Note on the Independence of Certain Statistics

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Let $x_1, x_2, \cdots\cdots, x_n$ be a sample from an n-variate normal population with the probability density function:

$$(2\pi)^{-\frac{n}{2}} |R^{-1}|^{\frac{1}{2}} e^{-\frac{1}{2}\sum_{i,j=1}^{n} r_{ij}x_i x_j},$$

where R denotes the matrix $(r_{ij})^{-1}$, which we assume to be symmetric and positive-definite, and let $\varphi=\sum a_{ij}x_i x_j$ and $\psi=\sum b_{ij}x_i x_j$ be real Hermitian forms of $x_1, x_2 \cdots\cdots, x_n$. Then the following theorem holds, as is known[1], with respect to the independence of φ and ψ.[2]

Theorem. *A necessary and sufficient condition that the statistics* $\varphi=\sum_{i,j=1}^{n} a_{ij}x_i x_j$ *and* $\psi=\sum_{i,j=1}^{n} b_{ij}x_i x_j$ *be mutually independent, is that the product of the matrices* $A=(a_{ij})$, R *and* $B=(b_{ij})$ *is the zero-matrix. i. e.,* $ARB=0$.

As the known proofs of this theorem seem to me not simple enough, I shall give my proof in the following lines.[2]

Proof. As R is a positive-definite and symmetric matrix, there exist a proper-orthogonal matrix T and a real diagonal matrix P with non-negative components so that $T'RT=P^2$.[3] Then simple calculations give for the moment generating function $f(z)$ of a Hermitian form $\sum_{i,j=1}^{n} c_{ij}x_i x_j$ the expression $f(z)=|E-2z\tilde{C}|^{-\frac{1}{2}}$, where E and $\tilde{C}$ denote the unit matrix and the matrix $P'T'(c_{ij})\,TP$ respectively, which is also symmetric. Therefore the mutual independence of φ and ψ is equivalent to the equation:

$$|E-2(z_1\tilde{A}+z_2\tilde{B})|^{-\frac{1}{2}}=|E-2z_1\tilde{A}|^{-\frac{1}{2}}|E-2z_2\tilde{B}|^{-\frac{1}{2}},$$

where z_1, z_2 are mutually independent variables, and A and B represent the matrices $P'T'ATP$ and $P'T'BTP$ respectively. From this equation it follows at once, that $ARB=0$ is a sufficient condition. To prove that this condition is also necessity, assume that φ and ψ are mutually independent. Then we see that the equation

$$(*) \qquad |E-(\lambda\tilde{A}+\mu\tilde{B})|=|E-\lambda\tilde{A}||E-\mu\tilde{B}|$$

holds identically in λ and μ. Now, let $\alpha_1, \alpha_2, \cdots\cdots, \alpha_r$ and $\beta_1, \beta_2, \cdots\cdots, \beta_s$ be the non-vanishing eigenvalues of $\tilde{A}$ and $\tilde{B}$ respectively, and let U be an orthogonal matrix such that $U'\tilde{A}U$ has a diagonal form

$$\begin{pmatrix} \alpha_1 & & & & & \\ & \alpha_2 & & & O & \\ & & \ddots & & & \\ & & & \alpha_r & & \\ & & & & 0 & \\ & O & & & & \ddots \\ & & & & & & 0 \end{pmatrix}.$$

The left-hand side of (*) can then be rewritten

$$\begin{vmatrix} 1-\lambda\alpha_1-\mu b^{**}_{11} & -\mu b^{**}_{12} & \cdots\cdots & \cdots\cdots & -\mu b^{**}_{1n} \\ -\mu b^{**}_{21} & 1-\lambda\alpha_2-\mu b^{**}_{22} & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \ddots & 1-\lambda\alpha_r-\mu b^{**}_{rr} & \cdots\cdots & \cdots\cdots \\ & & & 1-\mu b^{**}_{r+1\ r+1} & \\ -\mu b^{**}_{n1} & \cdots\cdots & \cdots\cdots & \ddots & 1-\mu b^{**}_{nn} \end{vmatrix}$$

where (b^{**}_{ij}) denotes the matrix $U'\tilde{B}U$. This is a polynomial in λ, which has

$$\alpha_1\cdots\cdots\alpha_r \begin{vmatrix} 1-\mu b^{**}_{r+1\ r+1} & \cdots\cdots & -\mu b^{**}_{r+1\ n} \\ \vdots & \ddots & \vdots \\ -\mu b^{**}_{n\ r+1} & \cdots\cdots & 1-\mu b^{**}_{nn} \end{vmatrix}$$

as the coefficient of λ^r, and this must be equal to $\alpha_1\cdots\cdots\alpha_r|E-\mu\tilde{B}|$ (identically in μ), as can be immediately seen from (*). This fact implies that the non-vanishing eigenvalues of the matrix

$$\tilde{B}_1=\begin{pmatrix} b^{**}_{r+1\,r+1}\cdots\cdots b^{**}_{r+1\,n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ b^{**}_{n\,r+1} \cdots\cdots b^{**}_{nn} \end{pmatrix}$$

are $\beta_1, \beta_2, \cdots\cdots\beta_s$. Consequently, there exists such an orthogonal matrix V_1 of degree $n-r$ that $V_1'\tilde{B}_1V_1$ has a diagonal form

$$\begin{pmatrix} \beta_1 & & & & & \\ & \beta_2 & & & O & \\ & & \ddots & & & \\ & & & \beta_s & & \\ & & & & 0 & \\ & O & & & & \ddots \\ & & & & & & 0 \end{pmatrix},$$

for the matrix $U'\tilde{B}U=$

(b_{ij}^*) is also symmetric. Set $V=\begin{pmatrix} E_r & 0 \\ 0 & V_1 \end{pmatrix}$, where E_r denots the unit matrix of degree r, and put $W=UV$. Then W is an orthogonal matrix and $W'\tilde{A}W$ and $W'\tilde{B}W$ have the form

$$\begin{pmatrix} \alpha_1 & & & & & O \\ & \alpha_2 & & & & \\ & & \ddots & & & \\ & & & \alpha_r & & \\ & & & & 0 & \\ O & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \quad \text{and} \quad \begin{array}{c} \begin{pmatrix} b_{11}^* & \cdots & b_{n1}^* & & & \\ \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & & & \\ \cdots & \beta_1 & & & & O \\ \cdots & & \ddots & & & \\ \cdots & & & \beta_s & & \\ & & & & 0 & \\ & & O & & & \ddots \\ \underbrace{b_{n1}^*}_{r} & & & & & & 0 \end{pmatrix} \end{array} \Big\} r.$$

respectively.

Now, with regards to the norm of $\tilde{B}$, *i. e.*, $tr(\tilde{B}'\tilde{B})$, we have

$$\sum{}'|b_{ij}^*|^2+\sum_{i=1}^{s}\left|\beta_i\right|^2=\sum_{i=1}^{s}\left|\beta_i\right|^2,$$

where $\sum'$ represents the summation running on such (ij) that at least one of i and j is equal or smaller than r, for the norm is invariant under the unitary transformation. Therefore, we must have

$$b_{11}^*=\cdots\cdots=b_{1n}^*=b_{21}^*=\cdots\cdots=b_{r+1,1}^*=\cdots\cdots=b_{r+1,r}^*=\cdots\cdots=b_{nr}^*=0,$$

that is,

$$W'\tilde{B}W=\begin{pmatrix} \overbrace{0 \quad\ddots\quad 0}^{r} & & & & O \\ & \beta_1 & & & \\ & & \ddots & & \\ & & & \beta_s & \\ & & & & 0 \\ O & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}.$$

Hence

$$W\tilde{A}\tilde{B}W=W\tilde{A}WW'\tilde{B}W=\begin{pmatrix} d_r & & & & O \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ O & & & & \ddots \\ & & & & & 0 \end{pmatrix}\begin{pmatrix} 0 & & & & & O \\ & \ddots & & & & \\ & & C & & & \\ & & & \beta_1 & & \\ & & & & \ddots & \\ & & & & & \beta_s \\ & & & & & & 0 \\ O & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}=0,$$

consequently

$$\tilde{A}\tilde{B}=0,$$

and finally we have

$$ARB=0.$$

Corollary. *A necessary and sufficient condition that the statistics* $\sum_{i,j=1}^{n} a_{ij}x_i x_j$ *and* $\sum_{i=1}^{n} a_i x_i$, *or* $\sum_{i=1}^{n} a_i x_i$ *and* $\sum_{i=1}^{n} b_i x_i$ *be mutually independent, is respectively,* $AR\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}=0$ *or* $(a_1, a_2, \cdots\cdots, a_n)\ R\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}=0.$

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1) A. T. Craig. Note on the independence of certain quadratic forms. Annals of Mathematical Statistics. Vol. 14. 1943.
proof at this stage.

H. Hotelling. Note on a matric theorem of A. T. Craig. Annals of Mathematical Statistics, Vol. 15 (1944).

2) I had this result in 1943, independently of A. T. Craig when H. Sakamoto asked me about the independence of quadratic forms, and informed it to him. He and some other colleagues of mine have searched for other proofs of this theorem (esp. M. Sugawara and S. Nabeya have found new proofs) or applied it to various problems. On seeing these recent investigations and those of some others, it seems to me of some use to publish my proof at this stage.

3) In this note we denote the transposed matrix of a matrix T, as usual, by T'.
