Absolute Moments in 2-dimensional Normal Distribution

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Let x and y be distributed according to the following 2-dimensional normal distribution,

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right\} dx\,dy.$$

It is our purpose to express absolute moments in terms of elementary functions. Putting $E(|x^m y^n|) = (m, n)$ for simplicity, we have

$$\begin{aligned}(m,n) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x^m y^n| \exp\left\{-\frac{1}{2(1-\rho^2)}\right. \\ &\qquad\qquad \times \left.\left(\frac{x^2}{\sigma_1^2} - 2\rho\frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right\} dx\,dy \\ &= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}(1-\rho^2)^{\frac{m+n+1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x^m y^n| \exp(-x^2+2\rho xy - y^2)\,dx\,dy \\ &= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}(1-\rho^2)^{\frac{m+n+1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x^m y^n| e^{-x^2-y^2}\sum_{k=0}^{\infty}\frac{(2\rho xy)^k}{k\,!}\,dx\,dy \\ &= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}(1-\rho^2)^{\frac{m+n+1}{2}} \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{m+1}{2}+k\right)\Gamma\left(\frac{n+1}{2}+k\right)}{(2k)\,!}(2\rho)^{2k} \\ &= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)(1-\rho^2)^{\frac{m+n+1}{2}} \\ &\qquad\qquad \times F\left(\frac{m+1}{2}, \frac{n+1}{2}; \frac{1}{2}; \rho^2\right) \\ &= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \rho^2\right).\end{aligned}$$

Here

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{1!\,\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)}z^2 + \cdots\cdots$$

is the hypergeometric function, which reduces to the polynomial of z if α or β is a non-positive integer and γ is positive. Thus, when at least one of the integers m, n is an even number, (m, n) reduces to the polynomial of ρ^2 multiplied by $\sigma_1^m\sigma_2^n$.

The case where both m and n are odd may be treated as follows. Put

$$x = \sqrt{2(1-\rho^2)}\,\sigma_1 r\cos\theta, \qquad y = \sqrt{2(1-\rho^2)}\,\sigma_2 r\sin\theta.$$

When $m - n = 2q$, where q is a non-negative integer, we have then

$$(m, n) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x^m y^n|\exp\left\{-\frac{1}{2(1-\rho^2)}\times\left(\frac{x^2}{\sigma_1^2} - 2\rho\frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right\}dx\,dy$$

$$= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}(1-\rho^2)^{\frac{m+n+1}{2}}\int_0^{2\pi}\int_0^{\infty} r^{m+n+1}|\cos^m\theta\sin^n\theta| \times\exp\left\{-r^2(1-2\rho\cos\theta\sin\theta)\right\}dr\,d\theta$$

$$= \frac{2^{\frac{m+n-2}{2}}\sigma_1^m\sigma_2^n}{\pi}\Gamma\left(\frac{m+n}{2}+1\right)(1-\rho^2)^{\frac{m+n+1}{2}} \times\int_0^{2\pi}\frac{|\cos^m\theta\sin^n\theta|}{(1-2\rho\cos\theta\sin\theta)^{\frac{m+n+2}{2}}}d\theta$$

$$= \frac{2^{\frac{m+n}{2}}\sigma_1^m\sigma_2^n}{\pi}\Gamma\left(\frac{m+n}{2}+1\right)(1-\rho^2)^{\frac{m+n+1}{2}} \times\int_0^{\frac{\pi}{2}}\left\{\frac{\cos^m\theta\sin^n\theta}{(1-2\rho\cos\theta\sin\theta)^{\frac{m+n+2}{2}}} + \frac{\cos^m\theta\sin^n\theta}{(1+2\rho\cos\theta\sin\theta)^{\frac{m+n+2}{2}}}\right\}d\theta$$

$$= \frac{2^{\frac{m-n}{2}}\sigma_1^m\sigma_2^n}{\pi}\Gamma\left(\frac{m-n}{2}+1\right)(1-\rho^2)^{\frac{m+n+1}{2}} \times\frac{d^n}{d\rho^n}\int_0^{\frac{\pi}{2}}\left\{\frac{\cos^{2q}\theta}{(1-2\rho\cos\theta\sin\theta)^{q+1}} - \frac{\cos^{2q}\theta}{(1+2\rho\cos\theta\sin\theta)^{q+1}}\right\}d\theta.$$

As the last integral may be calculated in the elementary fashion, (m, n) may be evaluated.

In the following we shall give the obatined formulae for the cases $m \geqq n$. The formula of (m, n) for $m \leqq n$, is obtained by exchanging σ_1 and σ_2 in the formula (n, m).

$$(1, 0) = \sqrt{\frac{2}{\pi}}\,\sigma_1,$$

$$(2, 0) = \sigma_1^2,$$

$$(1, 1) = \frac{2}{\pi}\left(\sqrt{1-\rho^2} + \rho\sin^{-1}\rho\right)\sigma_1\sigma_2,$$

$$(3, 0) = 2\sqrt{\frac{2}{\pi}}\,\sigma_1^3,$$

$$(2, 1) = \sqrt{\frac{2}{\pi}}\,(1 + \rho^2)\,\sigma_1^2\sigma_2,$$

$$(4, 0) = 3\sigma_1^4,$$

$$(3, 1) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(2 + \rho^2) + 3\rho\sin^{-1}\rho\right\}\sigma_1^3\sigma_2.$$

$$(2, 2) = (1 + 2\rho^2)\,\sigma_1^2\sigma_2^2,$$

$$(5, 0) = 8\sqrt{\frac{2}{\pi}}\,\sigma_1^5,$$

$$(4, 1) = \sqrt{\frac{2}{\pi}}\,(3 + 6\rho^2 - \rho^4)\,\sigma_1^4\sigma_2,$$

$$(3, 2) = 2\sqrt{\frac{2}{\pi}}\,(1 + 3\rho^2)\,\sigma_1^3\sigma_2^2,$$

$$(6, 0) = 15\sigma_1^6,$$

$$(5, 1) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(8 + 9\rho^2 - 2\rho^4) + 15\rho\sin^{-1}\rho\right\}\sigma_1^5\sigma_2,$$

$$(4, 2) = 3(1 + 4\rho^2)\sigma_1^4\sigma_2^2,$$

$$(3, 3) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(4 + 11\rho^2) + 3\rho(3 + 2\rho^2)\sin^{-1}\rho\right\}\sigma_1^3\sigma_2^3,$$

$$(7, 0) = 48\sqrt{\frac{2}{\pi}}\,\sigma_1^7,$$

$$(6, 1) = 3\sqrt{\frac{2}{\pi}}\,(5 + 15\rho^2 - 5\rho^4 + \rho^6)\,\sigma_1^6\sigma_2,$$

$$(5, 2) = 8\sqrt{\frac{2}{\pi}}\,(1 + 5\rho^2)\,\sigma_1^5\sigma_2^2,$$

$$(4, 3) = 6\sqrt{\frac{2}{\pi}}\,(1 + 6\rho^2 + \rho^4)\,\sigma_1^4\sigma_2^3,$$

$$(8, 0) = 105\sigma_1^8,$$

$$(7, 1) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(48 + 87\rho^2 - 38\rho^4 + 8\rho^6) + 105\rho\sin^{-1}\rho\right\}\sigma_1^7\sigma_2,$$

$$(6, 2) = 15(1 + 6\rho^2)\,\sigma_1^6\sigma_2^2,$$

$$(5, 3) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(16 + 83\rho^2 + 6\rho^4) + 15\rho(3 + 4\rho^2)\sin^{-1}\rho\right\}\sigma_1^5\sigma_2^3,$$

$$(4, 4) = 3(3 + 24\rho^2 + 8\rho^4)\,\sigma_1^4\sigma_2^4,$$

$$(9,0) = 384\sqrt{\frac{2}{\pi}}\,\sigma_1^9,$$

$$(8,1) = 3\sqrt{\frac{2}{\pi}}\,(35 + 140\rho^2 - 70\rho^4 + 28\rho^6 - 5\rho^8)\,\sigma_1^8\sigma_2,$$

$$(7,2) = 48\sqrt{\frac{2}{\pi}}\,(1 + 7\rho^2)\,\sigma_1^7\sigma_2^2,$$

$$(6,3) = 6\sqrt{\frac{2}{\pi}}\,(5 + 45\rho^2 + 15\rho^4 - \rho^6)\,\sigma_1^6\sigma_2^3,$$

$$(5,4) = 24\sqrt{\frac{2}{\pi}}\,(1 + 10\rho^2 + 5\rho^4)\,\sigma_1^5\sigma_2^4,$$

$$(10,0) = 945\sigma_1^{10},$$

$$(9,1) = \frac{6}{\pi}\left\{\sqrt{1-\rho^2}\,(128 + 325\rho^2 - 210\rho^4 + 88\rho^6 - 16\rho^8) + 315\rho\sin^{-1}\rho\right\}\sigma_1^9\sigma_2,$$

$$(8,2) = 105(1 + 8\rho^2)\,\sigma_1^8\sigma_2^2,$$

$$(7,3) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(96 + 741\rho^2 + 120\rho^4 - 12\rho^6) + 315\rho(1 + 2\rho^2)\sin^{-1}\rho\right\}\sigma_1^7\sigma_2^3,$$

$$(6,4) = 45(1 + 12\rho^2 + 8\rho^4)\sigma_1^6\sigma_2^4,$$

$$(5,5) = \frac{2}{\pi}\left\{\sqrt{1-\rho^2}\,(64 + 607\rho^2 + 274\rho^4) + 15\rho(15 + 40\rho^2 + 8\rho^4)\sin^{-1}\rho\right\}\sigma_1^5\sigma_2^5,$$

$$(11,0) = 3840\sqrt{\frac{2}{\pi}}\,\sigma_1^{11},$$

$$(10,1) = 15\sqrt{\frac{2}{\pi}}\,(63 + 315\rho^2 - 210\rho^4 + 126\rho^6 - 45\rho^8 + 7\rho^{10})\sigma_1^{10}\sigma_2,$$

$$(9,2) = 384\sqrt{\frac{2}{\pi}}\,(1 + 9\rho^2)\sigma_1^9\sigma_2^2,$$

$$(8,3) = 6\sqrt{\frac{2}{\pi}}\,(35 + 420\rho^2 + 210\rho^4 - 28\rho^6 + 3\rho^8)\sigma_1^8\sigma_2^3,$$

$$(7,4) = 48\sqrt{\frac{2}{\pi}}\,(3 + 42\rho^2 + 35\rho^4)\sigma_1^7\sigma_2^4,$$

$$(6,5) = 120\sqrt{\frac{2}{\pi}}\,(1 + 15\rho^2 + 15\rho^4 + \rho^6)\sigma_1^6\sigma_2^5,$$

$$(12, 0) = 10395\sigma_1^{12},$$

$$(11, 1) = \frac{6}{\pi}\left\{\sqrt{1-\rho^2}\,(1280 + 4215\rho^2 - 3590\rho^4 + 2248\rho^6 - 816\rho^8 + 128\rho^{10}) + 3465\rho\sin^{-1}\rho\right\}\sigma_1^{11}\sigma_2,$$

$$(10, 2) = 945(1 + 10\rho^2)\sigma_1^{10}\sigma_2^2,$$

$$(9, 3) = \frac{6}{\pi}\left\{\sqrt{1-\rho^2}\,(256 + 2639\rho^2 + 690\rho^4 - 136\rho^6 + 16\rho^8) + 315\rho(3 + 8\rho^2)\sin^{-1}\rho\right\}\sigma_1^9\sigma_2^3,$$

$$(8, 4) = 315(1 + 16\rho^2 + 16\rho^4)\sigma_1^8\sigma_2^4,$$

$$(7, 5) = \frac{6}{\pi}\left\{\sqrt{1-\rho^2}\,(128 + 1779\rho^2 + 1518\rho^4 + 40\rho^6) + 105\rho(5 + 20\rho^2 + 8\rho^4)\sin^{-1}\rho\right\}\sigma_1^7\sigma_2^5,$$

$$(6, 6) = 45(5 + 90\rho^2 + 120\rho^4 + 16\rho^6)\sigma_1^6\sigma_2^6.$$

In another paper we shall treat the 3-dimensional case by a unified but more complicated method.

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