\rho_{12}^3\rho_{13}^5\rho_{23}$$
$$+ \frac{R^2}{1-\rho_{23}^2}(189 - 131\rho_{12}^2 - 131\rho_{13}^2 - 116\rho_{12}\rho_{13}\rho_{23}$$
$$+ 35\rho_{12}^4 + 39\rho_{12}^2\rho_{13}^2 + 35\rho_{13}^4 + 40\rho_{12}^3\rho_{12}\rho_{23} + 40\rho_{12}\rho_{13}^3\rho_{23})$$
$$+ \frac{R^3}{(1-\rho_{23}^2)^2}(36 - 14\rho_{12}^2 - 14\rho_{13}^2 - 8\rho_{12}\rho_{13}\rho_{23}) + 8\frac{R^4}{(1-\rho_{23}^2)^3}\Big\}$$
$$+ (945\rho_{12} + 945\rho_{13}\rho_{23} + 3780\rho_{12}\rho_{13}^2 - 1260\rho_{13}^3\rho_{23} - 1890\rho_{12}\rho_{13}^4$$
$$+ 1134\rho_{13}^5\rho_{23} + 756\rho_{12}\rho_{13}^6 - 540\rho_{13}^7\rho_{23} - 135\rho_{12}\rho_{13}^8$$
$$+ 105\rho_{13}^9\rho_{23}) \sin^{-1}\rho_{12\cdot 3} + (945\rho_{13} + 945\rho_{12}\rho_{23} + 3780\rho_{12}^2\rho_{13}$$
$$- 1260\rho_{12}^3\rho_{23} - 1890\rho_{12}^4\rho_{13} + 1134\rho_{12}^5\rho_{23} + 756\rho_{12}^6\rho_{13}$$
$$- 540\rho_{12}^7\rho_{23} - 135\rho_{12}^8\rho_{13} + 105\rho_{12}^9\rho_{23}) \sin^{-1}\rho_{13\cdot 2} + (384\rho_{23}$$
$$+ 3456\rho_{12}\rho_{13}) \sin^{-1}\rho_{23\cdot 1}\Big]\sigma_1^9\sigma_2\sigma_3$$

$$(8,2,1) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big(105 + 840\rho_{12}^2 + 420\rho_{13}^2 + 105\rho_{23}^2 + 1680\rho_{12}\rho_{13}\rho_{23}$$
$$+ 2520\rho_{12}^2\rho_{13}^2 - 210\rho_{13}^4 - 420\rho_{13}^2\rho_{23}^2 - 1680\rho_{12}\rho_{13}^3\rho_{23} - 840\rho_{12}^2\rho_{13}^4$$
$$+ 84\rho_{13}^6 + 630\rho_{13}^4\rho_{23}^2 + 1008\rho_{12}\rho_{13}^5\rho_{23} + 168\rho_{12}^2\rho_{13}^6 - 15\rho_{13}^8$$
$$- 420\rho_{13}^6\rho_{23}^2 - 240\rho_{12}\rho_{13}^7\rho_{23} + 105\rho_{13}^8\rho_{23}^2\Big)\sigma_1^8\sigma_2^2\sigma_3$$

$$(7,3,1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\Big[\sqrt{R}\Big(102 + 723\rho_{12}^2 + 156\rho_{13}^2 + 36\rho_{23}^2 + 672\rho_{12}\rho_{13}\rho_{23}$$
$$+ 138\rho_{12}^4 + 948\rho_{12}^2\rho_{13}^2 + 18\rho_{12}^2\rho_{23}^2 - 58\rho_{13}^4 - 213\rho_{13}^2\rho_{23}^2 + 6\rho_{23}^4$$
$$- 72\rho_{12}^3\rho_{13}\rho_{23} - 828\rho_{12}\rho_{13}^3\rho_{23} - 36\rho_{12}\rho_{13}\rho_{23}^3 - 18\rho_{12}^6 + 72\rho_{12}^4\rho_{13}^2$$
$$- 215\rho_{12}^2\rho_{13}^4 + 72\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 10\rho_{13}^6 + 280\rho_{13}^4\rho_{23}^2 - 48\rho_{12}^3\rho_{13}^3\rho_{23}$$
$$+ 280\rho_{12}\rho_{13}^5\rho_{23} + 105\rho_{13}^6\rho_{23}^2 - 6\frac{R^3}{1-\rho_{23}^2}\Big) + (315\rho_{12} + 315\rho_{13}\rho_{23}$$
$$+ 630\rho_{12}^3 + 945\rho_{12}\rho_{13}^2 + 315\rho_{12}\rho_{23}^2 + 1890\rho_{12}^2\rho_{13}\rho_{23} - 315\rho_{13}^3\rho_{23}$$

$$
\begin{aligned}
&- 105\rho_{13}\rho_{23}^3 - 945\rho_{12}\rho_{13}^2\rho_{23}^2 + 1260\rho_{12}^3\rho_{13}^2 + 315\rho_{12}\rho_{13}^4 \\
&- 1260\rho_{12}^2\rho_{13}^3\rho_{23} + 189\rho_{13}^5\rho_{23} + 315\rho_{13}^3\rho_{23}^3 - 210\rho_{12}^3\rho_{13}^4 + 63\rho_{12}\rho_{13}^6 \\
&+ 945\rho_{12}\rho_{13}^4\rho_{23}^2 + 378\rho_{12}^2\rho_{13}^5\rho_{23} - 45\rho_{13}^7\rho_{23} - 315\rho_{13}^5\rho_{23}^3 \\
&- 315\rho_{12}\rho_{13}^6\rho_{23}^2 + 105\rho_{13}^7\rho_{23}^3)\sin^{-1}\rho_{12\cdot 3} + (210\rho_{13} + 630\rho_{12}\rho_{23} \\
&+ 1890\rho_{12}^2\rho_{13} + 630\rho_{12}^3\rho_{23} + 630\rho_{12}^4\rho_{13} - 126\rho_{12}^5\rho_{23} - 42\rho_{12}^6\rho_{13} \\
&+ 18\rho_{12}^7\rho_{23})\sin^{-1}\rho_{13\cdot 2} + (144\rho_{23} + 1008\rho_{12}\rho_{13} + 1008\rho_{12}^2\rho_{23} \\
&+ 1680\rho_{12}^3\rho_{13})\sin^{-1}\rho_{23\cdot 1}\Big]\sigma_1^7\sigma_2^3\sigma_3
\end{aligned}
$$

$$
\begin{aligned}
(7, 2, 2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} &(48 + 336\rho_{12}^2 + 336\rho_{13}^2 + 96\rho_{23}^2 + 1344\rho_{12}\rho_{13}\rho_{23} \\
&+ 1680\rho_{12}^2\rho_{13}^2)\,\sigma_1^7\sigma_2^2\sigma_3^2
\end{aligned}
$$

$$
\begin{aligned}
(6, 4, 1) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} &(45 + 540\rho_{12}^2 + 135\rho_{13}^2 + 90\rho_{23}^2 + 1080\rho_{12}\rho_{13}\rho_{23} + 360\rho_{12}^4 \\
&+ 1080\rho_{12}^2\rho_{13}^2 + 540\rho_{12}^2\rho_{23}^2 - 45\rho_{13}^4 - 270\rho_{13}^2\rho_{23}^2 - 15\rho_{23}^4 \\
&+ 1440\rho_{12}^3\rho_{13}\rho_{23} - 720\rho_{12}\rho_{13}^3\rho_{23} - 360\rho_{12}\rho_{13}\rho_{23}^3 + 360\rho_{12}^4\rho_{13}^2 \\
&- 180\rho_{12}^2\rho_{13}^4 - 1080\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 9\rho_{13}^6 + 270\rho_{13}^4\rho_{23}^2 + 135\rho_{13}^2\rho_{23}^4 \\
&- 480\rho_{12}^3\rho_{13}^3\rho_{23} + 216\rho_{12}\rho_{13}^5\rho_{23} + 720\rho_{12}\rho_{13}^3\rho_{23}^3 + 540\rho_{12}^2\rho_{13}^4\rho_{23}^2 \\
&- 90\rho_{13}^6\rho_{23}^2 - 225\rho_{13}^4\rho_{23}^4 - 360\rho_{12}\rho_{13}^5\rho_{23}^3 + 105\rho_{13}^6\rho_{23}^4)\,\sigma_1^6\sigma_2^4\sigma_3
\end{aligned}
$$

$$
\begin{aligned}
(6, 3, 2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} &(30 + 270\rho_{12}^2 + 180\rho_{13}^2 + 90\rho_{23}^2 + 1080\rho_{12}\rho_{13}\rho_{23} + 90\rho_{12}^4 \\
&+ 1080\rho_{12}^2\rho_{13}^2 + 270\rho_{12}^2\rho_{23}^2 + 720\rho_{12}^3\rho_{13}\rho_{23} - 6\rho_{12}^6 + 180\rho_{12}^4\rho_{13}^2 \\
&- 90\rho_{12}^4\rho_{23}^2 - 72\rho_{12}^5\rho_{13}\rho_{23} + 18\rho_{12}^6\rho_{23}^2)\,\sigma_1^6\sigma_2^3\sigma_3^2
\end{aligned}
$$

$$
\begin{aligned}
(5, 5, 1) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}} &\Big[\sqrt{R}\,(64 + 607\rho_{12}^2 + 72\rho_{13}^2 + 72\rho_{23}^2 + 576\rho_{12}\rho_{13}\rho_{23} \\
&+ 274\rho_{12}^4 + 333\rho_{12}^2\rho_{13}^2 + 333\rho_{12}^2\rho_{23}^2 - 16\rho_{13}^4 - 207\rho_{13}^2\rho_{23}^2 - 16\rho_{23}^4 \\
&+ 444\rho_{12}^3\rho_{13}\rho_{23} - 296\rho_{12}\rho_{13}^3\rho_{23} - 296\rho_{12}\rho_{13}\rho_{23}^3 - 489\rho_{12}^2\rho_{13}^2\rho_{23}^2 \\
&+ 115\rho_{13}^4\rho_{23}^2 + 115\rho_{13}^2\rho_{23}^4 + 340\rho_{12}\rho_{13}^3\rho_{23}^3 - 105\rho_{13}^4\rho_{23}^4) + (225\rho_{12} \\
&+ 225\rho_{13}\rho_{23} + 600\rho_{12}^3 + 450\rho_{12}\rho_{13}^2 + 450\rho_{12}\rho_{23}^2 + 1800\rho_{12}^2\rho_{13}\rho_{23} \\
&- 150\rho_{13}^3\rho_{23} - 150\rho_{13}\rho_{23}^3 + 120\rho_{12}^5 + 600\rho_{12}^3\rho_{13}^2 + 600\rho_{12}^3\rho_{23}^2 \\
&- 75\rho_{12}\rho_{13}^4 - 900\rho_{12}\rho_{13}^2\rho_{23}^2 - 75\rho_{12}\rho_{23}^4 + 600\rho_{12}^4\rho_{13}\rho_{23} - 600\rho_{12}^2\rho_{13}^3\rho_{23} \\
&- 600\rho_{12}^2\rho_{13}\rho_{23}^3 + 45\rho_{13}^5\rho_{23} + 300\rho_{13}^3\rho_{23}^3 + 45\rho_{13}\rho_{23}^5 - 600\rho_{12}^3\rho_{13}^2\rho_{23}^2 \\
&+ 450\rho_{12}\rho_{13}^4\rho_{23}^2 + 450\rho_{12}\rho_{13}^2\rho_{23}^4 + 600\rho_{12}^2\rho_{13}^3\rho_{23}^3 - 150\rho_{13}^5\rho_{23}^3 \\
&- 150\rho_{13}^3\rho_{23}^5 - 375\rho_{12}\rho_{13}^4\rho_{23}^4 + 105\rho_{13}^5\rho_{23}^5)\sin^{-1}\rho_{12\cdot 3}
\end{aligned}
$$

$$+ (120\rho_{13} + 600\rho_{12}\rho_{23} + 1200\rho_{12}^2\rho_{13} + 1200\rho_{12}^3\rho_{23} + 600\rho_{12}^4\rho_{13}$$
$$+ 120\rho_{12}^5\rho_{23}) \sin^{-1}\rho_{13\cdot 2} + (120\rho_{23} + 600\rho_{12}\rho_{13} + 1200\rho_{12}^2\rho_{23}$$
$$+ 1200\rho_{12}^3\rho_{13} + 600\rho_{12}^4\rho_{23} + 120\rho_{12}^5\rho_{13}) \sin^{-1}\rho_{23\cdot 1}\Big] \sigma_1^5\sigma_2^5\sigma_3$$

$$(5, 4, 2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (24 + 240\rho_{12}^2 + 120\rho_{13}^2 + 96\rho_{23}^2 + 960\rho_{12}\rho_{13}\rho_{23} + 120\rho_{12}^4$$
$$+ 720\rho_{12}^2\rho_{13}^2 + 480\rho_{12}^2\rho_{23}^2 + 960\rho_{12}^3\rho_{13}\rho_{23} + 120\rho_{12}^4\rho_{13}^2)\, \sigma_1^5\sigma_2^4\sigma_3^2$$

$$(5, 3, 3) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \Big[\sqrt{R}\, (32 + 166\rho_{12}^2 + 166\rho_{13}^2 + 88\rho_{23}^2 + 760\rho_{12}\rho_{13}\rho_{23}$$
$$+ 12\rho_{12}^4 + 444\rho_{12}^2\rho_{13}^2 + 54\rho_{12}^2\rho_{23}^2 + 12\rho_{13}^4 + 54\rho_{13}^2\rho_{23}^2 + 84\rho_{12}^3\rho_{13}\rho_{23}$$
$$+ 84\rho_{12}\rho_{13}^3\rho_{23} - 18\rho_{12}^4\rho_{23}^2 - 18\rho_{13}^4\rho_{23}^2) + (90\rho_{12} + 270\rho_{13}\rho_{23}$$
$$+ 120\rho_{12}^3 + 540\rho_{12}\rho_{13}^2 + 270\rho_{12}\rho_{23}^2 + 1080\rho_{12}^2\rho_{13}\rho_{23} + 180\rho_{13}^3\rho_{23}$$
$$+ 90\rho_{13}\rho_{23}^3 + 360\rho_{12}^3\rho_{13}^2 + 90\rho_{12}\rho_{13}^4 + 540\rho_{12}\rho_{13}^2\rho_{23}^2 + 360\rho_{12}^2\rho_{13}^3\rho_{23}$$
$$- 18\rho_{13}^5\rho_{23} - 60\rho_{13}^3\rho_{23}^3 - 90\rho_{12}\rho_{13}^4\rho_{23}^2 + 18\rho_{13}^5\rho_{23}^3) \sin^{-1}\rho_{12\cdot 3}$$
$$+ (90\rho_{13} + 270\rho_{12}\rho_{23} + 540\rho_{12}^2\rho_{13} + 120\rho_{13}^3 + 270\rho_{13}\rho_{23}^2$$
$$+ 180\rho_{12}^3\rho_{23} + 1080\rho_{12}\rho_{13}^2\rho_{23} + 90\rho_{12}\rho_{23}^3 + 90\rho_{12}^4\rho_{13} + 360\rho_{12}^2\rho_{13}^3$$
$$+ 540\rho_{12}^2\rho_{13}\rho_{23}^2 - 18\rho_{12}^5\rho_{23} + 360\rho_{12}^3\rho_{13}^2\rho_{23} - 60\rho_{12}^3\rho_{23}^3 - 90\rho_{12}^4\rho_{13}\rho_{23}^2$$
$$+ 18\rho_{12}^5\rho_{23}^3) \sin^{-1}\rho_{13\cdot 2} + (72\rho_{23} + 360\rho_{12}\rho_{13} + 360\rho_{12}^2\rho_{23} + 360\rho_{13}^2\rho_{23}$$
$$+ 48\rho_{23}^3 + 360\rho_{12}^3\rho_{13} + 360\rho_{12}\rho_{13}^3 + 720\rho_{12}\rho_{13}\rho_{23}^2 + 1080\rho_{12}^2\rho_{13}^2\rho_{23}$$
$$+ 120\rho_{12}^3\rho_{13}^3) \sin^{-1}\rho_{23\cdot 1}\Big] \sigma_1^5\sigma_2^3\sigma_3^3$$

$$(4, 4, 3) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (18 + 144\rho_{12}^2 + 108\rho_{13}^2 + 108\rho_{23}^2 + 864\rho_{12}\rho_{13}\rho_{23} + 48\rho_{12}^4$$
$$+ 432\rho_{12}^2\rho_{13}^2 + 432\rho_{12}^2\rho_{23}^2 + 18\rho_{13}^4 + 216\rho_{13}^2\rho_{23}^2 + 18\rho_{23}^4$$
$$+ 576\rho_{12}^3\rho_{13}\rho_{23} + 288\rho_{12}\rho_{13}^3\rho_{23} + 288\rho_{12}\rho_{13}\rho_{23}^3 + 432\rho_{12}^2\rho_{13}^2\rho_{23}^2$$
$$- 36\rho_{13}^4\rho_{23}^2 - 36\rho_{13}^2\rho_{23}^4 - 96\rho_{12}\rho_{13}^3\rho_{23}^3 + 18\rho_{13}^4\rho_{23}^4)\, \sigma_1^4\sigma_2^4\sigma_3^3$$

$$(10, 1, 1) = \frac{2}{\pi} \Big[\sqrt{1 - \rho_{23}^2} \Big\{ -1290 + 11652\rho_{12}^2 + 11652\rho_{13}^2 + 2235\rho_{23}^2$$
$$- 7554\rho_{12}\rho_{13}\rho_{23} - 11610\rho_{12}^4 + 2604\rho_{12}^2\rho_{13}^2 - 2052\rho_{12}^2\rho_{23}^2 - 11610\rho_{13}^4$$
$$- 2052\rho_{13}^2\rho_{23}^2 + 2760\rho_{12}^3\rho_{13}\rho_{23} + 2760\rho_{12}\rho_{13}^3\rho_{23} - 2196\rho_{12}\rho_{13}\rho_{23}^3$$
$$+ 7248\rho_{12}^6 + 5040\rho_{12}^4\rho_{13}^2 + 984\rho_{12}^4\rho_{23}^2 + 5040\rho_{12}^2\rho_{13}^4$$
$$+ 10440\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 7248\rho_{13}^6 + 984\rho_{13}^4\rho_{23}^2 - 1008\rho_{12}^5\rho_{13}\rho_{23}$$
$$- 9696\rho_{12}^3\rho_{13}^3\rho_{23} + 1344\rho_{12}^3\rho_{13}\rho_{23}^3 - 1008\rho_{12}\rho_{13}^5\rho_{23} + 1344\rho_{12}\rho_{13}^3\rho_{23}^3$$

$$- 2544\rho_{12}^{8} - 2400\rho_{12}^{6}\rho_{13}^{2} - 192\rho_{12}^{6}\rho_{23}^{2} - 5472\rho_{12}^{4}\rho_{13}^{4}$$

$$- 5760\rho_{12}^{4}\rho_{13}^{2}\rho_{23}^{2} - 2400\rho_{12}^{2}\rho_{13}^{6} - 5760\rho_{12}^{2}\rho_{13}^{4}\rho_{23}^{2} - 2544\rho_{13}^{8}$$

$$- 192\rho_{13}^{6}\rho_{23}^{2} + 192\rho_{12}^{7}\rho_{13}\rho_{23} + 5568\rho_{12}^{5}\rho_{13}^{3}\rho_{23} - 288\rho_{12}^{5}\rho_{13}\rho_{23}^{3}$$

$$+ 5568\rho_{12}^{3}\rho_{13}^{5}\rho_{23} - 384\rho_{12}^{3}\rho_{13}^{3}\rho_{23}^{3} + 192\rho_{12}\rho_{13}^{7}\rho_{23} - 288\rho_{12}\rho_{13}^{5}\rho_{23}^{3}$$

$$+ 384\rho_{12}^{10} + 384\rho_{12}^{8}\rho_{13}^{2} + 1152\rho_{12}^{6}\rho_{13}^{4} + 1152\rho_{12}^{6}\rho_{13}^{2}\rho_{23}^{2}$$

$$+ 1152\rho_{12}^{4}\rho_{13}^{6} + 1536\rho_{12}^{4}\rho_{13}^{4}\rho_{23}^{2} + 384\rho_{12}^{2}\rho_{13}^{8} + 1152\rho_{12}^{2}\rho_{13}^{6}\rho_{23}^{2}$$

$$+ 384\rho_{13}^{10} - 1152\rho_{12}^{7}\rho_{13}^{3}\rho_{23} - 1536\rho_{12}^{5}\rho_{13}^{5}\rho_{23} - 1152\rho_{12}^{3}\rho_{13}^{7}\rho_{23}$$

$$+ \frac{R^{2}}{1 - \rho_{23}^{2}}(1920 - 1854\rho_{12}^{2} - 1854\rho_{13}^{2} - 2052\rho_{12}\rho_{13}\rho_{23} + 936\rho_{12}^{4}$$

$$+ 1296\rho_{12}^{2}\rho_{13}^{2} + 936\rho_{13}^{4} + 1296\rho_{12}^{3}\rho_{13}\rho_{23} + 1296\rho_{12}\rho_{13}^{3}\rho_{23} - 192\rho_{12}^{6}$$

$$- 288\rho_{12}^{4}\rho_{13}^{2} - 288\rho_{12}^{2}\rho_{13}^{4} - 192\rho_{13}^{6} - 288\rho_{12}^{5}\rho_{13}\rho_{23} - 384\rho_{12}^{3}\rho_{13}^{3}\rho_{23}$$

$$- 288\rho_{12}\rho_{13}^{5}\rho_{23}) + \frac{R^{3}}{(1 - \rho_{23}^{2})^{2}}(240 - 174\rho_{12}^{2} - 174\rho_{13}^{2}$$

$$- 132\rho_{12}\rho_{13}\rho_{23} + 48\rho_{12}^{4} + 48\rho_{12}^{2}\rho_{13}^{2} + 48\rho_{13}^{4} + 48\rho_{12}^{3}\rho_{13}\rho_{23}$$

$$+ 48\rho_{12}\rho_{13}^{3}\rho_{23}) + \frac{R^{4}}{(1 - \rho_{23}^{2})^{3}}(60 - 24\rho_{12}^{2} - 24\rho_{13}^{2} - 12\rho_{12}\rho_{13}\rho_{23})$$

$$+ 15\frac{R^{5}}{(1 - \rho_{23}^{2})^{4}}\Big\} + (945\rho_{23} + 9450\rho_{12}\rho_{13})\sin^{-1}\rho_{23}\Big]\sigma_1^{10}\sigma_2\sigma_3$$

$$(9, 2, 1) = \frac{2}{\pi}\Big[\sqrt{1 - \rho_{13}^{2}}(384 + 3456\rho_{12}^{2} + 975\rho_{13}^{2} + 384\rho_{23}^{2} + 5022\rho_{12}\rho_{13}\rho_{23}$$

$$+ 6264\rho_{12}^{2}\rho_{13}^{2} - 630\rho_{13}^{4} - 1536\rho_{13}^{2}\rho_{23}^{2} - 5868\rho_{12}\rho_{13}^{3}\rho_{23} - 2736\rho_{12}^{2}\rho_{13}^{4}$$

$$+ 264\rho_{13}^{6} + 2304\rho_{13}^{4}\rho_{23}^{2} + 3600\rho_{12}\rho_{13}^{5}\rho_{23} + 576\rho_{12}^{2}\rho_{13}^{6} - 48\rho_{13}^{8}$$

$$- 1536\rho_{13}^{6}\rho_{23}^{2} - 864\rho_{12}\rho_{13}^{7}\rho_{23} + 384\rho_{13}^{8}\rho_{23}^{2}) + (945\rho_{13} + 1890\rho_{12}\rho_{23}$$

$$+ 7560\rho_{12}^{2}\rho_{13})\sin^{-1}\rho_{13}\Big]\sigma_1^{9}\sigma_2^{2}\sigma_3$$

$$(8, 3, 1) = \frac{2}{\pi}\Big[\sqrt{1 - \rho_{23}^{2}}\Big\{360 + 2040\rho_{12}^{2} + 312\rho_{13}^{2} - 195\rho_{23}^{2} + 3384\rho_{12}\rho_{13}\rho_{23}$$

$$+ 1800\rho_{12}^{4} + 8784\rho_{12}^{2}\rho_{13}^{2} + 504\rho_{12}^{2}\rho_{23}^{2} + 264\rho_{13}^{4} - 240\rho_{13}^{2}\rho_{23}^{2}$$

$$+ 150\rho_{23}^{4} - 1872\rho_{12}^{3}\rho_{13}\rho_{23} - 7200\rho_{12}\rho_{13}^{3}\rho_{23} - 816\rho_{12}\rho_{13}\rho_{23}^{3} - 408\rho_{12}^{6}$$

$$+ 1656\rho_{12}^{4}\rho_{13}^{2} - 72\rho_{12}^{4}\rho_{23}^{2} - 3528\rho_{12}^{2}\rho_{13}^{4} + 1296\rho_{12}^{2}\rho_{13}^{2}\rho_{23}^{2}$$

$$- 24\rho_{12}^{2}\rho_{23}^{4} - 216\rho_{13}^{6} + 1464\rho_{13}^{4}\rho_{23}^{2} - 72\rho_{13}^{2}\rho_{23}^{4} + 144\rho_{12}^{5}\rho_{13}\rho_{23}$$

$$- 480\rho_{12}^{3}\rho_{13}^{3}\rho_{23} + 4752\rho_{12}\rho_{13}^{5}\rho_{23} + 288\rho_{12}\rho_{13}^{3}\rho_{23}^{3} - 48\rho_{12}\rho_{13}\rho_{23}^{5}$$

$$+ 48\rho_{12}^{8} - 96\rho_{12}^{4}\rho_{13}^{4} + 288\rho_{12}^{4}\rho_{13}^{2}\rho_{23}^{2} + 768\rho_{12}^{2}\rho_{13}^{6} - 288\rho_{12}^{2}\rho_{13}^{4}\rho_{23}^{2}$$

$$+ 288\rho_{12}^{2}\rho_{13}^{2}\rho_{23}^{4} + 48\rho_{13}^{8} - 1344\rho_{13}^{6}\rho_{23}^{2} - 384\rho_{12}^{5}\rho_{13}^{3}\rho_{23}$$

$$- 576\rho_{12}^{3}\rho_{13}^{3}\rho_{23}^{3} - 1152\rho_{12}\rho_{13}^{7}\rho_{23} + 384\rho_{12}^{4}\rho_{13}^{4}\rho_{23}^{2} + 384\rho_{13}^{8}\rho_{23}^{2}$$

$$+ \frac{R^3}{1-\rho_{23}^2}(-144 + 24\rho_{12}^2 + 72\rho_{13}^2 + 48\rho_{12}\rho_{13}\rho_{23})$$

$$- 6\frac{R^4}{(1-\rho_{23}^2)^2}\Big\} + (315\rho_{23} + 2520\rho_{12}\rho_{13} + 2520\rho_{12}^2\rho_{23}$$

$$+ 5040\rho_{12}^3\rho_{13}) \sin^{-1}\rho_{23}\Big]\sigma_1^8\sigma_2^3\sigma_3$$

$$(8,2,2) = (105 + 840\rho_{12}^2 + 840\rho_{13}^2 + 210\rho_{23}^2 + 3360\rho_{12}\rho_{13}\rho_{23}$$

$$+ 5040\rho_{12}^2\rho_{13}^2)\sigma_1^8\sigma_2^2\sigma_3^2$$

$$(7,4,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{13}^2}(144 + 2016\rho_{12}^2 + 261\rho_{13}^2 + 288\rho_{23}^2 + 2772\rho_{12}\rho_{13}\rho_{23}$$

$$+ 1680\rho_{12}^4 + 2268\rho_{12}^2\rho_{13}^2 + 2016\rho_{12}^2\rho_{23}^2 - 114\rho_{13}^4 - 864\rho_{13}^2\rho_{23}^2$$

$$- 48\rho_{23}^4 + 4200\rho_{12}^3\rho_{13}\rho_{23} - 2184\rho_{12}\rho_{13}^3\rho_{23} - 1344\rho_{12}\rho_{13}\rho_{23}^3$$

$$+ 840\rho_{12}^4\rho_{13}^2 - 504\rho_{12}^2\rho_{13}^4 - 4032\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 24\rho_{13}^6 + 864\rho_{13}^4\rho_{23}^2$$

$$+ 480\rho_{13}^2\rho_{23}^4 - 1680\rho_{12}^3\rho_{13}^3\rho_{23} + 672\rho_{12}\rho_{13}^5\rho_{23} + 2688\rho_{12}\rho_{13}^3\rho_{23}^3$$

$$+ 2016\rho_{12}^2\rho_{13}^4\rho_{23}^2 - 288\rho_{13}^6\rho_{23}^2 - 816\rho_{13}^4\rho_{23}^4 - 1344\rho_{12}\rho_{13}^5\rho_{23}^3$$

$$+ 384\rho_{13}^6\rho_{23}^4) + (315\rho_{13} + 1260\rho_{12}\rho_{23} + 3780\rho_{12}^2\rho_{13} + 2520\rho_{12}^3\rho_{23}$$

$$+ 2520\rho_{12}^4\rho_{13}) \sin^{-1}\rho_{13}\Big]\sigma_1^7\sigma_2^4\sigma_3$$

$$(7,3,2) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{12}^2}(96 + 741\rho_{12}^2 + 672\rho_{13}^2 + 288\rho_{23}^2 + 3402\rho_{12}\rho_{13}\rho_{23}$$

$$+ 120\rho_{12}^4 + 3486\rho_{12}^2\rho_{13}^2 + 522\rho_{12}^2\rho_{23}^2 + 1176\rho_{12}^3\rho_{13}\rho_{23} - 12\rho_{12}^6$$

$$+ 252\rho_{12}^4\rho_{13}^2 - 228\rho_{12}^4\rho_{23}^2 - 168\rho_{12}^5\rho_{13}\rho_{23} + 48\rho_{12}^6\rho_{23}^2) + (315\rho_{12}$$

$$+ 630\rho_{13}\rho_{23} + 630\rho_{12}^3 + 1890\rho_{12}\rho_{13}^2 + 630\rho_{12}\rho_{23}^2 + 3780\rho_{12}^2\rho_{13}\rho_{23}$$

$$+ 2520\rho_{12}^3\rho_{13}^2) \sin^{-1}\rho_{12}\Big]\sigma_1^7\sigma_2^3\sigma_3^2$$

$$(6,5,1) = \frac{2}{\pi}\Big[\sqrt{1-\rho_{23}^2}(120 + 1800\rho_{12}^2 + 360\rho_{13}^2 + 135\rho_{23}^2 + 2250\rho_{12}\rho_{13}\rho_{23}$$

$$+ 1800\rho_{12}^4 + 3600\rho_{12}^2\rho_{13}^2 + 900\rho_{12}^2\rho_{23}^2 - 120\rho_{13}^4 - 720\rho_{13}^2\rho_{23}^2$$

$$- 30\rho_{23}^4 + 3600\rho_{12}^3\rho_{13}\rho_{23} - 2400\rho_{12}\rho_{13}^3\rho_{23} - 900\rho_{12}\rho_{13}\rho_{23}^3 + 120\rho_{12}^6$$

$$+ 1800\rho_{12}^4\rho_{13}^2 - 600\rho_{12}^2\rho_{13}^4 - 3600\rho_{12}^2\rho_{13}^2\rho_{23}^2 + 24\rho_{13}^6 + 840\rho_{13}^4\rho_{23}^2$$

$$+ 360\rho_{13}^2\rho_{23}^4 - 2400\rho_{12}^3\rho_{13}^3\rho_{23} + 720\rho_{12}\rho_{13}^5\rho_{23} + 2400\rho_{12}\rho_{13}^3\rho_{23}^3$$

$$+ 2400\rho_{12}^2\rho_{13}^4\rho_{23}^2 - 288\rho_{13}^6\rho_{23}^2 - 720\rho_{13}^4\rho_{23}^4 - 1440\rho_{12}\rho_{13}^5\rho_{23}^3$$

$$+ 384\rho_{13}^6\rho_{23}^4) + (225\rho_{23} + 1350\rho_{12}\rho_{13} + 2700\rho_{12}^2\rho_{23} + 3600\rho_{12}^3\rho_{13}$$

$$+ 1800\rho_{12}^4\rho_{23} + 720\rho_{12}^5\rho_{13}) \sin^{-1}\rho_{23}\Big]\sigma_1^6\sigma_2^5\sigma_3$$

$$(6,4,2) = (45 + 540\rho_{12}^2 + 270\rho_{13}^2 + 180\rho_{23}^2 + 2160\rho_{12}\rho_{13}\rho_{23} + 360\rho_{12}^4 + 2160\rho_{12}^2\rho_{13}^2 + 1080\rho_{12}^2\rho_{23}^2 + 2880\rho_{12}^3\rho_{13}\rho_{23} + 720\rho_{12}^4\rho_{13}^2)\,\sigma_1^6\sigma_2^4\sigma_3^2$$

$$\begin{aligned}(6,3,3) = \frac{2}{\pi}\Big[&\sqrt{1-\rho_{23}^2}\,(60 + 540\rho_{12}^2 + 540\rho_{13}^2 + 165\rho_{23}^2 + 2430\rho_{12}\rho_{13}\rho_{23} \\ &+ 180\rho_{12}^4 + 3240\rho_{12}^2\rho_{13}^2 + 270\rho_{12}^2\rho_{23}^2 + 180\rho_{13}^4 + 270\rho_{13}^2\rho_{23}^2 \\ &+ 1080\rho_{12}^3\rho_{13}\rho_{23} + 1080\rho_{12}\rho_{13}^3\rho_{23} - 12\rho_{12}^6 + 540\rho_{12}^4\rho_{13}^2 - 180\rho_{12}^4\rho_{23}^2 \\ &+ 540\rho_{12}^2\rho_{13}^4 - 180\rho_{13}^4\rho_{23}^2 - 216\rho_{12}^5\rho_{13}\rho_{23} - 216\rho_{12}\rho_{13}^5\rho_{23} + 48\rho_{12}^6\rho_{23}^2 \\ &+ 48\rho_{13}^6\rho_{23}^2) + (135\rho_{23} + 810\rho_{12}\rho_{13} + 810\rho_{12}^2\rho_{23} + 810\rho_{13}^2\rho_{23} \\ &+ 90\rho_{23}^3 + 1080\rho_{12}^3\rho_{13} + 1080\rho_{12}\rho_{13}^3 + 1620\rho_{12}\rho_{13}\rho_{23}^2 + 3240\rho_{12}^2\rho_{13}^2\rho_{23} \\ &+ 720\rho_{12}^3\rho_{13}^3)\sin^{-1}\rho_{23}\Big]\sigma_1^6\sigma_2^3\sigma_3^3\end{aligned}$$

$$\begin{aligned}(5,5,2) = \frac{2}{\pi}\Big[&\sqrt{1-\rho_{12}^2}\,(64 + 607\rho_{12}^2 + 320\rho_{13}^2 + 320\rho_{23}^2 + 2750\rho_{12}\rho_{13}\rho_{23} \\ &+ 274\rho_{12}^4 + 1660\rho_{12}^2\rho_{13}^2 + 1660\rho_{12}^2\rho_{23}^2 + 2500\rho_{12}^3\rho_{13}\rho_{23} + 120\rho_{12}^4\rho_{13}^2 \\ &+ 120\rho_{12}^4\rho_{23}^2) + (225\rho_{12} + 450\rho_{13}\rho_{23} + 600\rho_{12}^3 + 900\rho_{12}\rho_{13}^2 \\ &+ 900\rho_{12}\rho_{23}^2 + 3600\rho_{12}^2\rho_{13}\rho_{23} + 120\rho_{12}^5 + 1200\rho_{12}^3\rho_{13}^2 + 1200\rho_{12}^3\rho_{23}^2 \\ &+ 1200\rho_{12}^4\rho_{13}\rho_{23})\sin^{-1}\rho_{12}\Big]\sigma_1^5\sigma_2^5\sigma_3^2\end{aligned}$$

$$\begin{aligned}(5,4,3) = \frac{2}{\pi}\Big[&\sqrt{1-\rho_{13}^2}\,(48 + 480\rho_{12}^2 + 249\rho_{13}^2 + 288\rho_{23}^2 + 2340\rho_{12}\rho_{13}\rho_{23} \\ &+ 240\rho_{12}^4 + 1320\rho_{12}^2\rho_{13}^2 + 1440\rho_{12}^2\rho_{23}^2 + 18\rho_{13}^4 + 324\rho_{13}^2\rho_{23}^2 \\ &+ 48\rho_{23}^4 + 2160\rho_{12}^3\rho_{13}\rho_{23} + 360\rho_{12}\rho_{13}^3\rho_{23} + 600\rho_{12}\rho_{13}\rho_{23}^3 + 120\rho_{12}^4\rho_{13}^2 \\ &+ 720\rho_{12}^2\rho_{13}^2\rho_{23}^2 - 72\rho_{13}^4\rho_{23}^2 - 96\rho_{13}^2\rho_{23}^4 - 240\rho_{12}\rho_{13}^3\rho_{23}^3 + 48\rho_{13}^4\rho_{23}^4) \\ &+ (135\rho_{13} + 540\rho_{12}\rho_{23} + 1080\rho_{12}^2\rho_{13} + 180\rho_{13}^3 + 540\rho_{13}\rho_{23}^2 \\ &+ 720\rho_{12}^3\rho_{23} + 2160\rho_{12}\rho_{13}^2\rho_{23} + 360\rho_{12}\rho_{23}^3 + 360\rho_{12}^4\rho_{13} + 720\rho_{12}^2\rho_{13}^3 \\ &+ 2160\rho_{12}^2\rho_{13}\rho_{23}^2 + 1440\rho_{12}^3\rho_{13}^2\rho_{23})\sin^{-1}\rho_{13}\Big]\sigma_1^5\sigma_2^4\sigma_3^3\end{aligned}$$

$$\begin{aligned}(4,4,4) = (&27 + 216\rho_{12}^2 + 216\rho_{13}^2 + 216\rho_{23}^2 + 1728\rho_{12}\rho_{13}\rho_{23} + 72\rho_{12}^4 \\ &+ 864\rho_{12}^2\rho_{13}^2 + 864\rho_{12}^2\rho_{23}^2 + 72\rho_{13}^4 + 864\rho_{13}^2\rho_{23}^2 + 72\rho_{23}^4 \\ &+ 1152\rho_{12}^3\rho_{13}\rho_{23} + 1152\rho_{12}\rho_{13}^3\rho_{23} + 1152\rho_{12}\rho_{13}\rho_{23}^3 \\ &+ 1728\rho_{12}^2\rho_{13}^2\rho_{23}^2)\,\sigma_1^4\sigma_2^4\sigma_3^4\end{aligned}$$

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REFERENCE

S. NABEYA (1951): Absolute moments in 2-dimensional normal distribution. Annals of the Institute of Statistical Mathematics, Vol. III, No. 1.

EDITORIAL NOTE

This article consists of the results of calculations for the most part. We publish it here in expectation that it will be of some use.