On an Analytical Method in the Theory of Independent Random Variables*

By Kiyonori KUNISAWA

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Introduction. The object of this paper is to introduce an analytical method and to show that a systematic study of the theory of independent random variables is easily accessible. Such an attempt is connected with P. Lévy's *maximal concentration function*[1]:

$$Q_F(l)=\underset{-\infty<x<\infty}{Sup}\{F(x+l+0)-F(x-0)\}.$$

which has a fundamental rôle in his theory of the sums of independent random variables, where $F(x)$ is a probability distribution. Recently T. Kawata introduced a *mean concentration function*[2]:

$$C_F(l)=\frac{1}{2l}\int_{-\infty}^{\infty}\{F(x+l+0)-F(x-l-0)\}^2dx,\ l>0.$$

By this substitution of "maximal" by "mean" an analytical treatment of the probability theory has become easy. It is not difficult to see, by M. Plancherel's theorem and P. Lévy's inversion formula, that $C_F(l)$ can be also expressed by Féjer integral:

$$C_F(l)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin lt}{lt}\right)^2|f(t)|^2dt=2\int_0^{2l}(1-\frac{x}{2l})d\tilde{F}(x).$$

where $f(t)$ is the characteristic function of $F(x)$ and $\tilde{F}(x)=(1-F(-x))*F(x)$[3] is the symmetrized distribution of $F(x)$. In the present note we propose to adopt the following two functions

$$\Psi_F(l)=l\int_0^{\infty}e^{-lt}|f(t)|^2dt=\int_{-\infty}^{\infty}\frac{l^2}{l^2+x^2}d\tilde{F}(x),\ l>0$$

and

(*) This paper was read at the general meating of Math. Soc. Jap., June 4, 1946.

(1) P. L'évy, L'addition des variables aleatoires, Paris, 1937.

(2) T. Kawata, The function of the mean concentration function of a chance variable, Duke Math. Journ., 9, 1941.

(3) $F(x)*G(x)$ denotes the convolution of two distribution functions $F(x)$ and $G(x)$.

$$\Phi_F(l)=l\int_{-\infty}^{\infty} e^{-l|t|}\Re f(t)dt=\int \frac{l^2}{l^2+x^2}dF(x),\ l>0$$

where $\Re f(t)$ denotes the real part of $f(t)$. These two functions, based on Poisson integral, will turn out to be useful tools in an uniform treatment of our object. Hereafter in this paper, we shall call these functions $\Psi_F(l)$ and $\Phi_F(l)$ respectively the mean concentration function[(4)] and the typical function of a probability distribution.

The contents of the present paper is divided into six chapters. Chapter I is devoted to show seven elementary inequalities connected with $\Psi_F(l)$ and $\Phi_F(l)$ which play fundamental rôles in our note. Chapter II contains two uniform diminution theorems concerning $\Psi_F(l)$ and $\Phi_F(l)$. A theorem of this kind was, in terms of $Q_F(l)$, once given by P. Lévy-W. Doeblin[(5)]. In our case of $\Psi_F(l)$, the theorem is a little more general than that of $Q_F(l)$ and the proof is analytical and so perhaps more comprehensible. Regarding $\Phi_F(l)$, we cannot give a similar theorem under the same assumption. This fact may be attributed to the non-existence of the diminution property by the convolution of distribution functions. But under a stronger condition than the case of $\Psi_F(l)$, we can show an analogous theorem. Chapter III is devoted to exhibit our method in the convergence problem of the series of independent random variables. In the treatment of such a problem, P. Lévy[(6)] gave the method of the *limit maximal concentration function,* and A. Khintchine–A. Kolmogoroff[(7)] used the method of "equivalent series". These methods will be replaced by a useful one. In chapter IV we shall consider some problems concerning the law of large numbers from our viewpoint. Especially, the arguments of this law, which were shown by W. Feller[8)] and the others, will be much simplified by our method.

(4) Applying this function, the author has shown an outline of the proof of the large numbers; cf. K. Kunisawo, Mean concentration function and the law of large numbeıs, Proc. Imp. Acad. Tokyo, 20, pp. 627—630, 1944.

(5) P. Lévy—W. Doeblin, Sur les sommes de variables aléatoires indépendentes à disperseons bornées inferieurment, C. R. Paris, 202, pp. 2027—2029.

(6) P. Lévy, loc. cit. (1), or "Sur les séries dont terms sont des variables eventuelles independentes, St. Math., 3, pp. 119—155, 1931.

(7) A. Khintchine—A. Kolmogoroff, Über konvergenz von Reihen deren Glieder durch den Zufall bestimmt werden, Rec. Math., 32, pp. 668—688, 1935.

(8) W. Feller, Über das Gesetz der grössen Zahlen, Acta, Szeget 8, pp. 191—201, 1937.

In Chapter V, we shall deal with some problems connected with an infinitely divisible law. The definition of this law is known generally in the following two forms. The characteristic function $f(t)$ of a probability distribution $F(x)$ is called to depend on an infinitely divisible law, (1) if for any positive integer n there exists a system of characteristic functions $f_{n1}, f_{n2}, \cdots\cdots, f_{nn}$ satisfying $f(t)=f_{n1}(t)f_{n2}(t)\cdots\cdots f_{nn}(t)$ and the individual negligibility, i. e., $f_{nk}(t)\rightarrow 1(n\rightarrow\infty)$ uniformly for every finite interval and $k(1\leq k\leq n)$, or (2) if for any λ $(1>\lambda>0)$ $f^{\lambda}(t)$ is also the characteristic function of a probability distribution. In these definitions, the former clearly follows from the latter, but the converse is not so evident. It was proved by A. Khintchine(9). We shall, in § 1 of this chapter, give another proof from our angle. Next we shall consider to deduce the canonical form. Such a problem was discussed by A. Kolmogoroff(10), P. Lévy(11), K. Itô(12) and the others from the standpoint of stochastic processes. While A. Khintchine(13) directly gave an analytical deduction from the second definition quoted above. However, it seems that literatures have not contained a direct deduction from the first. It is the object of § 2 to show an answer to this problem. Let $X_{n1}, X_{n2}, \cdots\cdots X_{nm_n}$ $(n=1, 2, \cdots\cdots)$ be an individually negligible system of independent random variables. Then from § 1 of this chapter it is clear that the limit of the sums $X_{n1}+X_{n2}\cdots\cdots+X_{nm_n}$ as $n\rightarrow\infty$ has a distribution depending on an infinitely divisible law. This fact implies the following problem. What is the necessary and suffiicient condition that the distribution of the sum $X_{n1}+X_{n2}+\cdots\cdots+_{nm_n}$, should, in Bernoulli's sense, tend to that which depends on an infinitely divisible law? Such a problem was discussed by W. Doeblin(14), J. Marcinkiwicz(15) and B. Gnedenko(16) from the respective

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(9) A. Khintchine, Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze, Rec. Math., 2, pp. 79—117. 1937.

(10) A. Kolmogorof, Sulle forma generale di un processo stochastics omogenes, Atti. d. r. Accad. d. Lincei, S. 6, 15, pp. 805—808, pp. 866—868, 1932.

(11) P. Lévy, loc cit, (1), or Sur les intégrales dont les element sont des variables aléatoires independentes, Ann. d. r. Scuola Norm. d. Pisa; II 3, pp. 331—365, 1935, pp. 217—218, 1935.

(12) K. Itô, On stochastic processes (1), Jap. Journ. Math., 18, pp. 261—301, 1942.

(13) A. Khintchine, Deduction nouvelle d'une formula de M. Paul Lévy, Bull. Univ. Etat. Moscou, Ser. Int., S. A. Math. et Mechan. 1, pp. 5—9, 1935.

(14) W. Doeblin, Sur les sommes d'une grand nombre de variables aleatoires indépendent, Bull. Sci. Math., 63, pp. 35—64, 1939.

viewpoints. But the same idea as § 2 of this chapter can be also carried out in this case and so the argument is almost analogous to that of § 2. Further, it seems to the author that the present proof is considerably simpler than any previously given. It is the object of § 3 to show this fact. On the other hand, as an interesting property of an infinitely divisible law, it is known by A. Khintchine that it is representable as a partial limit law. However, A. Khintchine's proof of this fact is executed by a somewhat coercive calculation. We shall show another simple proof from our viewpoint.

In the last Chapter VI, we shall deal with some problems connected with the estimation of the magnitude of the sums of independent random variables. § 1 of this chapter is devoted to discuss the strong law of large numbers. Though the sufficient conditions for the validity of this law are known by A. Kolmogoroff(18), J. L. Doob(19), T. Kawata(20) and the others, these sufficient conditions are not the necessary conditions for the validity of this law. We shall here show that this problem can be answered if we deal with a sequence of independent random variables all having the same distribution. On the other hand, the following problem was touched upon by A. Khintchine. Let $\{X_n\}$ be a sequence of positive independent random variables all having the same distribution $F(x)$ such that

$$\int_0^\infty x dF(x) = \infty$$

then, what is the necessary and sufficient condition for the existence of a sequence of positve numbers $\{A_n \mid n=1, 2, \cdots\cdots\}$ satisfying

$$\lim_{n\to\infty} (X_1+X_2+\cdots\cdots+X_n)/A_n=1$$

(15) J. Marcinkiewicz, Quelques theoresm de la theorie des probabilités, Bull. Semin. Math. Univ. Wilno, 2, pp. 22–34, 1939,

(16) B. Gnedenko, Über die Konvergence der Verteilungsgesetzen von Summen von einander unabhangiger Summanden, C. R. URSS, 18, pp. 4–7, 1938, 22, 2, 1939.

(17) A. Khintchine, loc. cit. (9).

(18) A. Kolmogoroff, Grundbegrff der Wahrsheinlichkeitsrechnung, Ergeb. Math., II, 3 (1933), Springer.

(19) J. L. Doob, Probability and Statistics, Trans. Amer. Math. Soc. 36, pp. 759–775, 1934.

(20) T. Kawata, On the strong law of large numbers, Proc. Imp. Acad. Tokyo, 16, pp. 109–112, 1940.

(21) A. Khintchine, Su une legge dei grandi numeri generalzzato, Giorn. Ist. Ita. Attuari, 7, pp. 365–377, 1936.

with probability 1? In § 2, we shall show an answer to this problem. In § 2, we shall give an extension of P. Lévy—J. Marcinkiewicz's theorem. In other words, let $\{X_n \mid n=1, 2, \cdots\cdots\}$ be a sequence of independent random variables satisfying

$$P_r \{|X_n| > Z\} \leq CZ^{-a},\ P_r \{|X_n| > Z\} \geq cZ^{-a}$$
$$(Z \geq Z_0 > 0;\ 0 < a < 2;\ n=1, 2, \cdots\cdots),$$

where C and c are constants independent of n. P. Lévy for $0 < a \leq 1$[22] and J. Marcinkiewicz for $1 < a < 2$[23] showed the existence of a sequence $\{A_n\}$ such that with probability 1 $|X_1 + X_2 + \cdots\cdots + X_n| > A_n$ for infinitely many n's, or for at most finitely many n's. This theorem can be extended to more general case:

$$\frac{1}{Z^2}\int_{|x| \leq z} x^2 dF_n(x) \leq A \int_{|x| > z} dF_n(x) \qquad n=1, 2, \cdots\cdots$$
$$(Z > Z_0 > 0),$$

where A is a constant and $F_n(x)$ is the distribution function of X_n $(n=1, 2 \cdots\cdots)$.

The author here expresses his hearty thanks Prof. T. Kawata and Prof. S. Kakutani for their kind encouragement and valuable remarks given to him throughout the present works.

Contents.

(22) P. Lévy, loc. cit. (6).
(23) J. Marcinkiewicz, loc. cit. (14).

Chapter I. The fundamental inequalities.

§1. 1. The generalized mean concentration function. Given a function $m(x)$ defined on $(-\infty, \infty)$, which is even, non-negative, integrable and non-increasing (over $(0, \infty)$), furthermore satisfying the following conditions

$$m(0)=1 \tag{1.1.1}$$

and

$$m(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}\mu(t)dt, \quad (\mu(t)\geq 0). \tag{1.1.2}$$

Let $f(t)$ be the characteristic function of a probability distribution $F(x)$ i. e.,

$$f(t)=\int_{-\infty}^{\infty} e^{itx}dF(x).$$

Then we define a function $g_F(h)(h>0)$:

$$g_F(h)\equiv\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mu(ht)\,|\,f(t)\,|^2dt.$$

This function $g_F(h)$ is called a *generalized mean concentration function.* $g_F(h)$ has the following elementary properties.

1.° $g_F(h)\geq 0$ *and*

$$g_F(h)=\int_{-\infty}^{\infty} m(x/h)d\tilde{F}(x),$$

where $\tilde{F}(x)$ *denotes the symmetrized distribution of* $F(x)$, *i. e.,*

$$\tilde{F}(x)=\int_{-\infty}^{\infty} F(x-y)d(1-F(-y))=F(x)*(1-F(-x)).$$

For,

$$g_F(h)=\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu(ht)\,|f(t)|^2dt=\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu(ht)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{it(x-y)}dF(x)dF(y)\right\}dt$$

$$=\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dF(y)dF(x)\int_{-\infty}^{\infty}e^{it(x-y)}\mu(ht)dt$$

$$=\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dF(x)dF(y)\int_{-\infty}^{\infty}e^{it\frac{(x-y)}{h}}\mu(t)dt$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}m\left(\frac{x-y}{h}\right)dF(x)dF(y)=\int_{-\infty}^{\infty}m\left(\frac{x}{h}\right)d\int_{-\infty}^{\infty}F(x-y)d(1-F(-y))$$

$$=\int_{-\infty}^{\infty}m\left(\frac{x}{h}\right)d\tilde{F}(x).$$

2°.

$$\lim_{h\to\infty} g_F(h)=1.$$

For,

$$\lim_{h\to\infty} g_F(h)=\lim_{h\to\infty}\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu(ht)|f(t)|^2dt$$

$$=\lim_{h\to\infty}\int_{-\infty}^{\infty}m\left(\frac{x}{h}\right)d\tilde{F}(x)=\int_{-\infty}^{\infty}m(0)d\tilde{F}(x)=\int_{-\infty}^{\infty}d\tilde{F}(x)=1.$$

3°. *Under certain conditions on* $\mu(t)$(4)

$$\lim_{h\to 0} g_F(h)=\lim_{T\to 0}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2dt.$$

This fact follows from N. Wiener's formula.

4°. *$g_F(h)$ is a non-decreasing function of $h(>0)$, i. e., if $h>h'>0$, we have $g_F(h)\geq g_F(h')$.*

In fact,

$$g_F(h)-g_F(h')=\int_{-\infty}^{\infty}\left\{m\left(\frac{x}{h}\right)-m\left(\frac{x}{h'}\right)\right\}d\tilde{F}(x).$$

(4) S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, p. 30, 1923.

As $m(x)$ is non-increasing, we have $g_F(h)\geq g_F(h')$.

5°. $g_F(h)$ *diminishes by the convolution of distribution functions.*

Let $f_1(t)$, $f_2(t)$ be respectively, the characteristic functions of $F_1(x)$, $F_2(x)$, then the product $f_1(t)f_2(t)$ is that of F_1*F_2. Hence

$$g_{F_1*F_2}(h)=\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu(ht)|f_1(t)f_2(t)|^2dt$$

$$\leq\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu(ht)|f_i(t)|^2dt$$

$$=g_{F_i}(h)\qquad (i=1,\ 2).$$

Denote by $Q_F(h)$ the maximal concentration function of a distribution function $F_1(x)$ due to P. Lévy, i. e.,

$$Q_F(h)=\sup_{-\infty<x<\infty}\{F(x+h+0)-F(x-0)\}.$$

Then we have the following relation.

6°. *There exists a positive constant a satisfying* $m(a)>0$ *snch that*

$$2\left\{(1+\int_0^{\infty}m(x)dx\right\}Q_F(h)\geq g_F(h)\geq m(a)Q_F^2(ah)$$

In fact,

$$g_F(h)=\int_{-\infty}^{\infty}m\left(\frac{x}{h}\right)d\tilde{F}(x)=2\int_0^{\infty}m\left(\frac{x}{h}\right)d\tilde{F}(x)$$

$$\leq 2\sum_{k=0}^{\infty}m(k)\left\{\tilde{F}((k+1)h)-\tilde{F}(kh)\right\}$$

$$\leq 2\sum_{k=-\infty}^{\infty}m(k)Q_{\tilde{F}}(h)\leq 2\left\{m(0)+\int_0^{\infty}m(x)dx\right\}Q_{\tilde{F}}(h)$$

$$\leq 2\left\{1+\int_0^{\infty}m(x)dx\right\}Q_F(h),$$

where the inequality $Q_{\tilde{F}}(h)\leq Q_F(h)$ follows from the diminution of the maximal concentration function by the convolution of distribution functions. While, as $m(0)=1$, we can select a such that $m(a)>0$. Now, let

$$Q_F(ak)=\sup_{-\infty<x<\infty}\{F(x+ah+0)-F(x-0)\}$$

$$=F(\xi+ah/2+0)-F(\xi-ah/2-0).$$

For $\xi-ah/2\leq x\leq\xi+ah/2$ we have $F(x+ah)-F(x-ah)\geq Q_F(ah)$. Hence

$$g_F(h)=\int_{-\infty}^{\infty}m\Big(\frac{x}{h}\Big)d\tilde{F}(x)=\int_{-\infty}^{\infty}d(1-F(-x))\int_{-\infty}^{\infty}m\Big(\frac{x+y}{h}\Big)dF(y)$$

$$\geq\int_{-\infty}^{\infty}d(1-F(-x))\int_{-x-ah}^{-x+ah}m\Big(\frac{x+y}{h}\Big)dF(x)$$

$$=m(a)\int_{-\infty}^{\infty}\Big\{F(x+ah)-F(x-ah)\Big\}dF(x)$$

$$\geq m(a)\int_{\xi-ah/2}^{\xi+ah/2}\Big\{F(x+ah)-F(x-ah)\Big\}dF(x)$$

$$\geq m(a)Q_F^2(ak).$$

Thus, we obtain $g_F(h)\geq m(a)Q_F^2(ah)$.

As examples of the generalized mean concentration function, we can enumerate as follows.

1°. Put $m(x)=(1+x^2)^{-1}$, then $\mu(t)=\sqrt{\pi/2}\ exp(-|t|)$.
Hence

$$\Psi_F(h)\equiv\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{2}}e^{-|ht|}|f(t)|^2dt$$

$$=h\int_0^{\infty}e^{-ht}|f(t)|^2dt=\int_{-\infty}^{\infty}\frac{h^2}{h^2+x^2}d\tilde{F}(x). \tag{1.1.5}$$

2°. Put $m(x)=1-|x/2|$ for $|x|\leq 2$; $m(x)=0$ for $|x|>2$, then

$$\mu(t)=\sqrt{\frac{2}{\pi}}\Big(\frac{\sin t}{t}\Big)^2.$$

Hence

$$C_F(h)\equiv\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\Big(\frac{\sin ht}{ht}\Big)^2|f(t)|^2dt$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 ht}{ht^2}|f(t)|^2dt=2\int_0^{2h}\Big(1-\frac{x}{2h}\Big)d\tilde{F}(x).$$

This $C_F(h)$ is the function introduced by T. Kawata.[25]

(25) T. Kawata, loc. cit. (2).

3°. Let $m(x)=exp(-x^2/2)$, then $\mu(t)=exp(-t^2/2)$.
Hence

$$\Omega_F(h)=\frac{h}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-h^2t^2}|f(t)|^2dt.$$

$$=\int_{-\infty}^{\infty} e^{-x^2/2h^2}d\tilde{F}(x).$$

In the following lines we shall mean by Ψ_F the *mean concentration function of* $F(x)$.

§ **1. 2. The fundamental inequalities.** If we replace $|f(t)|^2$ in (1.1.5) by the real part of $f(t)$, we obtain the *typical function of* $F(x)$:

$$\Phi_F(h)=h\int_0^{\infty} e^{-ht}\Re f(t)dt=\int_{-\infty}^{\infty}\frac{h^2}{h^2+x^2}dF(x).$$

This $\Phi_F(h)$ is clearly non-negative, non-decreasing and continuous function of $h(h>0)$, and evidently

$$\lim_{h\to\infty}\Phi_F(h)=1.$$

Between $\Phi_F(h)$ and $\Psi_F(h)$, the following inequalities exists.

1°. *For any* $h>0$

(F. I. 1)[26] $$2(1-\Phi_F(h))\geq 1-\Psi_F(h).$$

For, as we see

$$1-|f(t)|^2=1-(\Re f(t))^2-(\Im f(t))^2$$
$$\leq 2(1-\Re f(t)),$$

we easily have (F. I. 1).

2°. *If*

(1.2.1) $$1-F(h/96)+F(-h/96)\leq 1/32,$$

we have

(F. I. 2) $$1-\Psi_F(h)\geq 10^{-6}(1-\Phi_{F'}(h)),$$

where $\Phi_{F'}(h)$ *is the typical function of* $F'(x)=F(x+ha)$ *and* $a=\int_{-1}^{1}xdF(hx)$.

(26) (F. I. 1) is the abbrebiation of "Fundamental Inequality 1".

Proof. In the first place we make a remark

$$|a|\leq\int_{-1}^{1}|x|\,dF(hx)\leq\int_{1>|x|>1/96}|x|dF(hx)+\int_{|x|\leq 1/96}|x|dF(hx)$$

$$\geq 1/32+1/96=1/24,$$

and for the median M of $F(x)$,

$$|M|\leq h/96. \tag{1.2.2}$$

Put

$$f'(t/h)=f(t/h)e^{-iat}=\int_{-\infty}^{\infty}e^{itx}dF(hx+ha),$$

then for $0<|t|\leq 2$

$$\left|(\Im f'(t/h))^2-\left(\int_{|x+a|\geq 1/48}\sin tx\,dF(hx+ha)\right)^2\right|$$

$$\leq\left|2\int_{|x+a|<1/48}\sin tx\,dF(hx+ha)\right|$$

$$\leq 4\left|\int_{|x+a|<1/48}xdF(hx+ha)\right|+\frac{8}{3}\int_{|x+a|<1/48}|x|^3dF(hx+ha)$$

$$\leq 4|a|\int_{|x|>1/48}dF(hx)+4\int_{|x|>1/48}dF(hx)+\frac{1}{6}\left(1+\left(\frac{1}{16}\right)^2\right)\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF(hx+ha)$$

$$\leq 25(1-F(h/48))+F(-h/48)/6+(1+(16)^{-2})6^{-1}(1-\Phi_{F'}(h)).$$

As we see easily, recalling (1.2.2),

$$\int_{|x|>1/48}dF(hx)\leq 2\int_{|x|>1/96}d\tilde{F}(hx)^{(27)},$$

we have

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(27) In fact,

$$Pr\{|X-M-(\bar{X}-M)|>1/96\}\geq Pr\{X-M-(\bar{X}-M)>1/96\frown(\bar{X}-M)\leq 0\}$$
$$+P_r\{|X-M-(\bar{X}-M)<-1/96\frown(\bar{X}-M)\geq 0\}$$
$$=\frac{1}{2}Pr\{|X-M|>1/96\}\geq\frac{1}{2}P_r\{|X|>1/48\},$$

where X and $\bar{X}$ are independent random variables both having the distribution $F(x)$.

$$(\Im f'(t/h))^2 \leq \frac{3}{25}\int_{|x|>1/96} d\tilde{F}(hx) + (1/5)(1-\Phi_{F'}(h)) + \left(\int_{|x+a|>1/48} \sin tx\, dF(hx+ha)\right)^2$$

$$\leq \frac{25}{3}(1+(96)^2)\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} d\tilde{F}(hx) + \frac{1}{5}(1-\Phi_{F'}(h))$$

$$+2\int_{|x+a|>1/48} dF(hx+ha)\int_{-\infty}^{\infty}(1-\cos tx)dF(hx+ha)$$

$$\leq 76800(1-\Psi_F(h)) + \frac{1}{5}(1-\Phi_{F'}(h)) + \frac{1}{16}(1-\Re f'(t/h)).$$

On the other hand, for $0\leq|t|\leq 2$

$$1-\Re f'(t/h) = \int_{-\infty}^{\infty}(1-\cos tx)dF(hx+ha)$$

$$= \int_{|x+a|\leq 1/96} + \int_{|x+a|>1/96}$$

$$\leq 2\int_{|x+a|\leq 1/96} x^2 dF(hx+ha) + 2\int_{|x-a|>1/96} dF(hx+ha)$$

$$\leq 2(1/96+1/24)^2 + 1/16 < 1/15.$$

Hence

$$14/15 < \Re f'(t/h).$$

Consequently

$$1-|f(t/h)|^2 = 1-(\Re f'(t/h))^2 - (\Im f'(t/h))^2$$

(1.2.3)
$$\geq (1-\Re f'(t/h))(1+14/15) - 76800(1-\Psi_F(h)).$$

$$-(1/5)(1-\Phi_{F'}(h)) - (1/15)(1-\Re f'(t/h))$$

$$\geq (9/5)(1-\Re f'(t/h)) - 76800(1-\Psi_F(h)) - (1/5)(1-\Phi_{F'}(h)).$$

As the inequality

$$1-\Re f'(2^i t) \leq 4^i(1-\Re f'(t))$$

follows from the elementary inequality $1-\cos(2^i x)\leq 4^i(1-\cos x)$, we see

$$1-\Phi_{F'}(h) = \int_0^{\infty} e^{-t}(1-\Re f'(t/h))dt = \int_0^1 + \sum_{i=0}^{\infty}\int_{2^i}^{2^{i+1}}$$

$$= \int_0^1 e^{-t}(1-\Re f'(t/h))dt + \sum_{i=0}^{\infty} 2^i \int_1^2 e^{-2^i t}(1-\Re f'(t/h))dt$$

$$(1.2.4) \quad \leq \int_0^1 (1-\Re f'(t/h))dt + \sum_{i=0}^{\infty} 2^i e^{-2^i} 4^i \int_1^2 (1-\Re f'(t/h))dt$$

$$\leq \sum_{i=0}^{\infty} 2^{3i} e^{-2^i} \int_0^2 (1-\Re f'(t/h))dt$$

$$\leq 3 \int_0^2 (1-\Re f'(t/h))dt.$$

Hence, by (1.2.3)

$$\int_0^2 (1-|f(t/h)|^2)dt \geq \frac{9}{5} \int_0^2 (1-\Re f'(t/h))dt - 153600(1-\Psi_F(h)) - \frac{2}{5}(1-\Phi_{F'}(h)).$$

Consequently, applying (1.2.4),

$$e^2 \int_0^{\infty} e^{-t}(1-|f(t/h)|^2)dt \geq \frac{9}{15}(1-\Phi_{F'}(h)) + 153600(1-\Psi_F(h)) - \frac{2}{5}(1-\Phi_{F'}(h)),$$

from which we obtain

$$(153600+e^2)(1-\Psi_F(h)) \geq \frac{1}{5}(1-\Phi_{F'}(h)).$$

Hence we can conclude (F. I. 2)

Next we can give the following inequality.

3°. *Let* α *be a positive number* $\left(0<\alpha<\frac{\lambda}{2}\right)$ *and let* $F(x)$ *be a distribution function satisfying*

$$(1.2.5) \qquad F(+0) \geq \lambda > 0, \quad F(-0) \leq 1-\lambda, \quad (0<\lambda<1)$$

and

$$(1.2.6) \qquad 1-F(h)+F(-h)<\alpha,$$

then we have

$$(\text{F. I. } 3) \qquad 1-\Psi_F(h) \geq K(\alpha, \lambda)(1-\Phi_F(h)),$$

where $K(\alpha, \lambda)$ *is a positive constant depening on* α *and* λ.

Proof. Denote by I′ and I″, respectively, the following sets:

$$\underset{x}{E}\left\{0<t\leq 2,\ \sin\frac{tx}{h}>0\right\} \text{ and } \underset{x}{E}\left\{0<t\leq 2,\ \sin\frac{tx}{h}<0\right\},$$

then we see that I′ contains the interval $(0, h)$ and I″ the interval $(-h, 0)$. Hence, recalling (1.2.5) and (1.2.6), for $0<t\leq 2$

$$(\Im f(t/h))^2=\left(\int_{-\infty}^{\infty}\sin\frac{tx}{h}\,dF(x)\right)^2\leq\left(\int_I \sin\frac{tx}{h}dF(x)\right)^2$$

$$\leq\int_I dF(x)\int_{-\infty}^{\infty}\sin^2\frac{tx}{h}dF(x)\leq 2(1-\lambda+\alpha)(1-\Re f(t/h)),$$

where I is I′ or I″. On the other hand, by (1.2.6)

$$1-\Re f(t/h)=\int_{-\infty}^{\infty}\left(1-\cos\frac{tx}{h}\right)dF(x)$$

$$=\int_{|x|\leq h}+\int_{|x|>h}\leq\frac{t^2}{2}+2\alpha,$$

from which we get $1-t^2/2-2\alpha\leq\Re f(t/h)$. Hence for $0\leq t\leq T$ $(2\lambda-4\alpha>T^2/2)$

$$1-|f(t/h)|^2=1-(\Re f(t/h))^2-(\Im f(t/h))^2$$

$$\geq(1-\Re f(t/h))(1+\Re f(t/h))-(2+2\alpha-2\lambda)(1-\Re f(t/h))$$

$$\geq\left(2\lambda-\frac{T^2}{2}-4\alpha\right)(1-\Re f(t/h)).$$

Hence we see

$$\int_0^T(1-|f(t/h)|^2)dt\geq\left(2\lambda-\frac{T^2}{2}-4\alpha\right)\int_0^T(1-\Re f(t/h))dt.$$

In the same manner as (1.2.4) we get

$$1-\Phi_F(h)\leq R(T)\int_0^T(1-\Re f(t/h))dt,$$

where $R(T)$ is a positive constant depending only on T, thus we have

$$e^T\int_0^{\infty}e^{-t}(1-|f(t/h)|^2)dt\geq\left(\left(2\lambda-\frac{T^2}{2}-4\alpha\right)/R(T)\right)(1-\Phi_F(h)),$$

which shows

$$1-\Psi_F(h)\geq\frac{1}{e^T R(T)}\left(2\lambda-\frac{T^2}{2}-4\alpha\right)(1-\Phi_F(h)).$$

Since T is a constant depending only on α and λ, putting

$$K(\alpha,\lambda)=\frac{1}{e^T R(T)}\left(2\lambda-\frac{T^2}{2}-4\alpha\right),$$

we have (F. I. 3).

Now we shall consider the relations of Ψ_F and Φ_F to the convolution of distribution functions. Let $F_k(x)$ $(k=1,2,\cdots\cdots,n)$ be an arbitrary system of distribution functions and $f_k(t)$ $(k=1,2,\cdots\cdots,n)$ be the characteristic function of $F_k(x)$ $(k=1,2,\cdots\cdots,n)$. Further denote by $F_1*F_2*\cdots\cdots*F_n$ the convolution of $F_1(x),\cdots\cdots,F_n(x)$. Then,

4°. *For any* $h>0$

(F. I. 4) $$1-\Psi_{F_1*\cdots\cdots*F_n}(h)\leq\sum_{k=1}^n(1-\Psi_{F_k}(h)).$$

For, this easily follows from the next inequality

$$1-\Pi_{k=1}^n|f_k(t)|^2\leq\sum_{k=1}^n(1-|f_k(t)|^2)$$

5°. *If there exist* $\delta>0$, $T>0$ *and* $h>0$ *such that for* $0\leq t\leq T$

(1.2.7) $$\Pi_{k=1}^n|f_k(t/h)|^2\geq\delta,$$

then we have

(F. I. 5) $$\sum_{k=1}^n(1-\Psi_{F_k}(h))\leq K(1-\Psi_{F_1*\cdots\cdots*F_n}(h)),$$

where K *is a constant depending only on* T *and* δ.

Proof. From (1.2.4) we have, by an elementary calculation,

(1.2.8) $$\sum_{k=1}^n(1-|f_k(t/h)|^2)\leq\frac{1}{\delta}(1-\Pi_{k=1}^n|f_k(t/h)|^2)$$

for $0\leq t\leq T$. And as the inequality $1-\cos(2^jx)\leq4^j(1-\cos x)$ clearly implies

$$1-|f(2^jt/h)|^2\leq4^j(1-|f(t/h)|),$$

we get in the same manner as (1.2.4)

$$1-\Psi_{F_k}(h)\leq M\int_0^T(1-|f_k(t/h)|^2)dt,$$

where

$$M = Max\{1, \sum_{i=0}^{\infty} 2^{3i} \exp(-2^{(i-1)}T)\}.$$

Hence from (1.2.8)

$$\sum_{k=1}^{n}(1-\Psi_{F_k}(h)) \leq M \sum_{k=1}^{n} \int_0^T (1-|f_k(t/h)|^2)dt$$

$$\leq \frac{Me^T}{\delta}\int_0^T e^{-t}(1-\Pi_{k=1}^{n}|f_k(t/h)|^2)dt$$

$$\leq \frac{Me^T}{\delta}\int_0^\infty e^{-t}(1-\Pi_{k=1}^{n}|f_k(t/h)|^2)dt \equiv K(1-\Psi_{F_1*\cdots\cdots F_n}(h)).$$

where $K \equiv M\, e^T/\delta$ is a constant depending only on T and δ.

6°. *We have*

(F. I. 6) $$\sum_{k=1}^{n}|f_k(t/h)\exp(-ia_k t)-1| \leq (t^2+2|t|+4)\sum_{k=1}^{n}(1-\Phi_{F_k}(h))$$

for every t and $h>0$, where

$$a_k = \int_{-1}^{1} x dF_k(hx) \quad (k=1, 2, \cdots\cdots, n).$$

Proof. We have

$$|f_k(t/h)\exp(-ia_k t)-1| = \left|\int_{-\infty}^{\infty}\left[\exp\left\{it\left(\frac{x}{h}-a_k\right)\right\}-1\right]dF_k(x)\right|$$

$$\leq \left|\int_{|x|\leq h}[\exp\{it(x-ha_k)/h\}-1]dF_k(x)\right| + 2\int_{|x|>h}dF_k(x)$$

$$\leq \left|\int_{|x|\leq h}\left(e^{it(x-ha_k)/h} - \frac{it(x-ha_k)}{h}-1\right)dF_F(x)\right|$$

$$+\frac{|t|}{h}\left|\int_{|x|\leq h}(x-ha_k)dF_k(x)\right| + 2\int_{|x|>h}dF_k(x)$$

$$\leq \frac{t^2}{2h^2}\int_{|x|\leq h}(x-ha_k)^2 dF_k(x) + |a_k||t|\int_{|x|>h}dF_k(x) + 2\int_{|x|>h}dF_k(x)$$

$$\leq (t^2+2|t|+4)\int_{-\infty}^{\infty}\frac{x^2}{(h^2+x^2)}dF_k(x).$$

Hence

$$\sum_{k=1}^{n}|f_k(t/h)\exp(-ia_k t)-1| \leq (t^2+2|t|+4)\sum_{k=1}^{n}(1-\Phi_{F_k}(h)).$$

7°. *Let $X_1, X_2, \cdots\cdots, X_n$ be any system of random variabes such that the expectation $E(X_k)$ of X_k is zero when it exists and define $a_k (k=1, 2, \cdots\cdots, n)$ as follows:*

$$a_k=0 \quad \textit{if } E(X_k) \textit{ exists,}$$

$$a_k=\int_{|x|\leq 1} x dF_k(x) \qquad \textit{otherwise.}$$

Then we have for every t

(F. I. 7) $\quad \sum_{k=1}^n |f_k(t/h)\exp(-ia_k t)-1| \leq (t^2+2|t|+4)\sum_{k=1}^n (1-\Phi_{F_k}(h)).$

Proof. It is sufficient to show that if $E(X_k)$ exists, we have

$$|f_k(t/h)-1(| \leq (t^2+4)(1-\Phi_{F_k}(h)).$$

In fact,

$$|f_k(t/h)-1|=\left|\int_{-\infty}^{\infty}(e^{itx/h}-1)dF_k(x)\right|$$

$$=\left|\int_{-\infty}^{\infty}\left(e^{itx/h}-\frac{itx}{h}-1\right)dF_F(x)\right|$$

$$\leq \frac{t^2}{2h^2}\int_{|x|\leq h} x^2 dF_k(x)+2\int_{|x|>h} dF_k(x) \leq (t^2+4)(1-\Phi_{F_k}(h)).$$

We shall find that the above stated inequalities will have fumdamental rôles in the present paper.

Chapter II. The uniform diminution theorems concerning a mean concentration function and a typical function.

§ 2. 1. The uniform diminution theorm concerning a mean concentration function. Given a system of random variables

$$\begin{array}{c} X_{11}, X_{12}, \cdots\cdots, X_{1m_1} \\ X_{21}, X_{22}, \cdots\cdots, X_{2m_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_{n1}, X_{n2}, \cdots\cdots, X_{nm_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \tag{2.1.1}$$

which will be denoted by denoted by $\|X_{nm}\|$. And let $F_{nm}(x)$ be the distribution function of X_{nm}. Hereafter we shall suppose the independency of the random variables in a same row but do not necessarily admit it in different rows. Then we have

Theorem 2. 1. 1. *Given two real numbers α and β $(0<\alpha<1, 0<\beta\leq 1)$, we can determine two positive numbers K and N—both depending only on α and β—having the following properties: if $m_n \geq N$ and if $F_{nm}(x)(m=1,2\cdots\cdots,m_n)$ is a system of distribution functions satisfying*

$$\frac{1}{m_n}\sum_{m=1}^{m_n}\Psi_{F_{nm}}(l_0)\leq\alpha, \tag{2.1.2}$$

where l_0 is an arbitrary but fixed positive number, then

$$\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(\sqrt{m_n}Kl_0)\leq\beta. \tag{2.1.3}$$

Let $f_{nm}(t)$ be the characteristic function of $F_{nm}(x)$ and let $\tilde{F}_{nm}(x)$ be the symmetrized distribution of $F_{nm}(x)$. Then

$$|\Pi_{m=1}^{m_n}f_{nm}(t)|^2\leq\left(\frac{1}{m_n}\sum_{m=1}^{m_n}|f_{nm}(t)|^2\right)^{m_n}=\left\{\int_{-\infty}^{\infty}e^{itx}\,d\left(\frac{1}{m_n}\sum_{m=1}^{m_n}\tilde{F}_{nm}(x)\right)\right\}^{m_n}$$

and consequently

$$\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(l)=l\int_0^{\infty}e^{-lt}|\Pi_{m=1}^{m_n}f_{nm}(t)|^2dt$$

$$\leq l\int_0^{\infty}e^{-lt}(g_n(t))^{m_n}dt=\int_0^{\infty}e^{-t}(g_n(t/l))^{m_n}dt\equiv\varphi_n(l),$$

where

$$g_n(t)\equiv\int_0^{\infty}e^{itx}d\left(\frac{1}{m_n}\sum_{m=1}^{m_n}\tilde{F}_{nm}(x)\right)\equiv\int_{-\infty}^{\infty}e^{itx}dG_n(x)$$

is the characteristic function of symmetric distribution $G_n(x)$. This fact implies that in order to prove (2.1.3) it is suffiiceint to show

$$\varphi_n(\sqrt{m_n}Kl_0)\leq\beta,\ m_n\geq N.$$

Lemma 2. 1. 1. *Let m be a given positive integer and put successively*

$$0<\delta\equiv\beta(1-\alpha)/8 \tag{2.1.6}$$

(2.1.7) $\epsilon=\delta\beta/2\sqrt{2\pi}(\log 3-\log\beta)$ (<1)

and

(2.1.8) $h=(3/\epsilon)^2$ (>1).

If there exist an integer n satisfying $m_n\geq mh$ and a real number a $(\geq\sqrt{m_n}\epsilon)$ such that

(2.1.9) $\delta\leq G_n^{m*}(3al_0)-G_n^{m*}(al_0)$, [8]

then we hvve

$$\varphi_n(\sqrt{m_n}\tau\, l_0)<\beta,\quad m_n\geq mh,$$

where

(2.1.10) $\tau=\delta\beta/2\sqrt{3\pi}=(\log 3-\log\beta)\epsilon$

is a constant depending only on α and β.

Proof. From (2.1.5) we have

$$(g_n(t))^m=\int_{-\infty}^{\infty}e^{itx}dG_n^{m*}(x)\,.$$

$$=1+\int_{|x|\leq\sqrt{h}\,a\epsilon l_0}(e^{itx}-1)dG_n^{m*}(x)+\int_{|x|>\sqrt{h}\,a\epsilon l_0}(e^{itx}-1)dG_n^{m*}(x).$$

since G_n^{m*} is symmetric, the last term is ≤ 0, and we see

$$(g_n(t))^m\leq 1-t^2\int_0^{\sqrt{h}\,a\epsilon l_0}x^2dG_n^{m*}(x)+\frac{t^3}{3}\int_0^{\sqrt{h}\,a\epsilon l_0}x^3dG_n^{m*}(x).$$

Let us put

(2.1.11) $T=\log 3-\log\beta=\tau/\epsilon$.

Then for every t $(0\leq t\leq T)$

(2.1.12) $$(g_n(t/\sqrt{h}\,a\tau l_0))^m\leq 1-\frac{t^2}{ha^2\tau^2l_0^2}\int_0^{\sqrt{h}\,a\epsilon l_0}x^2dG_n^{m*}(x)$$

$$+\frac{t^3}{3h^{3/2}a^3\tau^3l_0^3}\int_0^{\sqrt{h}\,a\epsilon l_0}x^3dG_n^{m*}(x).$$

---

(8) $G^{m*}(x)$ denotes the convolution $G*G*\cdots\cdots*G$ of m equal components G.

$$\leq 1-\frac{2t^2}{3ha^2\tau^2 l_0^2}\int_0^{\sqrt{ah}\,\epsilon l_0} x^2 dG_n^{m*}(x).$$

Since (2.1.9) and (2.1.8) imply

$$\delta \leq G_n^{m*}(3al_0)-G_n^{m*}(al_0)\leq \frac{1}{a^2 l_0^2}\int_{al_0}^{3al_0} x^2 dG_n^{m*}(x)$$

$$\leq \frac{1}{a^2 l_0^2}\int_0^{\sqrt{ha}\,\epsilon l_0} x^2 dG_n^{m*}(x),$$

so we see from (2.1.12)

$$g_n(t/\sqrt{h}\,a\tau l_0))^n \leq 1-\frac{2t^2\delta}{3h\tau^2}\leq \exp\left(-\frac{2t^2\delta}{3ht^2}\right),\quad 0\leq t\leq T.$$

If we now use the fact $m_n \geq mh$ then

$$(g_n(t/\sqrt{h}\,a\tau l_0))^{m_n}\leq (g_n(t/\sqrt{h}\,a\tau l_0))^{mh}\leq \exp\left(-\frac{2t^2\delta}{3\tau^2}\right),\quad 0\leq t\leq T.$$

and consequently

$$\varphi_n(\sqrt{h}\,a\tau l_0)=\int_0^\infty e^{-t}(g_n(t/\sqrt{h}\,a\tau l_0))^{m_n}dt$$

$$\leq \int_0^T e^{-t}(g_n(t/\sqrt{h}\,a\tau l_0))^{m_n}dt+\int_T^\infty e^{-t}dt$$

$$\leq \int_0^T e^{-t-\frac{2t^2\delta}{3\tau^2}}dt+e^{-T}\leq \int_0^\infty e^{-\frac{2t^2\delta}{3\tau^2}}dt+e^{-T}$$

$$\leq \frac{1}{2}\sqrt{\frac{3\pi}{2\delta}}\,\tau+\exp\{\log\beta-\log 3\}.$$

Thus the definition of δ and τ of (2.1.6) and (2.1.10) imply

$$\varphi_n(\sqrt{h}\,a\tau l_0)\leq \beta\sqrt{\delta}/4\sqrt{2}+\beta/3<\beta.$$

As $\sqrt{m_n}<\frac{3}{\epsilon}\sqrt{m_n}\epsilon<\sqrt{h}\,a$, we finally see

$$\varphi_n(\sqrt{m_n}\tau l_0)\leq \beta\sqrt{\delta}/4\sqrt{2}+\beta/3<\beta,\quad m_n\geq mh.$$

This completes the proof of the lemma.

Proof of Theorem 2.1.1. In the following lines we assume that

n is an arbitrary but fixed integer such that

$$m_n > N \equiv N(\alpha, \beta) \equiv h \log[1-\{(2-\beta)(1-\alpha-2\delta)/2(1-\alpha-4\delta)\}]/\log(\alpha+2\delta)$$

and δ, ϵ, τ and T are the same constant defined above.

The following two cases are possible:

$$\bar{G}_n(\sqrt{m_n}\epsilon l_0) \equiv 1 - G_n(\sqrt{m_n}\epsilon l_0)\begin{cases} \leq (1-\alpha-\delta)/2, & (2.1.14) \\ > (1-\alpha-\delta)/2. & (2.1.15) \end{cases}$$

1°. Assume (2.1.14). The from (2.1.2)

$$1-\alpha \leq 2\int_0^\infty \frac{x^2}{l_0^2+x^2} d\left(\frac{1}{m_n}\sum_{m=1}^{m_n}\tilde{F}_{nm}(x)\right)$$

$$\leq 2\int_0^\infty \frac{x^2}{l_0^2+x^2} dG_n(x) \leq \frac{2}{l_0^2}\int_0^{\sqrt{m_n}\epsilon l_0} x^2 dG_n(x) + 2\bar{G}_n(\sqrt{m_n}\epsilon l_0)$$

$$\leq \frac{2}{l_0^2}\int_0^{\sqrt{m_n}\epsilon l_0} x^2 dG_n(x) + 1 - \alpha - \delta.$$

Hence

$$\delta \leq \frac{2}{l_0^2}\int_0^{\sqrt{m_n}\epsilon l_0} x^2 dG_n(x).$$

Put

$$g_n(t) \equiv \int_{-\infty}^\infty e^{itx} dG_n(x) = 1 + \int_{|x| \leq \sqrt{m_n}\epsilon l_0} (e^{itx}-1) dG_n(x) + \int_{|x| < \sqrt{m_n}\epsilon l_0} (e^{itx}-1) dG_n(x),$$

then, by the same argument as in the proof of Lemma 2.1.1, we see

$$g_n(t/\sqrt{m_n}\tau l_0) \leq 1 - (2t^2/3m_n\tau^2 l_0^2)\int_0^{\sqrt{m_n}\epsilon l_0} x^2 dG_n(x)$$

$$\leq 1 - \frac{t^2\delta}{3m_n\tau^2} \leq \exp\left\{\frac{t^2\delta}{3m_n\tau^2}\right\}, \quad 0 \leq t \leq T,$$

therefore

$$\varphi_n(\sqrt{m_n}\tau l_0) = \int_0^\infty e^{-t}\left(g_n\left(\frac{t}{\sqrt{m_n}\tau l_0^2}\right)\right)^{m_n} dt$$

$$\leq \int_0^T e^{-t}(g_n(t/\sqrt{m_n}\tau l_0))^{m_n} dt + \int_T^\infty e^{-t} dt$$

$$\leq \int_0^T e^{-t-\frac{t^2\delta}{3\tau^2}}dt+e^{-T}\leq \frac{\sqrt{3\pi\tau}}{2\sqrt{\delta}}\ e^{\log \beta/3}\leq \beta\sqrt{\delta}/4+\beta/3<\beta.$$

2°. Let us suppose (2.1.15). It is sufficient to consider only the case which is

$$G_n^{m*}(3al_0)-G_n^{m*}(al_0)<\delta,\ m=1, 2,\cdots\cdots, [N/h], \tag{2.1.16}$$

for any $a>\sqrt{m_n}\epsilon$, where $[N/h]$ denotes the integral part of N/h. For, if there exist an $a(\geq\sqrt{m_n}\epsilon)$ and an m $(1\leq m\leq[N/h])$ such that (2.1.9) is true, by Lemma 2.1.1 $m_n\geq N\geq mh$ implies $\varphi_n(\sqrt{m_n}\tau l_0)<\beta$. We shall by induction prove the following inequality

$$\frac{1}{2}(1-\alpha-5\ \delta)\sum_{i=1}^{m}\eta^i<\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0) \tag{2.1.17}$$

$$m=1, 2,\cdots\cdots, [N/h],\ (\eta=\alpha+2\ \delta<1).$$

when $m=1$, we see from (2.1.15)

$$\eta(1-\alpha-4\ \delta)/2<(1-\alpha-\delta)/2<\bar{G}_n(\sqrt{m_n}\epsilon l_0).$$

Now assume (2.1.17) for a fixed m $([N/h]>m>1)$, then by (2.1.16) we see that for an arbitrary number a satisfying $y=2al_0>al_0\geq\sqrt{m_n}\epsilon l_0$,

$$\begin{aligned}&\bar{G}_n^{m*}(y-\sqrt{m_n}\epsilon l_0)-\bar{G}_n^{m*}(y+\sqrt{m_n}\epsilon l_0)\\&\quad=G_n^{m*}(y+\sqrt{m_n}\epsilon l_0)-G_n^{m*}(y-\sqrt{m_n}\epsilon l_0)\\&\quad\leq G_n^{m*}(3al_0)-G_n^{m*}(al_0)<\delta,\\&\qquad\qquad\qquad m=1, 2,\cdots\cdots, [N/h].\end{aligned}$$

Hence

$$\begin{aligned}&\bar{G}_n^{(m+1)*}(\sqrt{m_n}\epsilon l_0)\equiv 1-G_n^{(m+1)*}(\sqrt{m_n}\epsilon l_0)\\&=\int_{-\infty}^{\infty}\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0-y)dG_n(y)\\&=\int_0^{\infty}\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0-y)dG_n(y)+\int_0^{\infty}\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0+y)dG_n(y)\\&\geq\int_0^{2\sqrt{m_n}\epsilon l_0}\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0-y)dG_n(y)+\int_{2\sqrt{m_n}\epsilon l_0}^{\infty}\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0-y)dG_n(y)\end{aligned}$$

$$+\int_0^{2\sqrt{m_n}\epsilon l_0}\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0+y)dG_n(y)+\int_{2\sqrt{m_n}\epsilon l_0}^{\infty}(\bar{G}_n^{m*}(y-\sqrt{m_n}\epsilon l_0)-\delta)dG_n(y).$$

Whence, applying

$$\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0-y)+\bar{G}_n^{m*}(y-\sqrt{m_n}\epsilon l_0)=\bar{G}_n^{m*}(y-\sqrt{m_n}\epsilon l_0)+\bar{G}_n^{m*}(y-\sqrt{m_n}\epsilon l_0)=1,$$

which follows easily from the symmetry of $G_n^{m*}(x)$,

$$\bar{G}_n^{(m+1)*}(\sqrt{m_n}\epsilon l_0)\geq\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0)\left\{G_n(2\sqrt{m_n}\epsilon l_0)-\frac{1}{2}\right\}$$

$$+\bar{G}_n^{m*}(3\sqrt{m_n}\epsilon l_0)\left\{G_n(2\sqrt{m_n}\epsilon l_0)-\frac{1}{2}\right\}+(1-\delta)\bar{G}_n(2\sqrt{m_n}\epsilon l_0)$$

$$\geq\{\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0)+\bar{G}_n^{m*}(3\sqrt{m_n}\epsilon l_0)\}\left\{G_n(2\sqrt{m_n}\epsilon l_0)-\frac{1}{2}\right\}+(1-\delta)\bar{G}_n(2\sqrt{m_n}\epsilon l_0)$$

Putting $a=\sqrt{m_n}\epsilon$ in (2.1.16), we have

$$\bar{G}_n^{m*}(\sqrt{m_n}\epsilon l_0)-\delta<\bar{G}_n^{m*}(3\sqrt{m_n}\epsilon l_0)<\bar{G}_n^{m*}(2\sqrt{m_n}\epsilon l_0)$$

$$m=1,2,\cdots\cdots,[N/h].$$

Thus, by the assumption of induction,

$$\bar{G}_n^{(m+1)*}(\sqrt{m_n}\epsilon l_0)\geq\{(1-\alpha-4\delta)(\textstyle\sum_{i=1}^m\eta^i)-\delta\}\left\{G_n(2\sqrt{m_n}\epsilon l_0)-\frac{1}{2}\right\}$$

$$+(1-\delta)\bar{G}_n(2\sqrt{m_n}\epsilon l_0)$$

$$=\{(1-\alpha-4\delta)\textstyle\sum_{i=1}^m\eta^i-\delta\}\left\{G_n(2\sqrt{m_n}\epsilon l_0)+\bar{G}_n(2\sqrt{m_n}\epsilon l_0)-\frac{1}{2}\right\}$$

$$+\{1-(1-\alpha-4\delta)\textstyle\sum_{i=1}^m\eta^i\}\bar{G}_n(2\sqrt{m_n}\epsilon l_0)$$

$$\geq\frac{1-\alpha-4\delta}{2}\textstyle\sum_{i=1}^m\eta^i-\frac{\delta}{2}+\{1-\sum_{i=1}^m\eta^i+(\alpha+4\delta)\sum_{i=1}^m\eta^i\}\frac{(1-\alpha-3\delta)}{2}$$

$$=\frac{1-\alpha-4\delta}{2}\textstyle\sum_{i=1}^m\eta^i+\{\eta^{m+1}-\eta+2\delta\sum_{i=1}^m\eta^i\}\frac{(1-\alpha-3\delta)}{2}+\frac{1+\alpha-4\delta}{2}$$

$$\geq\frac{1-\alpha-4\delta}{2}\textstyle\sum_{i=1}^m\eta^i+\{1+\eta^{m+1}-\eta+2\delta\sum_{i=1}^m\eta^i\}\frac{(1-\alpha-4\delta)}{2}$$

$$\geq\frac{1-\alpha-4\delta}{2}\textstyle\sum_{i=1}^{m+1}\eta^{i+1}+\{1-\eta+2\delta\sum_{i=1}^m\eta^i\}\frac{(1-\alpha-4\delta)}{2}$$

$$\geq \frac{(1-\alpha-4\delta)}{2}\sum_{i=1}^{m+1}\eta^{i+1},$$

which shows the required inequality. Accordingly, putting $m=[N/h]$ in (2.1.17), we obtain

$$\bar{G}_n^{[N/h]*}(\sqrt{m_n}\epsilon l_0)\geq \frac{(1-\alpha-4\delta)}{2}\sum_{i=1}^{[N/h]}\eta^i$$

$$=\frac{(1-\alpha-4\delta)}{2}(1-\eta^{([N/h]+1)})/(1-\eta)\geq \frac{(1-\alpha-4\delta)}{2}\frac{1-\eta^{N/h}}{1-\eta}.$$

By (2.1.13), we easily see

$$\bar{G}^{[N/h]*}(\sqrt{m_n}\epsilon l_0)\geq \frac{(1-\alpha-4\delta)}{2}\sum_{i=1}^{[N/h]}\eta^i=\frac{(2-\beta)}{4}. \tag{2.1.18}$$

Put

$$\lambda=\sqrt{\epsilon^2\beta/2(1-\beta)}. \tag{2.1.19}$$

By (2. 1. 18) and the diminution of the mean concentration functions by the convolution of distribution functions

$$1-\varphi_n(\sqrt{m_n}\lambda l_0)=2\int_0^\infty \frac{x^2}{(\sqrt{m_n}\lambda l_0)^2+x^2}dG_n^{m_n*}(x)$$

$$\geq 2\int_0^\infty \frac{x^2}{(\sqrt{m_n}\lambda l_0)^2+x^2}dG_n^{[N/h]*}(x)\geq 2\int_{\sqrt{m_n}\epsilon l_0}^\infty \frac{x^2}{(\sqrt{m_n}\lambda l_0)^2+x^2}dG^{[N/h]*}(x)$$

$$\geq \frac{2\epsilon^2}{\epsilon^2+\lambda^2}\bar{G}_n^{[N/h]*}(\sqrt{m_n}\epsilon l_0)\geq \frac{\epsilon^2(2-\beta)}{2(\epsilon^2+\lambda^2)}.$$

By (2.1.19)

$$\frac{(2-\beta)\epsilon^2}{2(\epsilon^2+\lambda^2)}=\frac{(2-\beta)\epsilon^2}{2(\epsilon^2+\epsilon^2\beta/2(1-\beta))}=1-\beta.$$

Whence we have

$$\varphi_n(\sqrt{m_n}\lambda l_0)\geq\beta,\quad n\geq N.$$

From the definition of λ it follows that λ is a constant depending only on α and β. Thus putting $K=Min(\lambda, \tau)$, we can conclude

$$\varphi_{F_{n1}*\cdots\cdots*F_{nm_n}}(\sqrt{m_n}Kl_0)\leq\beta$$

for n satisfying (2.1.4).

Corollary 3. 1. 1. *Given a sequence of mutually independent random variables* $\{X_k|k=1,2,\cdots\cdots\}$ *and given two positive numbers* α *and* β, $(0<\alpha<1,\ 0<\beta\leq 1)$, *then there exist two positive numbers* K *and* N—*both depending only on* α *and* β—*having the following properties if* $n\geq N$ *and if* $\{F_1(x), F_2(x),\cdots\cdots, F_n(x)\}$ *is a system of the distribution function of* $X_1, X_2,\cdots\cdots, X_n$ *satisfying*

$$\Psi_{F_m}(l_0)\leq\alpha, \quad (m=1,2,\cdots\cdots,n)$$

where l_0 *is an arbitrary but fixed positive number, then*

$$\Psi_{F_1*\cdots\cdots*F_n}(\sqrt{n}\,Kl_0)\leq\beta.$$

If we replace Ψ_F in this corollary by Q_F, we have P. Lévy—W. Doeblin's theorem(29). However, in our case, we could decide the values of the constants N and K which have not been concretely given by them.

§**2. 2. The uniform diminution theorem concerning a typical function.** A typical function $\Phi_F(h)$ does not necessarily diminish by the convolution of distribution functions. For example, put $F(x)=0$ for $x<-1$, $=1/2$ for $-1\leq x<1$ and $=1$ for $1\leq x$. Then the characteristic function $f(t)$ is $\cos t$, Hence $\Re f(t)=\cos t$. Consequently

$$\Phi_F(h)=h\int_0^\infty e^{-ht}\Re f(t)dt=h\int_0^\infty e^{-ht}\cos t\,dt=\frac{h^2}{h^2+1}$$

and

$$\Phi_{F*F}(h)=h\int_0^\infty e^{-ht}\Re f(t)dt=h\int_0^\infty e^{-ht}\cos^2 t\,dt$$

$$=h\int_0^\infty e^{-ht}\frac{(\cos(2t)+1)}{2}dt=\frac{1}{2}\left(\frac{h^2}{h^2+4}+1\right).$$

Therefore we have for $h<\sqrt{2}$

$$\Phi_F(h)=\frac{h^2}{h^2+1}<\frac{1}{2}\left(\frac{h^2}{h^2+4}\right)=\Phi_{F*F}(h).$$

However, we can show an analogous theorem to the previous section.

Given a system of random variables $||X_{nm}||$ defined by (2.1.1) and let $F_{nm}(x)$ be the distribution of X_{nm}. And furthermore, applying

(29) P. Lévy—W. Doeblin, loc. cit. 5).

the continuity and the monotone of $\Phi_F(h)$, define the following function $D_n(\alpha)$:

$$\alpha=\Phi_{F_{n1}*\cdots\cdots*F_{nm_n}}(D_n(\alpha)), \quad (1>\alpha>0). \tag{2.2.1}$$

Then we have

Theorem 2. 2. 1. *Given two real numbers α and β $(3/4<\alpha\leq 1,\ 7/8<\beta\leq 1)$, we can determine two positive numbers K and N—both depending only on α and β—having the following properties: if $m_n>N$, $D_n(\beta)\geq l_0$ and if $F_{nm}(x)$ $(m=1,2,\cdots\cdots,m_n)$ is a set of distribution function each median of which is zero, and satisfy*

$$\Phi_{F_{nm}}(l_0)=\alpha,\quad m=1,2,\cdots\cdots,m_n, \tag{2.2.2.}$$

where l_0 is an arbitrary but fixed positive number, then we have

$$D_n(\beta)\geq\sqrt{m_n}\,l_0K \quad m_n>N \tag{2.2.3}$$

and consequently

$$\Phi_{F_{n1}*\cdots\cdots*F_{nm_n}}(\sqrt{m_n}\,l_0K)\leq\beta. \tag{2.2.4}$$

Proof. From (F. I. 1),

$$\begin{aligned}1-\beta&=1-\Phi_{F_{n1}*\cdots\cdots*F_{nm_n}}(D_n(\beta))\geq\frac{1}{2}(1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(D_n(\beta)))\\ &\geq\frac{1}{4}\left\{\int_{|x|\leq D_n(\beta)}x^2/D_n^2(\beta)d\tilde{F}_{n_1}*\cdots\cdots*\tilde{F}_{nm}+\int_{|x|>D_n(\beta)}d\tilde{F}_{n_1}*\cdots\cdots*\tilde{F}_{nm_n}(x)\right\}.\end{aligned} \tag{2.2.5}$$

While

$$1-\Pi_{m=1}^{m_n}\left|f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2=\int_{-\infty}^{\infty}\left(1-\cos\frac{tx}{D_n(\beta)}\right)d\tilde{F}_{n_1}*\cdots\cdots*\tilde{F}_{nm_n}$$

$$\leq t^2\int_{|x|\leq D_n(\beta)}\frac{x^2}{2D_n^2(\beta)}d\tilde{F}_{n_1}*\cdots\cdots*\tilde{F}_{nm_n}+2\int_{|x|>D_n(\beta)}d\tilde{F}_{n_1}*\cdots\cdots*\tilde{F}_{nm_n}.$$

Hence we have for $0\leq t\leq 2$, applying $1\geq\beta\geq 7/8$ and (2.2.5)

$$1-\Pi_{m=1}^{m_n}\left|f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2$$

$$\leq 2\left\{\int_{|x|\leq D_n(\beta)} x^2/D_n^2(\beta)d\widetilde{F}_{n1}*\cdots\cdots*\widetilde{F}_{nm_n}(x)+\int_{|x|>D_n(\beta)} d\widetilde{F}_{n1}*\cdots\cdots*\widetilde{F}_{nm_n}(x)\right\}\leq 8(1-\beta)<1.$$

Whence we obtain for $0\leq t\leq 2$

$$0<\delta=1-8(1-\delta)\leq \Pi_{m=1}^{m_n}|f_{nm}(t/D_n(\beta))|^2.$$

By appealing to (F. I. 5) and (2.2.5) we see

$$\sum_{m=1}^{m_n}(1-\Psi_{F_{nm}}(D_n(\beta)))\leq K(1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(D_n(\beta))) \tag{2.2.6}$$

$$\leq 2(1-\beta)C$$

where C is a constant. On the other hand, from the assumption (2.2.2) and $D_n(\beta)\geq l_0$

$$1-\alpha=1-\Phi_{F_{nm}}(l_0)\geq 1-\Phi_{F_{nm}}(D_n(\beta))$$

$$\geq\int_{|x|>\gamma D_n(\beta)} dF_{nm}(x).\frac{\gamma^2}{1+\gamma^2},$$

where γ satisfies $1/4>(1+\gamma^2)(1+\alpha)/\gamma^2(\equiv\xi)$. Hence by (F. I. 3), we see

$$1-\Psi_{F_{nm}}(\gamma D_n(\beta))\geq K(\xi)(1-\Phi_{F_{nm}}(\gamma D_n(\beta)))$$

$$\geq K(\xi)/(\gamma^2+1)\left\{\int_{|x|\leq D_n(\beta)} x^2/D_n^2(\beta)dF_{nm}(x)+\int_{|x|>D_n(\beta)} dF_{nm}(x)\right\}$$

$$\geq K(\xi)/(\gamma^2+1)\left\{\int_{|x|\leq D_n(\beta)} x^2/(D_n^2(\beta)+x^2)dF_{nm}(x)+\int_{|x|>D_n(\beta)} x^2/(D_n^2(\beta)+x^2)dF_{nm}(x)\right.$$

$$\geq K(\xi)/(\gamma^2+1)\{1-\Phi_{F_{mn}}(D_n(\beta))\},\quad n=1,2,\cdots\cdots;\ m=1,2,\cdots\cdots,\ m_n.$$

While

$$1-\Psi_{F_{nm}}(\gamma D_n(\beta))\leq\frac{1}{\gamma^2}\int_{|x|\leq D_n(\beta)} x^2/D_n^2(\beta)d\widetilde{F}_{nm}(x)+\int_{|x|>D_n(\beta)} d\widetilde{F}_{nm}(x)$$

$$\leq\frac{2}{\gamma^2}\int_{|x|\leq D_n(\beta)} x^2/(D_n^2(\alpha)+x^2)d\widetilde{F}_{nm}(x)+2\int_{|x|>D_n(\beta)} x^2/(D_n^2(\beta)+x^2)d\widetilde{F}_{nm}(x)$$

$$\leq\{2(\gamma^2+1)/\gamma^2\}\{1-\Psi_{F_{nm}}(D_n(\beta))\},\ n=1,2,\cdots\cdots,\ m=1,2,\cdots\cdots,\ m_n.$$

Since ξ and γ depend only on α, we may put $K(\xi)\gamma^2/2(\gamma^2+1)^2=K'(\alpha)$. Thus we have

$$1-\Psi_{F_{nm}}(D_n(\beta))\geq K'(\alpha)\{1-\Phi_{F_{nm}}(D_n(\beta))\}$$

$$n=1, 2, \cdots\cdots; \ m=1, 2, \cdots\cdots, m_n.$$

Hence by (2.2.6) and (2.2.2) we obtain

$$\frac{2(1-\beta)C}{K'(\alpha)}\geq\sum_{m=1}^{m_n}\{1-\Phi_{F_{mn}}(D_n(\beta))\}$$

$$\geq\sum_{m=1}^{m_n}\frac{l_0^2}{D_n^2(\beta)+l_0^2}\left\{\int_{|x|\leq l_0}x^2/l_0^2dF_{nm}(x)+\int_{|x|>l_0}dF_{nm}(x)\right\}$$

$$\geq\frac{l_0^2}{D_n^2(\beta)+l_0^2}\sum_{m=1}^{m_n}\{1-\Phi_{F_{nm}}(l_0)\}\geq\frac{m_n l_0}{D_n^2(\beta)+l_0^2}(1-\alpha).$$

Hence

$$D_n(\beta)\geq\sqrt{m_n}\,l_0\sqrt{\frac{(1-\alpha)K(\alpha)}{2(1-\beta)C}-\frac{1}{m_n}}$$

Thus if $m_n>N(\alpha, \beta)\equiv4(1-\beta)C/(1-\alpha)K'$, we have $D_n(\beta)\geq\sqrt{m_n}\,l_0K$, where

$$K\equiv\sqrt{\frac{(1-\alpha)K'(\alpha)}{4(1-\beta)C}}.$$

Consequently we have

$$\Phi_{F_{n1}*\cdots\cdots*F_{nm_n}}(\sqrt{m_n}\,l_0K)\leq\beta,$$

where K is a constant depending only on α and β. This completes the proof of this theorem.

Chapter III. The convergence problems concerning the series of independent random variables

§ 3. 1. Convergence criteria of the series of independent random variables. Let $\{X_k|k=1, 2, \cdots\cdots\}$ be a sequence of independent random variables and let $F_k(x)$ be the distribution of $X_k(k=1, 2, \cdots\cdots)$. Put

$$\Psi_{nN}(h)=\sum_{k=n+1}^{N}(1-\Psi_{F_k}(h)).$$

As $\Psi_{nN}(h)$ increases with N and decreases when n increases, the equations

$$\Psi_n = \lim_{N\to\infty} \Psi_{nN}(h),$$

$$\Psi = \lim_{n\to\infty} \Psi_n(h)$$

define a limit function $\Psi(h)$.

Theorem 3. 1. 1. *The function $\Psi(h)$ is independent of h, namely identically zero or infinite.*

Proof. The series with positive terms such as

$$((3.1.1) \qquad \sum_{k=1}^{\infty}\{1-\Psi_{F_k}(h)\}$$

is convergent or divergent. If (3.1.1) is convergent for a fixed number $h>0$, then (3.1.1) is also convergent for h' $(h'\neq h,\ \infty>h'>0)$. In fact, if $h'\geq h$, it is evident. Hence it is sufficient to consider the case $h'<h$,

$$\infty > \sum_{k=1}^{\infty}\{1-\Psi_{F_k}(h)\} = \left(\frac{h'}{h}\right)^2 \sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{(h')^2+(h'x/h)^2}d\widetilde{F}_k(x)$$

$$\geq \left(\frac{h'}{h}\right)^2 \sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{(h')^2+x^2}d\widetilde{F}_k(x) = \left(\frac{h'}{h}\right)^2 \sum_{k=1}^{\infty}\{1-\Psi_{F_k}(h').$$

Consequently if (3.1.1) is convergent for some $h>0$, then holds identically. In the same way, we can show that if (3.1.1) is divergent for a number h, then (3.1.1) is also divergent for h' $(h\neq h',\ \infty>h'>0)$ and $\Psi(h)=\infty$ for every $h>0$.

Theorem 3. 1. 2. *If $\Psi(h)\equiv 0$, there exists a sequence $\{a_k|k=1, 2,\cdots\cdots\}$ so that $\sum(X_k-a_k)$ converges in probability, and if $\Psi(h)\equiv\infty$, then there exists no sequnce having the above property.*

Proof. First suppose $\Psi(h)\equiv 0$, then we have for any $\epsilon>0$ $\sum(1-\Psi_{F_k}(\epsilon))<\infty$. Now let $X_1, \bar{X}_1, X_2, \bar{X}_2,\cdots\cdots$ be mutually independent random variables, X_k and $\bar{X}_k$ having the same distribution function $F_k(x)$ $(k=1, 2,\cdots\cdots)$, and put $\widetilde{X}_k=X_k-{}_k\bar{X}$, $\widetilde{S}_{nN}=\widetilde{X}_{n+1}+\cdots\cdots+\widetilde{X}_N$. Then by (F.I. 5)

$$(3.1.2) \qquad \sum_{k=n+1}^{N}\{1-\Psi_{F_k}(\epsilon)\} \geqq \{1-\Psi_{F_{n+1}*\cdots\cdots*F_N}(\epsilon)\} \geqq \frac{1}{2}P_r\{|\widetilde{S}_{nN}|\geqq\epsilon\}.$$

Whence, denoting by M_{nN} the median of $S_{nN}=X_{n+1}+\cdots\cdots+X_N$,

$$P_\gamma\{|\tilde{S}_{nN}|>\epsilon\}=P_\gamma\{\tilde{S}_{nN}>\epsilon\}+P_\gamma\{\tilde{S}_{nN}<-\epsilon\}$$
$$\geq P_\gamma\{|S_{nN}-M_{nN}-\bar{S}_{nN}-M_{nN})<-\epsilon\}\frown(\bar{S}_{nN}-M_{nN}\leq 0)\}$$
$$+P_\gamma\{(S_{nN}-M_{nN}-(\bar{S}_{nN}-M_{nN})<-\epsilon)\frown(\bar{S}_{nN}-M_{nN}\geq 0)\}$$
$$\geq\frac{1}{2}P_\gamma\{|S_{nN}-M_{nN}|>\epsilon\}..$$

By (3.1.2) and the convergence of $\sum(1-\Psi_{F_k}(\epsilon))$, we have for any $\epsilon>0$

$$\lim_{n,N\to\infty}P_\gamma\{|S_{nN}-M_{nN}|\geq\epsilon\}=0, \tag{3.1.3}$$

from which we can show the existence of a sequence $\{a_k\,|\,k=1,2,\cdots\cdots\}$ so that $\sum(X_k-a_k)$ converges in probability. In fact, select two sequences $\{\epsilon_k>0|k=1,2,\cdots\cdots;\ \sum\epsilon_k<\infty\}$ and $0=N_0<N_1<N_2<\cdots\cdots<N_k<N_{k+1}<\cdots\cdots$ such that for $n,\ N\geq N_k$

$$P_\gamma\{|S_{nN}-M_{nN}|\geq\epsilon_k\}\leq\epsilon_k.$$

Now define a sequence $\{a_k|k=1,2,\cdots\cdots\}$ successively so that the medians of $\sum_{i=n}^{N_k}(X_i-a_i)$ are zero $(n=N_{k-1}+1,\ N_{k-1}+2,\cdots\cdots,\ N_k;\ k=1,2,\cdots\cdots)$,[30] Then we have for $N_p<n<N_{p+1}$ and $N_q<N\leq N_{q+1}$ $(n<N,\ p\leq q)$

$$P_\gamma\{|\textstyle\sum_{k=1}^{N}(X_k-a_k)-\sum_{k=1}^{n}(X_k-a_k)|>\sum_{k=p-1}^{q}\epsilon_k\}$$
$$=\mathrm{P}_\gamma\{|\textstyle\sum_{n+1}^{N_{p+1}}+\cdots\cdots+\sum_{N_{q-1}+1}^{N_q}+\sum_{N_{q+1}}^{N_{q+1}}-\sum_{N+1}^{N_{q+1}}|>\sum_{k=p-1}^{q}\epsilon_k\}$$
$$\leq\mathrm{P}_\gamma\{|\textstyle\sum_{n+1}^{N_{p+1}}|\geq\epsilon_{p-1}\}+\cdots\cdots+\mathrm{P}_\gamma\{|\sum_{N+1}^{N_{q+1}}|>\epsilon_{q-1}\}+\mathrm{P}_\gamma\{|\sum_{N_q+1}^{N_{q+1}}|\geq\epsilon_q\}$$
$$\leq\epsilon_{p-1}+\epsilon_p+\cdots\cdots+\epsilon_{q-1}+\epsilon_q.$$

Hence the fact $\sum\epsilon_k<\infty$ implies the convergence in probability of $\sum(X_k-a_k)$.

In order to show the converse, it is sufficient to prove that if there exists a sequence $\{a_k|k=1,2,\cdots\cdots\}$ so that $\sum(X_k-a_k)$ converges in probability, we have $\sum(1-\Psi_{F_k}(1))<\infty$. As $\sum(X_k-a_k)$ converges in probabilty, we have

$$\Pi_{k=n+1}^{N}f_k(t)\exp(-ia_kt)\to 1$$

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[30] That is, in the first place, define a_{N_k} $(k=1,2,\cdots\cdots)$ so that the medians of $X_{N_k}-a_{N_k}(k=1,2,\cdots\cdots)$ are zero, and next define a_{N_k-1} $(k=1,2,\cdots\cdots)$ so that the medians of $X_{N_k}-a_{N_k}+X_{N_k-1}-a_{N_k-1}$ $(k=1,2,\cdots\cdots)$ are zero, etc.

$(n,\ N\to\infty)$ uniformly for every finite interval. Whence we see $\Pi_{n+1}^{N}|f_k(t)|^2\to 1$ $(n,\ N\to\infty)$ in the same sense as above. Therefore there exists a number N_1 such that $\Pi_{k=n+1}^{N}|f(t)|^2\geq 1/2$ for $n,\ N\geq N_1$, and $0\leq t\leq 2$. Hence by (F. I. 5) we get

$$\sum_{k=n+1}^{N}\{1-\Psi_{F_k}(1)\}\leq K\{1-\Psi_{F_{n+1}*\cdots\cdots*F_N}(1)\}$$

for $n,\ N\geq N_1$, where K is a constant, from which by $\Psi_{F_{n+1}*\cdots\cdots*F_N}(1)\to 1$ $(n,\ N\to\infty)$ we have $\sum_{k=1}^{\infty}(1-\Psi_{F_k}(1))<\infty$. This completes the proof of Theorem 3. 1. 2.

Theorem 3. 1. 2 can be readily reworded as follows:

Corollary 3. 1. 1. *A necessary and sufficient condition that there should exist a sequence* $\{a_k|k=1, 2,\cdots\cdots\}$ *so that* $\sum(X_k-a_k)$ *converges in probability is*

$$\sum_{k=1}^{\infty}\{1-\Psi_{F_k}(h)\}<\infty$$

for every $h>0$.

Here we remark another criterion of convergence of the sums of independent random variables according to the notion of P. Lévy [31] and T. Kawata [32]. As $\Psi_{F_{n+1}*\cdots\cdots*F_N}(h)$ increases with n and decreases when N increases, we can define

$$\Psi_n^*(h)=\lim_{N\to\infty}\ \Psi_{F_{n+1}*\cdots\cdots*F_N}(h)$$

$$\Psi^*(h)=\lim_{n\to\infty}\ \Psi_n^*(h).$$

Concerning $\Psi^*(h)$, there exists the following theorem.

Theorem 3. 1. 3. $\Psi^*(h)$ *is identically* 0 *or* 1. *If* $\Psi^*(h)\equiv 1$, *we have a sequence* $\{a_k|k=1, 2,\cdots\cdots\}$ *so that* $\sum(X_k-a_k)$ *converges in probability. If* $\Psi^*(h)\equiv 0$, *then there does not exist such a seqnence.*

§ 3. 2. The three series theorem and the related theorems. In the first place, we shall consider the following theorem.

Theorem 3. 2. 1. *A necessary and sufficient condition that*

$$\sum\{X_k-a_k\} \tag{3.2.1}$$

should converge in probablity and

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(31) P. Lévy, loc. cit. 6).

(32) T. Kawata, loc. cit. 2).

(3.2.2) $$X_k \to 0 \ (k \to \infty)$$

in probability is that

(3.2.3) $$\sum\{1-\Phi_{F'_k}(1)\}<\infty,$$

and

(3.2.4) $$a_k \to 0, \ (k \to \infty)$$

where $F'_k(x)=F_k(x+a_k)$ *and* $a_k=\int_{-1}^{1} x\,dF_k(x)$.

Proof. Sufficiency. The tendency to zero in probability of $\{X_k\}$ as $k\to\infty$ follows easily from (3.2.4) and $1-\Phi_{F'_k}(1)\to 0(k\to\infty)$. By (F.I.6)

(3.2.5) $$\sum_{k=n+1}^{N}|f'_k(t)\exp(-ita'_k)-1|\leq(t^2+2|t|+4)\sum_{k=n+1}^{N}\{1-\Phi_{F'_k}(1)\}$$

Since we see by (3.2.2) and (3.2.5) that

$$|\log \Pi_{k=n+1}^{N} f(t)\exp(-ita'_k)-\sum_{k=n+1}^{N}\{f'_k(t)\exp(-ita'_k)-1\}$$
$$\leq\sum_{k=n+1}^{N}|f'_k(t)\exp(-ita'_k)-1|^2$$

for sufficiently large n and $N(n<N)$, we have

(3.2.6) $$\Pi_{k=n+1}^{N} f'_k(t)\exp(-ita'_k)\to 1, \ (n, \ N\to\infty)$$

uniformly for every finite interval. But we have

(3.2.7) $$f'_k(t)\exp(-ita'_k)=f_k(t)\exp\{-it(a_k+a'_k)\}$$

and

$$|a'_k|=\left|\int_{-1}^{1} x\,dF'_k(x)\right|=\left|\int_{-1+a_k}^{1+a_k}(x-a_k)dF_k(x)\right|$$

$$\leq\left|a_k-a_k\int_{|x-a_k|\leq 1}dF_k(x)\right|+\left|\int_{|x-a_k|\leq 1}x\,dF_k(x)-a_k\right|$$

$$\leq|a_k|\int_{|x-a_k|>1}dF_k(x)+(1+|a_k|)\int_{|x|>1-|a_k|}dF_k(x)$$

$$\leq|a_k|\int_{|x|>1}dF'_k(x)+(1+|a_k|)\int_{|x|>1-|a_k|}dF'_k(x).$$

By the assumption (3.2.4) we have $|a_k|<1/4$ for sufficiently large $k\geq N_0>0$. Hence

$$|a'_k|\leq\frac{1}{4}\int_{|x|>1}dF'_k(x)+\frac{5}{4}\int_{|x|>1/2}dF'_k(x).$$

Hence by $\sum\{1-\Phi_{F'_k}(1)\}<\infty$, we see

$$\sum_{k=1}^{\infty}|a'_k|<\infty, \tag{3.2.8.}$$

Since by (3.2.6) and (3.2.7) we have

$$P_\gamma\{|\sum_{k=n+1}^{N}\{X_k-a_k-a'_k\}|>\epsilon\}\to 0,\ (n,\ N\to\infty)$$

for any $\epsilon>0$, we obtain the convergence in probablity of $\sum\{X_k-a_k-a'_k\}$ and so $\sum\{X_k-a_k\}$.

Necessity. By the tendency to zero in probablility of $\{X_k\}$, we see (3.2.4). Hence by (F. I. 2) for sufficiently large n and $N(N>n\geq N_0)$

$$\sum_{k=n+1}^{N}\{1-\Psi_{F_k}(1)\}\geq R\sum_{k=n+1}^{N}\{1-\Phi_{F'_k}(1)\},$$

where R is a constant. While by Corollary 3. 1. 1 we have $\sum\{1-\Psi_{F_k}(1)\}<\infty$, from which we obtain the convergence of $\sum\{1-\Phi'_{F_k}(1)\}$.

Theorem 3. 2. 2. *The necessary and sufficient condition of Theorem* 3. 2. 1. *can be replaced by the following*

$$\sum\{1-\Phi_{F_k^*}(1)\}<\infty \tag{3.2.9}$$

and

$$M_k\to 0\ (k\to\infty), \tag{3.2.10}$$

where M_k is the median of $F_k(x)$ and $F_k^*(x)=F_k(x+M_k)$.

The proof is almost analogous to that of the previous theorem except the following fact. Under the assumption (3.2.8) and (3.2.9), by (F. I. 6) we see

$$\sum_{k=n+1}^{N}|f_k^*(t)\exp(-ita_k^*)-1|\leq(t^2+2|t|+4)\sum_{k=n+1}^{N}\{1-\Phi_{F_k^*}(1)\},$$

where

$$f_k^*(t)=\int_{-\infty}^{\infty}e^{itx}dF_k^*(x),\quad a_k^*=\int_{-1}^{1}xdF_k^*(x).$$

Hence we have for sufficiently large n and N $(N>n\geq N_0)$

$$|\log \Pi_{k=n+1}^{N} f_k^*(t) \exp(-ita_k^*) - \sum_{k=n+1}^{N} \{ita_k^*) - 1\}|$$
$$\leq \sum_{k=n+1}^{N} |f_k^*(t) \exp(-ita_k^*) - 1|^2.$$

Hence

$$\Pi_{k=n+1}^{N} f_k^*(t) \exp(-ita_k^*) \to 1, \ (n, \ N \to \infty)$$

uniformly for every finite interval and so for any $\epsilon > 0$

(3.2.11) $$P_Y\{|\sum_{k=n+1}^{N}(X_k - M_k - a_k^*)| \geq \epsilon\} \to 0, \ (n, \ N \to \infty).$$

On the other hand

$$|M_k + a_k^* - a_k| \leq |M_k| \int_{|x-M_k|>1} dF_k(x) + (1+|M_k|) \int_{|x|>1-|M_k|} dF_k(x)$$
$$\leq |M_k| \int_{|x|>1} dF_k^*(x) + (1+|M_k|) \int_{|x|>1-|M_k|} dF_k^*(x)$$

and as by (3.2.10) we have $|M_k| < 1/4$ for sufficiently large $k \geq N_0 > 0$ we get

$$|M_k + a_k^* - a_k| \leq \frac{1}{2}\{1 - \Phi_{F_k^*}(1)\} + \frac{25}{4}\{1 - \Phi_{F_k^*}(1)\}$$

(3.2.12) $$\leq 7\{1 - \Phi_{F_k^*}(1)\}.$$

Hence we see $\sum |M_k + a_n^* - a_k| < \infty$, from which by (3.2.11) we can conclude the convergence in probability of $\sum (X_k - a_k)$.

Theorem 3. 2. 3. *A sufficient condition that* (3.2.1) *should converge in probability is*

(3.2.13) $$\sum \{1 - \Phi_{F_k}(1)\} < \infty,$$

and it is also necessary if the condition

(3.2.14) $$F_k(+0) \geq \lambda > 0, \ F_k(-0) \leq 1 - \lambda, \ (0 < \lambda < 1)$$
$$k = 1, 2 \cdots\cdots$$

is satisfied.

Proof. Sufficiency. By (F. I. 6) we see

(3.2.14) $$\sum_{k=n+1}^{N} |f_k(t) - 1| \leq (t^2 + 2|t| + 4) \sum_{k=n+1}^{N} \{1 - \Phi_{F_k}(1)\},$$

where $f'_k(t)=f_k(t)\exp(-ita_k)$ and $a_k=\int_{-1}^{1} x dF_k(x)$. Hence (3.2.13) implies that

$$\sum_{k=n+1}^{N}|f'_k(t)-1|\to 0,\ (n,\ N\to\infty)$$

uniformly for every finite interval. Since

$$|\log \Pi_{k=n+1}^{N} f'_k(t)-\sum_{k=n+1}^{N}(f'_k(t)-1)|\leq\sum_{k=n+1}^{N}|f'_k(t)-1|^2$$

for sufficiently large values of n and N, we see $\Pi_{k=n+1}^{N} f'_k(t)\to 1\ (n,\ N\to\infty)$ uniformly for every finite interval. Hence for any $\epsilon>0$

$$P_\gamma\{|\sum_{k=n+1}^{N}(x_k-a_k)|>\epsilon\}\to 0,\ (n,\ N\to\infty)$$

which shows the convergence in probability of $\sum(X_k-a_k)$.

Necessity. By Corollary 3.1.1, the convergence in probability of (3.2.1) implies $\sum\{1-\Psi_{F_k}(1)\}<\infty$. Hence for any $\epsilon>0$

$$\frac{1}{2}P_\gamma\{|\tilde{X}_k|\geq\epsilon\}\leq 1-\Psi_{F_k}(\epsilon)\to 0,\ (k\to\infty),$$

where $\tilde{X}_k=X_k-\bar{X}_k$, X_k and $\bar{X}_k$ being independent random variables which have the same distribution $F_k(x)$. Hence as

$$\begin{aligned} P_\gamma\{|\tilde{X}_k|\geq\epsilon\} &= P_\gamma\{|X_k-\bar{X}_k|\geq\epsilon\} \\ &= P_\gamma\{X_k-\bar{X}_k\geq\epsilon)\frown(\bar{X}_k\leq 0)\}+P_\gamma\{(X_k-\bar{X}_k\leq-\epsilon)\frown(\bar{X}_k\geq 0)\} \\ &\geq P_\gamma\{(X_k\geq\epsilon)\frown(\bar{X}_k\leq 0)\}+P_\gamma\{(X_k\leq-\epsilon)\frown(\bar{X}_k\geq 0)\} \\ &\geq\lambda P_\gamma\{|X_k|\geq\epsilon\}, \end{aligned}$$

we see $P_\gamma\{|X_k|\geq\epsilon\}\to 0\ (k\to\infty)$ for any $\epsilon>0$. Hence by (F. I. 3) for sufficiently large values of n and N $(n<N)$

$$\sum_{k=n+1}^{N}\{1-\Psi_{F_k}(1)\}\geq R\sum_{k=n+1}^{N}\{1-\Phi_{F_k}(1)\},$$

where R is a constant, which completes the proof.

According to A. Khintcine—A. Kolomogoroff(33), it is known that the simultaneous convergence of the following *three series*

(33) A. Khintchine—A. Kolmogoroff, loc. cit. 7).

(3.2.16) $$\sum_{k=1}^{\infty}\int_{|x|>1} dF_k(x),\ \sum_{k=1}^{\infty}\left\{\int_{|k|\leq 1} x^2 dF_k(x)-\left(\int_{|k|\leq 1} x dF_k(x)\right)^2\right\}$$

and

(3.2.17) $$\sum_{k=1}^{\infty}\int_{|x|\leq 1} x dF_k(x)$$

is a necessary and sufficient condition for the convergence in probability of $\sum X_k$. By appealing to our typical function, the above condition can be replaced by the following equivalent condition

(3.2.18) $$\sum\{1-\Phi_{F'_k}(1)\}<\infty$$

with the convergence of (3.2.17), where $F'_k(x)=F_k(x+a_k)$ and $a_k=\int_{-1}^{1} x\, dF_k(x)$. That is,

Theorem 3. 2. 4. *A necessary and sufficient condition for the convergence in probability of* $\sum X_k$ *is the existence of* (3.2.18) *together with the convergence of* (3.2.17).

Proof. Sufficiency follows clearly from Theorem 3. 2. 1. Conversely the convergence of $\sum\{1-\Phi_{F'_k}(1)\}$ is obtained from the proof of Theorem 3.2.1. Hence we have, using again the proof of the same theorem $\left(\text{putting } a_k=\int_{-1}^{1} x\, dF(x)\right)$,

$$\log \Pi_{k=n+1}^{N} f_k(t) \exp(-ita_k)=\log \Pi_{k=n+1}^{N} f_k(t)-it\left(\sum_{k=n+1}^{N} a_k\right)$$

$$\to 0,\ (n,\ N\to\infty)$$

uniformly for every finite interval. While by the assumption we see $\log \Pi_{k=n+1}^{N} f_k(t)\to 0$ $(n,\ N\to\infty)$ in the same sense as above. Thus we can conclude the convergence of $\sum a_k$.

Corresponding to Theorem 3.2.2, we can give

Theorem 3. 2. 5. *A necessary and sufficient condition for the convergence in probability of* $\sum X_k$ *is the existence of* (3.2.8) *and* (3.2.9) *together with the convergence of* $\sum\int_{-1}^{1} x\, dF(x)$.

The proof is almost clear from Theorem 3. 2. 2.

Chapter IV. The law of large numbers.

§ 4. 1. The law of large numbers. Let $||X_{nm}||$ be a system of random variables defined by (2.1.1). Then it is called that $||X_{nm}||$ obeys *the law of large numbers* if there exist two sequences of real numbers $\{0<A_n|n=1,2,\cdots\cdots\}$ and $\{-\infty<B_n<\infty\,|\,n=1,2,\cdots\cdots\}$ such that

(4.1.1) $$P_\gamma\{|(X_{n1}+X_{n2}+\cdots\cdots+X_{nm_n})/A_n-B_n|\geq\eta\}\to 0,\ (n\to\infty)$$

for any $\eta>0$.

Therem 4. 1. 1. *A necessary and sufficient condition that* $||X_{nm}||$ *should obey the law of large numbers is that there exists a sequence* $\{A_n>0|$ $n=1,2,\cdots\cdots\}$ *satisfyiug*

(4.1.2) $$\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(A_n)\}\to 0,\ (n\to\infty)$$

Proof. Necessity. From (4.1.1) we see

(4.1.3) $$\Pi_{m=1}^{m_n}|f_{nm}(t/A_n)|^2\to 1,\ (n\to\infty)$$

uniformly for every finite interval, and so there exists $N_0>0$ such that $\Pi_{m=1}^{m_n}|f_{nm}(t/A_n)|^2\geq 1/2$ for $n>N_0$ and $0\leq t\leq 2$. Hence by (F. I. 5) we get

$$\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(A_n)\}\leq K\{1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(A_n)\}$$

for $n\geq N_0$, where K is a constant. Whence (4.1.2) clearly follows from (4.1.3).

Sufficiency. It is easily shown that (4.1.2) implies that

(4.1.4) $$\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(\eta A_n)\}\to 0,\ (n\to\infty)$$

for any $\eta>0$. Let X_{nm}, $\bar{X}_{nm}(n=1,2,\cdots\cdots;\ m=1,2,\cdots\cdots m_n)$ be mutually independent random variables and suppose X_{nm} and $\bar{X}_{nm}$ have the same distribution $F_{nm}(n=1,2,\cdots\cdots;\ m=1,2,\cdots\cdots m_n)$. And furthermore put $\tilde{X}_{nm}=X_{nm}-\bar{X}_{nm}$. Then by apealing to (F. I. 4), we have

(4.1.5) $$\sum_{m=1}^{m_n}\{1-\Psi_{F_{mn}}(\eta A_n)\}\geq\{1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(\eta A_n)\}$$

$$\geq\frac{1}{2}P_\gamma\{|\sum_{m=1}^{m_n}\tilde{X}_{nm}|\geq\eta A_n\}.$$

Hence, denoting by M_n the median of $F_{n1}*F_{n2}*\cdots\cdots*F_{nm_n}$, from

$$\sum_{m=1}^{m_n}\tilde{X}_{nm}=\sum_{m=1}^{m_n}X_{nm}-M_n-(\sum_{m=1}^{m_n}\bar{X}_{nm}-M_n),$$

we have

$$P_\gamma\{|\sum_{m=1}^{m_n}\tilde{X}_{nm}|\geq\eta A_n\}$$

$$\geq P_\gamma\{(\sum_{m=1}^{m_n}X_{nm}-M_n-(\sum_{m=1}^{m_n}\bar{X}_{nm}-M_n)\geq\eta A_n)\frown(\sum_{m=1}^{m_n}\bar{X}_{nm}-M_n<0)\}$$

$$+P_\gamma\{(\sum_{m=1}^{m_n}X_{nm}-M_n-(\sum_{m=1}^{m_n}\bar{X}_{nm}-M_n)\geq\eta A_n)\frown(\sum_{m=1}^{m_n}\bar{X}_{nm}-M_n>0)\}$$

$$\geq\frac{1}{2}P_\gamma\{|\sum_{m=1}^{m_n}X_{nm}-M_n|\geq\eta A_n\}.$$

Consequently (4.1.1) follows from (4.1.4) and (4.1.5).

§ **4. 2. Some special cases of the law of large numbers.** In the first place we consider the following theorem.

Theorem 4. 2. 1. *A necessary and sufficient condition that there should exist a sequence* $\{A_n>0|\,n=1,2,\cdots\cdots\}$ *so that for any* $\epsilon>0$

$$P_\gamma\left\{\left|\sum_{m=1}^{m_n}\left(X_{nm}-\int_{-A_n}^{A_n}xdF_{nm}(x)\right|\geq\epsilon A_n\right\}\rightarrow0,\ (n\rightarrow\infty)\tag{4.2.1}$$

and for any $\epsilon>0$

$$P_\gamma\{|X_{nm}|\geq\epsilon A_n\}\rightarrow0,\ (n\rightarrow\infty)\tag{4.2.2}$$

uniformly for $1\leq m\leq m_n$ *is that*

$$\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(A_n)\}\rightarrow0,\ (n\rightarrow\infty)\tag{4.2.3}$$

and

$$a_{nm}\equiv\int_{-1}^{1}xdF_{nm}(A_nx)\rightarrow0\ (n\rightarrow\infty)\tag{4.2.4}$$

uniformly for $1\leq m\leq m_n$, *where* $F'_{nm}(x)=F_{nm}(x+A_na_{nm})$.

Proof. Necessity. By the assumption (4.2.1) we see

$$\Pi_{m=1}^{m_n}f'_{nm}(t/A_n)\rightarrow1,\ (n\rightarrow\infty),$$

uniformly for every finite interval, where

$$f'_{nm}(t/A_n)=f_{nm}(t/A_n)\exp(-ita_{nm})=\int_{-\infty}^{\infty}e^{itx/A_n}dF'(x).$$

Hence, since $\Pi_{m=1}^{m_n}|f_{nm}(t/A_n)|^2\rightarrow1$ $(n\rightarrow\infty)$ in the same sense as above, we have $N_0>0$ such that $\Pi_{m=1}^{m_n}|f_{nm}(t/A_n)|^2\geq 1/2$ for $n>N_0$ and $0\leq t\leq 2$. Therefore, recalling (4.2.2), by appealing to (F. I. 2) and (F. I. 4) we obtain that for $n\geq N_0$

$$\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(A_n)\}\leq R\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(A_n)\}$$
$$\leq RK\{1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(A_n)\},$$

where K and R are constants. Hence (4.2.3) follows from

$$1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(A_n)=\int_0^\infty e^{-t}(1-\Pi_{m=1}^{m_n}|f_{nm}(t/A_n)|^2)dt.$$
$$\to 0,\ (n\to\infty).$$

Sufficiency. By (4.2.4), there exists, for any $\epsilon>0$, $N_0>0$ so that $|a_{nm}|<\epsilon/2$ uniformly for $1\leq m\leq m_n$ $(n>N_0)$. Next it follows from (4.2.3) that $1-\Phi_{F'_{nm}}\to 0$ $(n\to\infty)$ uniformly for $1\leq m\leq m_n$. Hence for any $\epsilon>0$

$$1-\Phi_{F'_{nm}}(A_n)\geq\{\epsilon^2/(\epsilon^2+4)\}\int_{|x|>\epsilon/2}dF'_{nm}(A_nx)=\{\epsilon^2/(\epsilon^2+4)\}\int_{|x|>\epsilon-\epsilon/2}dF'_{nm}(A_nx)$$
$$\geq\{\epsilon^2/(\epsilon^2+4)\}\int_{|x+a_{nm}|>\epsilon}dF_{nm}(A_nx+A_na_{nm})=\{\epsilon^2/(\epsilon^2+4)\}\int_{|x|>\epsilon}dF_{nm}(A_nx),$$

which shows (4.2.2) uniformly for $1\leq m\leq m_n$ as $n\to\infty$. By (F. I. 6), we see

$$\sum_{m=1}^{m_n}|f'_{nm}(t/A_n)\exp(-ita'_{nm})-1|\leq(t^2+2|t|+4)\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(A_n)\},\tag{4.2.5}$$

where $f'_{nm}(t/A_n)=f_{nm}(t/A_n)\exp(-ita_{nm})$ and $a'_{nm}=\int_{-1}^{1}x\,dF'_{nm}(A_nx)$. Since we see by (4.2.3) and (4.2.5)

$$|\log\Pi_{m=1}^{m_n}f'_{nm}(t/A_n)\exp(-ita'_{nm})-\sum_{m=1}^{m_n}\{f'_{nm}(t/A_n)\exp(-ita'_{nm})-1\}|$$
$$\leq\sum_{m=1}^{m_n}|f'_{nm}(t/A_n)\exp(-ita'_{nm})-1|^2$$

for sufficiently large n, we get

$$\Pi_{m=1}^{m_n}f'_{nm}(t/A_n)\exp(-ita'_{nm})\to 1,\ (n\to\infty)\tag{4.2.6}$$

uniformly for every finite interval. We have

$$f'_{nm}(t/A_n)\exp(-ita'_{nm})=f_{nm}(t/A_n)\exp\{-it(a_{nm}+a'_{nm})\}\tag{4.2.7}$$

and for $n \geq N_0 > 0$

$$|a'_{nm}| = \left| \int_{-1}^{1} x dF'_{nm}(A_n x) \right|$$

$$\leq \left| a_{nm} - a_{nm} \int_{|x-a_{nm}| \leq 1} dF_{nm}(x) \right| + \left| \int_{|x-a_{nm}| \leq 1} x dF_{nm}(A_n x) - a_{nm} \right|$$

$$\leq |a_{nm}| \int_{|x|>1} dF'_{nm}(A_n x) + 2(1+|a_{nm}|) \int_{|x|>1-|a_{nm}|} dF_{nm}(A_n x)$$

$$\leq 2\epsilon(1-\Phi_{F'_{nm}}(A_n)) + 4(1+\epsilon)(1-2\epsilon+2\epsilon^2)(1-\Phi_{F'_{nm}}(A_n)).$$

From (4.2.3) we obtain

(4.2.8) $$\sum_{m=1}^{m_n} |a'_{nm}| \to 0, \quad (n \to \infty).$$

Since by (4.2.6) and (4.2.7) we have

$$P_\gamma\{|\sum_{m=1}^{m_n}(X_{nm}/A_n - a_{nm} - a'_{nm})| > \epsilon\} \to 0, \quad (n \to \infty)$$

for any $\epsilon > 0$, we get, by applying (4.2.8), (4.2.1).

Corollary 4. 2. 1. (*A. Bobroff*[34]). *Suppose that* $||X_{nm}||$ *defined by* (2.1.1) *is a system of positive random variables. Then it is necessary and sufficient for the existence of* $\{A_n > 0 | n = 1, 2, \cdots\cdots\}$ *satisfying that for any* $\epsilon > 0$

(4.2.9) $$P_\gamma\{|(\sum_{m=1}^{m_n} X_{nm})/A_n - 1| > \epsilon\} \to 0, \quad (n \to \infty)$$

and

(4.2.10) $$P_\gamma\{X_{nm} > \epsilon A_n\} \to 0, \quad (n \to \infty)$$

uniformly for $1 \leq m \leq m_n$ *is that there exists a sequence of positive numbers* $\{C_n > 0 | n = 1, 2, \cdots\cdots\}$ *satisfying*

(4.2.11) $$\sum_{m=1}^{m_n}\{1 - F_{nm}(C_n)\} \underset{n\to\infty}{\to} 0, \quad \sum_{m=1}^{m_n} \frac{1}{C_n} \int_0^{C_n} (1 - F_{nm}(x)) dx \underset{n\to\infty}{\to} \infty$$

Proof[35]. (4.2.11) can be replaced by

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[34] A. Bobroff, Über relative Stablität von Summen positive zufälliger Grössen (Russian), Uchenye Zapiski Moskov, Gos. Univ., Nomografija, pp. 191—202, 1939.

[35] Recently T. Kawata showed another proof of this theorem in a different angle and the author also gave another proof. They have not been published.

(4.2.12) $$\sum_{m=1}^{m_n}\{1-F_{nm}(C_n)\}\underset{n\to\infty}{\longrightarrow}0,\quad \sum_{m=1}^{m_n}\frac{1}{C_n}\int_0^{C_n} xdF_{nm}(x)\underset{n\to\infty}{\longrightarrow}\infty.$$

In fact, by partial integration, we have

$$\sum_{m=1}^{m_n}\frac{1}{C_n}\int_0^{C_n}(1-F_{nm}(x))dx=\sum_{m=1}^{m_n}\left\{(1-F_{nm}(C_n))+\frac{1}{C_n}\int_0^{C_n} xdF_{nm}(x)\right\}$$

which shows the equivalency of (4.2.11) and (4.2.12). First, we suppose (4.2.9) and (4.2.10) uniformly for $1\leq m\leq m_n$, then (4.2.9) is a special case of (4.2.1) satisfying

(4.2.13) $$\frac{1}{A_n}\sum_{m=1}^{m_n}\int_0^{A_n} xdF_{nm}(x)=1+o(1),\quad (n\to\infty).$$

Hence we have (4.2.3) and (4.2.4) uniformly for $1\leq m\leq m_n$, from which for any $\eta>0$

$$\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(A_n)\}\geq\frac{\eta^2}{1+\eta^2}\sum_{m=1}^{m_n}\{1-F'_{nm}(\eta A_n)\}\to 0,\quad (n\to\infty)$$

and for sufficiently large n

$$|a_{nm}|=\left|\int_0^1 xdF_{nm}(A_n x)\right|\leq\eta$$

uniformly for $1\leq m\leq m_n$. Hence for sufficiently large n

$$\begin{aligned}\sum_{m=1}^{m_n}\{1-F'_{nm}(\eta A_n)\}&=\sum_{m=1}^{m_n}\{1-F_{nm}(\eta A_n+A_n a_{nm})\}\\&\geq\sum_{m=1}^{m_n}\{1-F_{nm}(2\eta A_n)\},\end{aligned}$$

from which there exists a sequence $\{\eta_n>0|n=1,2,\cdots\cdots\}$ so that

(4.2.14) $$\sum_{m=1}^{m_n}\{1-F_{nm}(\eta_n A_n)\}\underset{n\to\infty}{\longrightarrow}0\quad\text{and}\quad\eta_n\underset{n\to\infty}{\longrightarrow}0.$$

Then we have

(4.2.15) $$\frac{1}{\eta_n A_n}\sum_{m=1}^{m_n}\int_0^{\eta_n A_n} xdF_{nm}(x)\to\infty,\ n\to\infty.$$

For, otherwise, there should exist a subsequence $\{n_i|i=1,2,\cdots\cdots\}$ and a constant M satisfying

$$M\geq\frac{1}{\eta_{n_i}A_{n_i}}\sum_{m=1}^{m_{n_i}}\int_0^{\eta_{n_i}A_{n_i}} xdF_{n_i m}(x),\quad i=1,2,\cdots\cdots$$

Hence

$$\eta_{n_i}M+\sum_{m=1}^{m_{n_i}}\{1-F_{n_im}(\eta_{n_i}A_{n_i})\}$$

$$\geq\sum_{m=1}^{m_{n_i}}\left\{\int_0^{\eta_{n_i}A_{n_i}}x/A_{n_i}dF_{n_im}(x)+\int_{\eta_{n_i}A_{n_i}}^{\infty}dF_{n_im}(x)\right\}$$

$$>\sum_{m=1}^{m_{n_i}}\left\{\int_0^{\eta_{n_i}A_{n_i}}x/A_{n_i}dF_{n_im}(x)+\int_{\eta_{n_i}A_{n_i}}^{A_{n_i}}x/A_{n_i}dF_{n_im}(x)\right\}$$

$$=\sum_{m=1}^{m_{n_i}}\int_0^{A_{n_i}}x/A_{n_i}dF_{n_im}(x)\to 0,\quad (i\to\infty),$$

which is contrary to (4.2.13). Thus putting $\eta_nA_n=C_n$ in (4.2.14) and (4.2.15), we obtain (4.2.12).

Conversely suppose (4.2.12). Then putting

$$A_n=\sum_{m=1}^{m_n}\int_0^{C_n}xdF_{nm}(x),$$

we see

$$\frac{A_n}{C_n}=\sum_{m=1}^{m_n}\int_0^{C_n}x/C_ndF_{nm}(x)\to\infty,$$

$$\sum_{m=1}^{m_n}\frac{1}{A_n}\int_0^{A_n}xdF_{nm}(x)=\sum_{m=1}^{m_n}\left\{\frac{1}{A_n}\int_0^{C_n}xdF_{nm}(x)+\frac{1}{A_n}\int_{C_n}^{A_n}xdF_{nm}(x)\right\}\to 1,\ (n\to\infty),$$

and furthermore

$$a_{nm}=\frac{1}{A_n}\int_0^{A_n}xdF_{nm}(x)=\frac{1}{A_n}\int_0^{C_n}xdF_{nm}(x)+\frac{1}{A_n}\int_{C_n}^{A_n}xdF_{nm}(x)=o(1),\ (n\to\infty)$$

uniformly for $1\leq m\leq m_n$. (The latter half of the above relations is nothing but (4.2.4)). Hence

$$\sum_{m=1}^{m_m}\{1-\Phi_{F'_{nm}}(A_n)\}\leq\sum_{m=1}^{m_n}\left\{\int_0^{A_n-A_na_{nm}}x^2/A_n^2dF'_{nm}(x)+\int_{A_n-A_na_{nm}}^{\infty}dF'_{nm}(x)\right.$$

$$\leq\sum_{m=1}^{m_n}\left\{\int_0^{A_n}(x-A_na_{nm})^2/A_n^2dF_{nm}(x)+\int_{A_n}^{\infty}dF_{nm}(x)\right\}$$

$$\leq\sum_{m=1}^{m_n}\left\{\int_0^{A_n}x^2/A_n^2dF_{nm}(x)+\int_{A_n}^{\infty}dF_{nm}(x)\right\}$$

$$\leq \sum_{m=1}^{m_n} \left\{ \int_0^{C_n} x^2/A_n^2 dF_{nm}(x) + \int_{C_n}^{A_n} x^2/A_n^2 dF_{nm}(x) + \int_{A_n}^{\infty} dF_{nm}(x) \right\}$$

$$\leq C_n/A_n + \sum_{m=1}^{m_n} \left\{ \int_{C_n}^{\infty} dF_{nm}(x) + \int_{A_n}^{\infty} dF_{nm}(x) \right\}$$

$$\to 0, \quad (n \to \infty).$$

which shows (4.2.3).

Corresponding to Theorem 4.2.1, we obtain the following theorem:

Theorem 4. 2. 2. *The necessary and sufficient condition of Theorem 4. 2. 1 can be replaced by the following conditions:*

$$\sum_{m=1}^{m_n} \{1 - \Phi_{F^*_{nm}}(A_n)\} \to 0, \quad (n \to \infty) \tag{4.2.16}$$

and

$$M_{nm}/A_n \to 0 \quad (n \to \infty) \tag{4.2.17}$$

uniformly for $1 \leq m \leq m_n$, *where* M_{nm} *is the median of* $F_{nm}(x)$ *and* $F^*_{nm}(x) = F_{nm}(x + M_{nm})$.

The proof is almost analogous to that of Theorem 4. 2. 1 except the following fact. Under the assumption (4.2.16) and (4.2.17), by (F. I. 6) we see

$$\sum_{m=1}^{m_n} |f^*_{nm}(t/A_n) \exp(-ita^*_{nm}) - 1| \leq (t^2 + 2|t| + 4) \sum_{m=1}^{m_n} \{1 - \Phi_{F^*_{nm}}(A_n)\},$$

where

$$f^*_{nm}(t) = \int_{-\infty}^{\infty} e^{itx} dF^*_{nm}(A_n x), \quad a^*_{nm} = \int_{-1}^{1} x dF^*_{nm}(A_n x).$$

Hence we have for sufficiently large values of n

$$|\log \Pi_{m=1}^{m_n} f^*_{nm}(t/A_n) \exp(-ita^*_{nm}) - \sum_{m=1}^{m_n} \{f^*_{nm}(t/A_n) \exp(-ita^*_{nm}) - 1\}|$$

$$\leq \sum_{m=1}^{m_n} |f^*_{nm}(t/A_n) \exp(-ita^*_{nm}) - 1|^2.$$

Hence

$$\Pi_{m=1}^{m_n} f^*_{nm}(t/A_n) \exp(-ita^*_{nm}) \to 1, \quad (n \to \infty)$$

uniformly for every finite interval and so for any $\epsilon > 0$

$$P_\gamma\{|\sum_{m=1}^{m_n} (X_{nm} - M_{nm})/A_n - a^*_{nm}| > \epsilon\} \to 0, \quad (n \to \infty). \tag{4.2.18}$$

While

$$\left|\frac{M_{nm}}{A_n}+a^*_{nm}-a_{nm}\right| \le \frac{|M_{nm}|}{A_n}\int_{|x-M_{nm}|>A_n} dF_{nm}(x)+\left|\frac{1}{A_n}\int_{|x-M_{nm}|\le A_n} xdF_{nm}(x)-\frac{1}{A_n}\int_{|x|\le A_n} xdF_{nm}(x)\right|$$

$$\le \frac{|M_{nm}|}{A_n}\int_{|x|>A_n} dF^*_{nm}(x)+\left(1+\frac{|M_{nm}|}{A_n}\right)\int_{|x|\ge A_n-|M_{nm}|} dF_{nm}(x)$$

$$\le \frac{|M_{nm}|}{A_n}\int_{|x|>A_n} dF^*_{nm}(x)+\left(1+\frac{|M_{nm}|}{A_n}\right)\int_{|x|\ge A_n-2|M_{nm}|} dF^*_{nm}(x)$$

and as by (4.2.10), we have $|M_n|<A_n/4$ uniformly for $1\le m\le m_n$ and for sufficiently large values of $n\ge N_0>0$, we get for $n\ge N_0$

$$\left|\frac{M_{nm}}{A_n}-a^*_{nm}-a_{nm}\right| \le \frac{1}{2}\{1-\Phi_{F^*_{nm}}(A_n)\}+\frac{25}{4}\{1-\Phi_{F^*_{nm}}(A_n)\}$$

$$\le 7\{1-\Phi^*_{nm}(A_n)\}.$$

Hence, it follows from (4.2.16) that

$$\sum_{m=1}^{m_n}\left|\frac{M_{nm}}{A_n}+a^*_{nm}-a_{nm}\right|\to 0,\quad (n\to\infty),$$

from which we can easily conclude (4.2.1).

Theorem 4. 2. 3. *A sufficient condition that* (4.2.1) *should exist is*

$$\sum_{m=1}^{m_n}\{1-\Phi_{F_{nm}}(A_n)\}\to 0,\quad (n\to\infty), \tag{4.2.19}$$

and it is also necessary if the condition

$$F_{nm}(+0)\ge\lambda>0,\ F_{nm}(-0)\le 1-\lambda,\quad (0<\lambda<1) \tag{4.2.20}$$

$$1\le m\le m_n,\quad n=1,2,\cdots\cdots$$

is satisfied.

Proof. sufficiency. By (F. I. 6) we see

$$\sum_{m=1}^{m_n}|f_{nm}(t/A_n)\exp(-ita_{nm})-1|\le(t^2+2|t|+4)\sum_{m=1}^{m_n}\{1-\Phi_{F_{nm}}(A_n)\}. \tag{4.2.21}$$

Hence (4.2.19) implies that

$$\sum_{m=1}^{m_n}|f_{nm}(t/A_n)\exp(-ita_{nm})-1|\to 0,\quad (n\to\infty)$$

uniformly for every finite interval. Since

$$|\log \Pi_{m=1}^{m_n} f_{nm}(t/A_n)\exp(-ita_{nm})-\sum_{m=1}^{m_n}\{f_{nm}(t/A_n)\exp(-ita_{nm})-1\}|$$

$$\leq \sum_{m=1}^{m_n}|f_{nm}(t/A_n)\exp(-ita_{nm})-1|^2$$

for sufficiently large n, we see $\Pi_{m=1}^{m_n} f_{nm}(t/A_n)\exp(-ita_{nm})\to 1$ $(n\to\infty)$ uniformly for every fiuite interval, which shows that (4.2.1) holds.

Necessity. By Theorem 4. 1. 1, (4.2.1) implies $\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(A_n)\}\to 0$ uniformly for $1\leq m\leq m_n$, where $X_{nm}=X_{nm}-\bar{X}_{nm}$, X_{nm} and $\bar{X}_{nm}$ being mutually independent random variables which have the same distribution $F_{nm}(x)$. Hence in the same manner as in the proof of Theorem 3. 2. 3, we obtain $P_\gamma\{|\tilde{X}_{nm}|\geq\epsilon\}\geq\lambda P_\gamma\{|X_{nm}|\geq\epsilon\}$. Hence we see that $P_\gamma\{|X_{nm}|\geq\epsilon\}\to 0$ $(n\to\infty)$ uniformly for $1\leq m\leq m_n$ and for any $\epsilon>0$, from which by (F. I. 3) for sufficiently large n

$$\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(A_n)\}\geq R\sum_{m=1}^{m_n}\{1-\Phi_{F_{nm}}(A_n)\}$$

where R is a constant, which completes the proof.

Corollary 4. 2. 3. (*W. Feller*)[36]. *A sufficient condition for the existence of* $\{A_n>0|n=1, 2,\cdots\cdots\}$ *satisfying* (4.2.1) *for any* $\epsilon>0$ *is*

$$\sum_{m=1}^{m_n}\int_{|x|>A_n} dF_{nm}(x)\to 0,\quad (n\to\infty)$$

(4.2.22)

$$\frac{1}{A_n^2}\sum_{m=1}^{m_n}\int_{|x|\leq A_n} x^2 dF_{nm}(x)\to 0,\quad (n\to\infty)$$

and it is also necessary if (4.2.20) *is satisfied.*

Chapter V. The infinitely divisible law.

§ 5. 1. The definition of the infinitely divisible law. The definition of the infinitely divisible law may be generally described in the following two forms:

(I) A random variable X depends on the infinitely divisible law I, if for any positive integer n there exist independent random vari-

(36) W. Feller, loc. cit. (8).

ables X_{nm} $(1 \leq m \leq n)$ such that

$$X = X_{n1} + X_{n2} + \cdots\cdots + X_{nn} \tag{5.1.1}$$

and for any $\epsilon > 0$

$$P_Y\{|X_{nm}| > \epsilon\} \to 0, \quad (n \to \infty) \tag{5.1.2}$$

uniformly for $1 \leq m \leq n$.

(II) A random variable X depends on the infinitely divisible law I if for any positive integer n there exist independent random variables X_{nm} $(1 \leq m \leq n)$ having the same distribution function such that (5.1.1) exists.

The latter is immediately reworded by appealing to the characteristic function as follows:

(III) The characteristic function $f(t)$ of a probability distribution desends on the infinitely divisible law I if for any λ $(1 > \lambda > 0)$ $f^{\lambda}(t)$ is also the characteristic function of a probability distribution.

It is evident that (I) follows from (II), however the converse is not so clear. It has been shown by A. Khintchine(7), i. e.,

Theorem 5. 1. 1. *(III) and so (II) follows from (I).*

Before we shall show another proof of this theorem, we need the following lemma.

Given a system of random variables $||X_{nm}||$ defined by (2.1.1), then it is called to be individually negligible if for any $\epsilon > 0$ (5.1.2) exists uniformly for $1 \leq m \leq m_n$.

Lemma 5. 1. 1.(8) *Let $||X_{nm}||$ be an individually negligible system of random variables having, respectively, distribution functions $F_{nm}(X)$ $(n = 1, 2, \cdots\cdots;\ m = 1, 2, \cdots\cdots m)$ and if there exist $\delta > 0$ and $T > 0$ such that for $0 \leq t \leq T$*

$$\Pi_{m=1}^{m_n} |f_{nm}(t)|^2 \geq \delta, \quad n = 1, 2, \cdots\cdots, \tag{5.1.3}$$

then we have

$$\left|\log \Pi_{m=1}^{m_n} f_{nm}(t) - \sum_{m=1}^{m_n} [f_{nm}(t) \exp\{-it(a_{nm} + a'_{nm})\} - 1 + it(a_{um} + a'_{nm})]\right| \underset{n \to \infty}{\to} 0. \tag{5.1.4}$$

(7) A. Khintchine, loc. cit. (8)

(8) This result holds even if we replace a_{nm} with the median M_{nm} of $F_{nm}(x)$ and a'_{nm} with $a^*_{nm} = \int_{-1}^{1} x dF_{nm}(x + M_{nm})$

and

(5.1.5) $$\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\leq(t^2+2|t|+4)C$$

uniformly for every finite interval and for an absolute constant C and for every t, where $a_{nm}=\int_{-1}^{1}xdF_{nm}(x)$, $a'_{nm}=\int_{-1}^{1}xdF'_{nm}(x)$, $F'_{nm}(x)=F_{nm}(x+a_{nm})$ *and* $f_{nm}(t)$ *is the characteristic function of* F_{nm}.

Proof[39]. The fact that $\|X_{nm}\|$ is individually negligible implies that $f_{nm}(t)\to 1$, $(n\to\infty)$ uniformly for every finite interval and $m(1\leq m\leq m_n)$, $a_{nm}\to 0$, $(n\to\infty)$ uniformly for $m(1\leq m\leq m_n)$. Hence we have easily $a'_{nm}\to 0$, $(n\to\infty)$, from which

(5.1.6) $$f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}\to 1$$

uniformly for every finite interval. While from (5.1.3), by appealing to (F. I. 2) and (F. I. 5), we have $N_0>0$ so that for $n\geq N_0$

$$\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(1)\}\leq R\sum_{m=1}^{m_n}\{1-\Psi_{F_{nm}}(1)\}$$
$$\leq RK\{1-\Psi_{F_{n1}*\cdots\cdots*F_{nm_n}}(1)\}$$

where R and K are constants. While by (F. I. 6) we see

(5.1.7) $$\begin{aligned}&\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\\ &\leq(t^2+2|t|+4)\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(1)\}\leq(t^2+2|t|+4)C\end{aligned}$$

for every t $(-\infty<t<\infty)$, where $C=RK$. Hence for sufficiently large $n>N_2>0$ we get

$$\begin{aligned}&\Big|\sum_{m=1}^{m_n}\log f_{nm}(t)-\sum_{m=1}^{m_n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]\Big|\\ &\leq\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|^2\\ &\leq\mathop{Max}_{1\leq m\leq m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\\ &\leq\mathop{Max}_{1\leq m<m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\\ &\qquad\times C(t^2+2|t|+4)\end{aligned}$$

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[39] In the proof of this theorem, we may also replace a_{nm} with the median M_{nm} of $F_{nm}(x)$ and α'_{nm} with $a^*_{nm}=\int_{-1}^{1}xdF_{nm}(a+M_{nm})$.

for every t $(-\infty<t<\infty)$. Thus, applying (5.1.6), we can conclude (5.1.4).

Prooe of Theorem 5.1.1. From (I) there exists a system of characteristic functions $\{f_{nm}(t)\}$ $(n=1, 2, \cdots\cdots; m=1, 2, \cdots\cdots, n)$ such that $f(t)=\Pi_1^n f_{nm}(t)$, $n=1, 2, \cdots\cdots$. Hence we can take two constants $\delta>0$ and $T_0>0$ satisfying $|f(t)|^2=\Pi_{m=1}^n|f_{nm}(t)|^2\geq\delta$, $n=1, 2, \cdots\cdots$ for $0\leq t\leq T$. Furthermore from the individual negligibility of $||X_{nm}||$ $(n=1, 2, \cdots\cdots; m=1, 2, \cdots\cdots, n)$, applying Lemma 5.1.1, we have

$$(5.1.7)\qquad |\log[f(t)\exp\{-it\textstyle\sum_{m=1}^n(a_{nm}+a'_{nm})\}] \\ -\textstyle\sum_{m=1}^{n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1]|\underset{n\to\infty}{\to}0$$

uniformly for every finite interval of t. Next put

$$(5.1.8)\qquad f_{nm}^{(\lambda)}(t)=\lambda f_{nm}(t)\exp\{-it(a_{nm} \\ +a'_{nm})\}+1-\lambda,\ n=1, 2, \cdots\cdots;\ m=1, 2, \cdots\cdots, n,$$

then this is the characteristic function of $\lambda\, F_{nm}(x+a_{nm}+a'_{nm})+(1-\lambda)\epsilon(x)$, where $\epsilon(x)$ is the distribution function having the jump 1 at 0. From (5.1.8) and the individual negligibility of $||X_{nm}||$, we see

$$(5.1.9)\qquad |f_{nm}^{(\lambda)}(t)-1|=\lambda|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\to 0,\quad (n\to\infty)$$

uniformly for every finite interval of t and $m(1\leq m\leq n)$. Furthermore, (5.1.5) of the same Lemma implies

$$\textstyle\sum_{m=1}^n|f_{nm}^{(\lambda)}(t)-1|\leq\lambda\sum_{m=1}^n|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1| \\ \leq\lambda C(t^2+2|t|+4).$$

Hence for sufficiently large values of $n\geq N_0$

$$|\textstyle\sum_{m=1}^n\log f_{nm}^{(\lambda)}(t)-\sum_{m=1}^n(f_{nm}^{(\lambda)}(t)-1)|\leq\sum_{m=1}^n|f_{nm}^{(\lambda)}(t)-1|^2 \\ \leq\lambda C(t^2+2|t|+4)\,\mathrm{Max}\ |f_{nm}^{(\lambda)}(t)-1|.$$

Whence we have, by (5.1.9),

$$|\textstyle\sum_{m=1}^n\log f_{nm}^{(\lambda)}(t)-\sum_{m=1}^n(f_{nm}^{(\lambda)}(t)-1)|\to 0,\quad (n\to\infty)$$

uniformly for every finite interval and $1\leq m\leq n$. Therefore, from (5.1.7) and $f_{nm}^{(\lambda)}(t)-1=\lambda[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1]$, it follows that

$$|\lambda\log[f(t)\exp\{-it\textstyle\sum_{m=1}^n(a_{nm}+a'_{nm})\}]-\sum_{m=1}^n\log f_{nm}^{(\lambda)}(t)|$$

$$=|\lambda \log f(t)-\sum_{m=1}^{n}\log [f_{nm}^{(\lambda)}(t)\exp\{it\,\lambda(a_{nm}+a'_{nm})\}]|\underset{n\to\infty}{\to}0$$

uniformly for every finite interval. From the continuity theorem of P. Lévy we know that $f^{\lambda}(t)$ is a characteristic function.

§ **5. 2. A deduction of the canonical form of the infinitely divisible law.** A deduction of the canonical form of the infinitely divisible law from the definition (III) was given by A. Khintchine[40], we shall here show a deduction from (I).

Theorem 5. 2. 1. *The canonical form of the law defined by the definition (I) is uniquely representable by the characteristic function $f(t)$ having the following form:*

$$\log f(t)=itA+\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x), \tag{5.2.1}$$

where $G(x)$ is a bounded and non-decreasing function satisfying $G(-\infty)=0$ and A $(-\infty<A<\infty)$ is a constant.

Proof. By the definition (I) there exists a system of characteristic function $f_{n1}(t), f_{n2}(t),\cdots\cdots, f_{nn}(t)(n=1, 2,\cdots\cdots)$ which is individually negligible and $f(t)=\prod_{m=1}^{n}f_{nm}(t)$ $(n=1, 2,\cdots\cdots)$. Hence we have two constants $\delta>0$ and $T_0>0$ such that

$$|f(t)|^2=\prod_{m=1}^{n}|f_{nm}(t)|^2\geq\delta, \quad n=1, 2,\cdots\cdots$$

for $|t|\leq T_0$. Whence, by appealing to Lemma 5. 1. 1, we have

$$|\log f(t)-\sum_{m=1}^{n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]|\underset{n\to\infty}{\to}0 \tag{5.2.2}$$

uniformly for every finte interval. Put

$$\varphi_n(t)=\sum_{m=1}^{n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})\}]$$

$$=\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG_n(x)+it\sum_{m=1}^{n}\left(\int_{-\infty}^{\infty}\frac{x}{1+x^2}dF''_{nm}(x)+a_{nm}+a'_{nm}\right),$$

where

$$G_n(x)\equiv\sum_{m=1}^{n}\int_{-\infty}^{x}\frac{u^2}{1+u^2}dF''_{nm}(u)$$

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(40) A. Khintchine, loc. cit. (13).

and $F''_{nm}(x)=F_{nm}(x+a_{nm}+a'_{nm})$. Then by (5.1.5) of Lemma 5.1.1

$$|\varphi_n(t)-it\sum_{m=1}^{n}(a_{nm}+a'_{nm})|\le C(t^2+2|t|+4),\quad n=1,2,\cdots\cdots,$$

where C is a constant. On the other hand by (5.2.2) we see

$$\Re\varphi_n\to\Re\log f(t),\quad (n\to\infty) \tag{5.2.3}$$

uniformly for every fintie interval. Hence $|\Re\varphi_n(t)|$ $(n=1,2,\cdots\cdots)$ and $|\log f(t)|$ are dominated by $C(t^2+2|t|+4)$. This fact and (5.2.3) imply that

$$\begin{aligned}\delta C&\ge-\int_0^\infty e^{-t}\Re\varphi_n(t)dt=\int_{-\infty}^\infty dG_n(x)=G_n(\infty)\\ &=\sum_{m=1}^{n}\int_{-\infty}^\infty\frac{x^2}{1+x^2}dF''_{nm}(x)\underset{n\to\infty}{\to}-\int_0^\infty e^{-t}\Re\log f(t)dt.\end{aligned}\tag{5.2.4}$$

Now put

$$G(\infty)=-\int_0^\infty e^{-t}\Re\log f(t)dt,$$

then we have $G_n(\infty)\to G(\infty)(n\to\infty)$ and clearly $G_n(-\infty)=G(-\infty)(=0;\ n=1,2,\cdots\cdots)$ and a bounded non-decreasing function $G(x)$ $(G(-\infty)=0)$ such as $G_{n_i}(x)\to G(x)$ $(i\to\infty)$ at the continuity points of $G(x)$. As $(\exp(itx)-1-itx/(1+x^2))\,(1+x^2)/x^2$ is bounded and continuous function of x for fixed t, applying Helly's theorem, we obtain

$$\begin{aligned}&\int_{-\infty}^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG_{n_i}(x)\\ &\underset{i\to\infty}{\to}\int_{-\infty}^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x}dG(x)\end{aligned}$$

uniformly for every finite interval. Hence by (5.2.2) we have a constant A such that

$$\sum_{m=1}^{n_i}\left\{\int_{-\infty}^\infty\frac{x}{1+x^2}dF''_{nm}(x)+a_{nm}+a'_{nm}\right\}\to A\quad(i\to\infty).\tag{5.2.5}$$

However, we must here remark that the above representation of logf(t) is unique. In fact, put

$$\Lambda(t)\equiv\log f(t+1)+\log f(t-1)-2\log f(t)$$

$$= \int_{-\infty}^{\infty} e^{iut}\{e^{iu}+e^{-iu}-2\}\frac{1+u^2}{u^2}dG(u)$$

$$=2\int_{-\infty}^{\infty} e^{iut}(\cos u-1)\frac{1+u^2}{u^2}dG(u)$$

Furthermore, put

$$\text{H}(u)=-\int_{-\infty}^{u} 2(\cos u-1)\frac{1+u^2}{u^2}dG(u), \tag{5.2.6}$$

then $\text{H}(u)$ is non-decreasing and

$$\int_{-\infty}^{\infty} dH(u)=-\Lambda(0)=-\Re\log f(1).$$

Hence

$$\Lambda(t)=-\int_{-\infty}^{\infty} e^{iut}dH(u).$$

From the unicity theorem of Fourier transform, We see $H(u)$ is unique and also $G(u)$ by (5.2.6), from which follows the uniqueness of A. Thus we completed the proof of Theorem 5. 2. 1.

§ **5. 3. A criterion of convergence of the distributions of the sums of independent random variables to that of an infinitely divisible law.** Given a system of random variables $||X_{nm}||$ defined by (2.1.1) and suppose that it is individually negligible. Furthermore, let $F_{nm}(x)$ be the distribution function of the random variables X_{nm}, and denote by $f_{nm}(t)$ its characteristic function ($n=1, 2,\cdots\cdots$; $m=1, 2,\cdots\cdots, m_n$). First we show

Theorem 5. 3. 1. *Let* $||X_{nm}||$ *be an individually negligible system of random variables and if* $S=\sum_{m=1}^{m_n}X_{nm}$ *converges in Bernoulli's sense to a distribution function. Then this limit function depends on the infinitely divisible law I.*

This theorem was proved by A. Khintchine[41]. But another proof can be carried out in the same manner as in the proof of Theorm 5. 1. 1, and so we omit the proof.

The following theorem was shown by B. Gnedenko[42], but it has some point of difference from B. Gnedenko's theorem. We shall

(41) A. Khintchine, loc. cit. (9).

(42) B. Gnedenko, loc. cit. (16).

show another proof from our point of view.

Theorem 5. 3. 2. *Civen the infinitely divisible law I, with the characteristic function $f(t)$ which has the canonical form* (5.2.1), *and let $\|X_{nm}\|$ be an individually negligible system of random variables, then a necessary and sufficient condition for the convergence of the sums $\sum_{m=1}^{m_n} X_{nm}$ to the distribution function of $f(t)$ in Bernoulli's sense is that there exist a bounded non-decreasing function $G(u)$ $(G(-\infty)=0)$ and a constant A such that*

$$\sum_{m=1}^{m_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF''_{nm}(x) \to G(\infty), \quad (n\to\infty) \tag{5.3.1}$$

$$\sum_{m=1}^{m_n} \left\{ \int_{-\infty}^{\infty} \frac{x}{1+x^2} dF''_{nm}(x) + a_{nm} + a'_{nm} \right\} \to A, \quad (n\to\infty) \tag{5.3.2}$$

and

$$\sum_{m=1}^{m_n} \int_{-\infty}^{u} \frac{x^2}{1+x^2} dF''_{nm}(x) \to G(u), \quad (n\to\infty) \tag{5.3.3}$$

at the continuity points of $G(u)$, where $F''_{nm}(x)=F_{nm}(x+a_{nm}+a'_{nm})$ $a_{nm}=\int_{-1}^{1} x dF_{nm}(x)$ and $a'_{nm}=\int_{-1}^{1} x dF_{nm}(x+a_{nm})$.

Proof[48]. In the first place we remark that from the individual negligibility of $\|X_{nm}\|$ we see

$$f_{nm}(t)\to 1, \quad (n\to\infty) \tag{5.3.4}$$

uniformly for every finite interval and $1\leq m\leq m_n$.

1. *Necessity.* By the assumption we have

$$\Pi_{m=1}^{m_n} f_{nm}(t)\to f(t), \quad (n\to\infty) \tag{5.3.5}$$

uniformly for every finite interval. Hence there exist $\delta>0$ and $T_0>0$ so that for $|t|\leq T_0$

$$\Pi_{m=1}^{m_n} |f_{nm}(t)|^2 \geq \delta, \quad n=1, 2, \cdots\cdots.$$

Whence, by Lemma 5. 1. 1, we get

(48) Even if we replace a_{nm} with the median M_{nm} of $F_{nm}(x)$ and a'_{nm} with $a^*_{nm}=\int_{-1}^{1} x dF_{nm}(x+M_{nm})$, this theorem holds, the proof of which is almost analogous to that of Theorem 5.3.2.

$$|\sum_{m=1}^{m_n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]-\log f(t)|$$

(5.3.6)

$$=\left|\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG_n(x)+it\sum_{m=1}^{m_n}\left(\int_{-\infty}^{\infty}\frac{x}{1+x^2}dF''_{nm}(x)+a_{nm}+a'_{nm}\right)\right.$$

$$\left.-\log f(t)\right|\to 0,\quad (n\to\infty)$$

uniformly for every finite interval, where

$$G_n(x)=\sum_{m=1}^{m_n}\int_{-\infty}^{x}\frac{u^2}{1+u^2}dF''_{nm}(u).$$

And also by (5.1.5) of the same Lemma, we see $\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|$ is dominated by $C(t^2+2|t|+4)$ where C is an absolute constant. These facts imply

$$8C\geq\sum_{m=1}^{m_n}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF''_{nm}(x)$$

(5.3.7)

$$=-\int_0^{\infty}e^{-t}\sum_{m=1}^{m_n}[\Re(f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\})-1]dt$$

$$\to-\int_0^{\infty}e^{-t}\Re\log f(t)dt=\int_0^{\infty}e^{-t}\left\{\int_{-\infty}^{\infty}(1-\cos tx)\frac{1+x^2}{x^2}dG(x)\right\}dt$$

$$=\int_{-\infty}^{\infty}dG(x)=G(\infty),$$

which shows (5.3.1). Next by the compactness and the fact

$$8C\geq\sum_{m=1}^{m_n}\int_{-\infty}^{u}\frac{x^2}{1+x^2}dF''_{nm}(x),\quad n=1,2,\cdots\cdots$$

we have a subsequence $\{n_i\}$ and $G^*(x)$ ($G^*(\infty)=G(\infty)$, $G^*(-\infty)=G(-\infty)=0$) satisfying

$$G_{n_i}(u)\equiv\sum_{m=1}^{m_{n_i}}\int_{-\infty}^{u}\frac{x^2}{1+x^2}dF''_{n_i m}(x)\to G^*(u),\ (i\to\infty)$$

at the continuity point of $G^*(u)$. Hence we have, applying Helly's theorem

$$\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG_{n_i}(x)$$

$$\underset{i\to\infty}{\to}\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dG^*(x),$$

from which by (5.3.6) we obtain a constant A^* so that

$$\sum_{m=1}^{m_{n_i}}\left\{\int_{-\infty}^{\infty}\frac{x}{1+x^2}dF_{n_i m}(x)+a_{n_i m}+a'_{n_i m}\right\}\to A^*.$$

However, by the unicity of the representation of logf (t) we see $G^*(x)=G(x)$ and $A=A^*$. Thus we can conclude (5.3.2) and (5.3.3).

2. *Sufficiency.* By the condition (5.3.1) we have a constant M such that

$$\sum_{m=1}^{m_n}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF''_{nm}(x)\leq M,\quad n=1,2,\cdots\cdots$$

and from the individual negligibility of $\|X_{nm}\|$, appealing to (F. I. 1) aud (F. I. 2), it follows that for $n\geq N_0\geq 0$

$$\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(1)\}\leq R\sum_{m=1}^{m_n}\{1-\Psi_{F''_{nm}}(1)\}\leq 2R\sum_{m=1}^{m_n}\{1-\Phi_{F''_{mn}}(1)\}\leq 2RM,$$

where $F'_{nm}(x)=F_{nm}(x+a_{nm})$, $a'_{nm}=\int_{-1}^{1}xdF_{nm}(x)$ and R is a constant. Whence by recalling (F. I. 6), denoting $a'_{nm}=\int_{-1}^{1}xdF'_{nm}(x)$, we see

$$\begin{aligned}&\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\\&\qquad\leq(t^2+2|t|+4)\sum_{m=1}^{m_n}\{1-\Phi_{F'_{nm}}(1)\}\leq(t^2+2|t|+4)2RM.\end{aligned}$$

But by (5.3.4) we have

$$\begin{aligned}&\left|\sum_{m=1}^{m_n}\log f_{nm}(t)-\sum_{m=1}^{m_n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]\right|\\&\qquad\leq\sum_{m=1}^{m_n}|f_{nm}(t)\exp\{-it(a_{nm}+a_{nm})\}-1|^2\end{aligned}$$

for sufficiently large values of n. Hence (5.3.4) imply

(5.3.8) $\sum_{m=1}^{m_n}\log f_{nm}(t)-\sum_{m=1}^{m_n}[f_{nm}(\mathrm{t})\exp\{it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]$

uniformly for every finite interval. On the other hand, by (5.3.1), (5.3.2), (5.3.3) and Helly's theorem,

$$\sum_{m=1}^{m_n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]$$

$$=\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG_n(x)$$

$$+it\left(\sum_{m=1}^{m_n}\left(\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF_{nm}(x)+a_{nm}+a'_{nm}\right)\right)$$

$$\rightarrow\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x}\right)\frac{1+x^2}{x^2}dG(x)+itA$$

in the same sense as above, where

$$G_n(x)=\sum_{m=1}^{m_n}\int_{-\infty}^{x}\frac{u^2}{1+u^2}dF''_{nm}(u).$$

Hence we have, by (5.3.8),

$$\log\Pi_{m=1}^{m_n}f_{nm}(t)=\sum_{m=1}^{m_n}\log f_{nm}(t)$$

$$\rightarrow\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x)+itA=\log f(t)$$

uniformly for every finite interval, which completes the proof of Theorem 5. 3. 2.

§ **5. 4. The partial limit law.** Let $\{X_n|n=1,2,\cdots\cdots\}$ be a sequence of independent random variables all having the same distribution. Then if there exist two sequences of real numbers $\{A_n>0|n=1,2,\cdots\cdots\}$, $\{\infty>B_n>-\infty|n=1,2,\cdots\cdots\}$ and a sequence of positive integers $\{n_m|m=1,2,\cdots\cdots\}$ such that the distribution of

$$\frac{1}{A_{n_i}}\left(\sum_{m=1}^{n_i}X_m-B_{n_i}\right)\tag{5.4.1}$$

tends to a distribution $F(x)$ in Bernoulli's sense as $i\rightarrow\infty$, it is called that $F(x)$ depends on a partial limit law. Any partial limit law is, by Theorem 5. 3. 1, an infinitely divisible law. And the converse is also true. This latter fact was first given by A. Khintchine(44) and next W. Doeblin(45) gave another proof. But there remains in his proof much to be desired. Here we show an proof in W. Doeblin's line more strictly.

(44) A. Khintchine, loc. cit. (9).

(45) W. Doeblin, Étude de l'ensemble de puissances d'une loi de probablité, St. Math., 9 (1), pp. 71—96, 1939.

Theorem 5. 4. 1. *Let $F(x)$ be the distribution function depending on an infinitely divisible law, then we can select a law L_0 so that the distribution of* (5.4.1) *tends to $F(x)$ in Bernoulli's sense as $i\to\infty$, where $X_1, X_2,\cdots\cdots, X_m, \cdots\cdots$ mutually independent random variables all depending on L_0.*

Proof[(46)]. Let $f(t)$ be the characteristic function corresponding to $F(x)$. Then from the definition (I) there exists a system of characteristic functions $f_{n1}(t), f_{n}(t),\cdots\cdots, f_{nn}(t)$ $(n=1, 2.\cdots\cdots)$ such that $f(t)=\Pi_{m=1}^{n}f_{nm}(t)$ and $f_{nm}(t)\to 1$, $(n\to\infty)$ uniformly for every finite interval and $m(1\leq m\leq n)$. Hence by Lemma 5. 1. 1 we get

$$(5.4.2) \qquad |\log f(t)-\textstyle\sum_{m=1}^{n}[f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1+it(a_{nm}+a'_{nm})]\underset{n\to\infty}{\to} 0$$

uniformly every finite interval and for every $t(-\infty<t<\infty)$

$$(5.4.3) \qquad \textstyle\sum_{m=1}^{n}|f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1|\leq(t^2+2|t|+4)C$$

where C is a constant and a_{nm} and a'_{nm} are the same as is the previous section. Now we consider the following function:

$$\log\varphi(t)\equiv\textstyle\sum_{n=1}^{\infty}2^{-n^2}\sum_{m=1}^{n}\displaystyle\int_{-\infty}^{\infty}(e^{itx}-1)dG_{nm}(x/A_n), \quad A_n=2^{n^3},$$

where

$$G_{nm}(x)=\begin{cases} F''_{nm}(x) \ \textit{for} \ |x|\leq n, \\ F''_{nm}(n) \ \textit{for} \ x>n, \\ F''_{nm}(-n) \ \textit{for} \ x<-n, \end{cases}$$

and $F''_{nm}(x)=F_{nm}(x+a_{nm}+a'_{nm})$. Then $\varphi(t)$ is clearly the characteristic function of a probability distribution, in fact, $\varphi(t)$ depends on an infinitely divisible law. Since, in the equality

$$2^{n^2}\log\varphi(t/A_n)$$

$$=2^{n^2}\textstyle\sum_{k=1}^{n-1}2^{-k^2}\sum_{m=1}^{k}\displaystyle\int_{-\infty}^{\infty}(e^{itx/A_n}-1)dG_{km}(x/A_k) \quad (\equiv I_1)$$

$$+\textstyle\sum_{m=1}^{n}\displaystyle\int_{-\infty}^{\infty}(e^{itx/A_n}-1)dG_{nm}(x/A_n) \quad (\equiv I_2)$$

---

(46) In the proof of Theorem 5. 4. 1, we may also repace a_{nm} with the median M_{nm} of $F_{nm}(x)$ and a'^{nm} with a^*_{nm}.

$$+2^{n^2}\sum_{k=n+1}^{\infty} 2^{-k^2}\sum_{m=1}^{k}\int_{-\infty}^{\infty}(e^{itx/A_n}-1)dG_{km}(x/A_k)\ (\equiv I_3),$$

we have

$$|I_1|\leq 2^{n^2}\sum_{k=1}^{n-1}2^{-k^2}|t|\frac{A_k}{A_n}\sum_{m=1}^{k}\int_{-\infty}^{\infty}|x|dG_{km}(x)$$

$$\leq 2^{n^2}\sum_{k=1}^{n-1}2^{-k^2}k^2\frac{A_k}{A_n}|t|$$

$$\leq 2^{n^2+(n-1)^3-n^3}\sum_{k=1}^{n-1}2^{-k^2}k^2|t|=o(|t|),\quad (n\to\infty)$$

and

$$|I_3|\leq 2^{n^2}2\sum_{k=n+1}^{\infty}k2^{-k^2}=o(1),\quad (n\to\infty),$$

we get

$$\left|2^{n^2}\log\varphi(t/A_n)-\sum_{m=1}^{n}\int_{-\infty}^{\infty}(e^{itx/A_n}-1)dG_{nm}(x/A_n)\right|$$

(5.44)

$$=\left|2^{n^2}\log\varphi(t/A_n)-\sum_{m=1}^{n}\int_{-\infty}^{\infty}(e^{itx}-1)dG_{nm}(x)\right|$$

$$\to 0,\quad (n\to\infty)$$

uniformly for every finite interval. On the other hand, by (5.4.2) and (5.4.3) we have

$$8C\geq -\sum_{m=1}^{n}\int_0^{\infty}e^{-t}[\Re f_{nm}(t)\exp\{-it(a_{nm}+a'_{nm})\}-1]dt$$

$$=\sum_{m=1}^{n}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF''_{nm}(x)$$

$$\to -\int_0^{\infty}e^{-t}\Re\log f(t)dt,\quad (n\to\infty).$$

Hence we have a positive number N so that

$$\frac{1}{2}\sum_{m=1}^{n}\int_{|x|\geq N}dF''_{nm}(x)\leq\sum_{m=1}^{n}\int_{|x|\geq N}\frac{x^2}{1+x^2}dF''_{nm}(x)<\frac{\epsilon}{2}$$

for any $\epsilon>0$. Then we see

$$\sum_{m=1}^{n}\int_{|x|\geq n} dF''_{nm}(x) \leq \sum_{m=1}^{n}\int_{|x|\geq N} dF''_{nm}(x) < \epsilon$$

for $n \geq N$. Therefore,

$$\Big|\sum_{m=1}^{n}\Big\{\int_{-\infty}^{\infty}(e^{itx}-1)dG_{nm}(x)+it(a_{nm}+a'_{nm})\Big\} - \sum_{m=1}^{n}\int_{-\infty}^{\infty}\{e^{itx}-1+it(a_{nm}+a'_{nm})\}dF''_{nm}(x)\Big| \to 0;$$

apylying (5.4.2), we find

$$\Big|\sum_{m=1}^{n}\Big\{\int_{-\infty}^{\infty}(e^{itx}-1)dG_{nm}(x)+it(a_{nm}+a'_{nm})\Big\} - \log f(t)\Big| \underset{n\to\infty}{\to} 0$$

uniformly for every finite interval. Consequently (5.4.4) implies

$$|2^{n^2}\log[\varphi(t/A_n)\exp\{it(a_{nm}+a'_{nm})/2^{n^2}\}] - \log f(t)| \underset{n\to\infty}{\to} 0$$

in the same sense as above. Thus we see that the distribution of

$$\sum_{m=1}^{2^{n^2}}\{X_m/2^{n^3}+(a_{nm}+a'_{nm})/2^{n^2}\}$$

tends to $F(x)$ in Bernoulli's sense, which completes the proof of Theorem 5. 4. 1.

Chapter VI. On the estimation of the magnitudes of the sums of independent random variables.

§ **6. 1. The strong law of large numbers.** Let $\{X_k | k=1, 2, \cdots\cdots\}$ be a seuqence of independent random variables and denote by $F_k(x)$ the distribution function of $X_k (k=1, 2, \cdots\cdots)$. Then it is called that $\{X_k\}$ obeys the *strong law of large numbers* if we have

(6.1.1) $$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - \int_{-k}^{k} x dF_k(x)\Big) = 0 \text{ and } \lim_{n\to\infty} X_n/n = 0$$

with probability 1.

Sufficient conditions for the validity of this law where given by many authors. Here we give a theorem showing a sufficient condition from our viewpoit without proof.

Theorem 6. 1. 1. *Let $\{X_n | n=1,2,\cdots\cdots\}$ be a sequence of independent random variables, then a sufficient condition for the existence of the strong law of large numbers is that there exists either of the following condi-*

tions :

(6.1.2) $$\sum_{n=1}^{\infty}\{1-\Phi_{F_n}(n)\}<\infty$$

and

(6.1.3) $$\sum_{n=1}^{\infty}\{1-\Phi_{F'_n}(n)\}<\infty,$$

where $F_n(x)$ *is the distribution function of* X_n $(n=1, 2,\cdots\cdots)$, $F'_n(x)=F_n(x+na_n)$ *and* $a_n=\int_{-1}^{1} xdF_n\ (nx)$

Next we shall search for the case in which (6.1.2) is a necessary and sufficient condition.

Theorem 6. 1. 2. *Let* $\{X_n|n=1,2,\cdots\cdots\}$ *be a sequence of independent random variables all having the same distribution* $F(x)$, *then a necessary and sufficient condition that the strong law of large numbers should hold true for* $\{X_n\}$ *is*

(6.1.4) $$\sum_{n=1}^{\infty}\{1-\Phi_F(n)\}<\infty$$

We need the following lemma for the proof.

Lemma 6. 1. 1. *Let* $F(x)$ *be a given distribution and let* $\{\tau_n|n=1, 2,\cdots\cdots\}$ *be a non-decreasing sequence of positive numbers. Then*

$$\sum_{n=1}^{\infty}\{1-\Phi_F(n\tau_n)\}\ \textit{and}\ \sum_{n=1}^{\infty}\int_{|x|\geq n\tau_n} dF(x)$$

are simultaneously convergent or divergent.

Proof. As

$$\sum_{n=1}^{\infty}\int_{|x|>n\tau_n} dF(x)<\infty$$

follows clearly from

$$\sum_{n=1}^{\infty}\{1-\Phi_F(n\tau_n)\}<\infty,$$

it is sufficient to show the converse. Without loss of generality we suppose $\tau_1\geq 1$. Now put

$$\int_{|x|\geq n\tau_n} dF(x)=\delta_n,$$

then we have, by the assumption, $\delta=\delta_1+\delta_2+\cdots\cdots\delta_n+\cdots\cdots$. If

$$\int_{-\infty}^{\infty} x^2 dF(x) < \infty,$$

then we see

$$\sum_{n=1}^{\infty}\{1-\Phi_F(n)\} \geq \sum_{n=1}^{\infty}\frac{1}{n^2}\int_{-\infty}^{\infty} x^2 dF(x) < \infty.$$

Hence we may consider the case:

$$\int_{-\infty}^{\infty} x^2 dF(x) = \infty.$$

By partial integration,(47)

$$\int_{|x|\leq n\tau_n} x^2 \; dF(x) = -n^2\tau_n^2\int_{|x|>n\tau_n} dF(x) + 2\int_0^{n\tau_n} v dv \int_{|x|>v} dF(x)$$

$$\leq 2\int_0^{n\tau_n} v dv \int_{|x|>v} dF(x) \leq 2\sum_{k=1}^{n} k\tau_k \int_{|x|>(k-1)\tau_{k-1}} dF(x)$$

$$= 2\sum_{k=1}^{n} k\tau_k\delta_{k-1}.$$

Hence

$$\sum_{n=1}^{N} \frac{1}{(n\tau_n)^2}\int_{|x|\leq n\tau_n} x^2 dF(x) \leq 2\sum_{n=1}^{N}\frac{1}{(n\tau_n)^2}\sum_{k=1}^{n} k\tau_k\delta_{k-1}$$

$$\leq 2\sum_{n=1}^{N}\frac{1}{n^2}\sum_{k=1}^{n} k\delta_{k-1} \leq 2\sum_{n=1}^{N}\frac{1}{n^2}\sum_{k=1}^{n}(k-1)\delta_{k-1} + 2\delta\sum_{n=1}^{N}\frac{1}{n^2}$$

$$\leq 2\sum_{k=1}^{N}(k-1)\delta_{k-1}\sum_{n=k}^{N}\frac{1}{n^2} + 4\delta$$

$$\leq 2\sum_{k=1}^{N}\delta_{k-1} + 4\delta \leq 6\delta < \infty.$$

Thus

$$\sum_{n=1}^{\infty}\{1-\Phi_F(n\tau_n)\} \leq \sum_{n=1}^{\infty}\left\{\frac{1}{(n\tau_n)^2}\int_{|x|\leq n\tau_n} x^2 dF(x) + \int_{|x|>n\tau_n} dF(x)\right\} < \infty.$$

Proof of Theorem 6. 1. 2. As sufficiency follows from Theorem

(47) Cf. H. Cramér, Random variables and probability distributions, Cambridge tracts, 36, p. 41, 1937.

6. 1. 1, it remains to show its necessity. Since by the assumption :

$$P_\gamma\{X_n/n\to 0\}=1,$$

we see for infinitely many n's

$$|X_n|<n$$

with probability 1. Hence, applying Borel-Cantelli's theorem, we get

$$\sum\int_{|x|>n} dF(x)<+\infty.$$

Hence we obtain, by appealing to Lemma 6. 1. 1, (6. 1. 4).

§ **6. 2. A problem of A. Khintchine.** Let $\{X_n|n=1, 2,\cdots\cdots\}$ be a sequence of positive mutually independent random variables all having the same distribution function F(x) (with the corresponding characteristic function $f(t)$) such that

$$\int_0^\infty x dF(x)=\infty \tag{6.2.1}$$

and denote by S_n the n-th partial sum of $\{X_n\}$. Then the following problem arises, which was proposed by A. Khintchine[48], i. e., what is the necessary and sufficient condition for the existence of $\{\kappa(n)\}$ such that

$$\lim_{n\to\infty}\frac{S_n}{n\kappa(n)}=1$$

with probablity 1? The object of the present section is to give an answer to this problem under the assumption

$$\int_0^{\lambda x} v dF(v)\leq C(\lambda)\int_0^x v dF(v), \tag{6.2.2}$$

where $\lambda>1$ and $C(\lambda)\to 1$, as $\lambda\to 1$. Then we can formulate our theorem as follows:

Theorem 6. 2. 1. *A necssary and sufficient condition for the validity of*

$$P_\gamma\{lim\ S_n/n\ \kappa(n)=1\}=1$$

[48] A. Khintchine, loc. cit. (21).

is the existence of a sequence $\{C_n>0|n=1,2,\cdots\cdots\}$ *such that*

(6.2.3) $$\cdots\cdots<C_n<C_{n+1}<\cdots\cdots\to\infty,\quad (n\to\infty)$$

(6.2.4) $$\sum_{n=1}^{\infty}\{1-\Phi_F(C_n)\}<\infty$$

and

(6.2.5) $$\frac{n}{C_n}\int_0^{C_n}xdF(x)\to\infty,\quad (n\to\infty),$$

where $\kappa(n)$ *monotonically tends to* ∞ *as* $n\to\infty$, *and satisfies, for* $C(\lambda)$ *defined by* (6.2.2),

(6.2.6) $$\kappa(\lambda n)\leq C(\lambda)\kappa(n),\quad (n\geq n_0>0\,;\ \lambda>1).$$

For the proof of Theorem 6. 2. 1 we need the following lemma.

Lemma 6. 2. 1. *Let* $\{X_n|n=1,2,\cdots\cdots\}$ *be a sequence of positive mutually independent random variables all having the same distribution* $F(x)$ *satisfying* (6.2.1), *and let* $\{C_n|n=1,2,\cdots\cdots\}$ *be the sequence defined by* (6.2.3), (6.2.4) *and* (6.2.5). *Then, for any* $\epsilon>0$, *if* n *is sufficiently large, we have*

$$P_r\left\{\left|S_n-n\int_0^{C_n}xdF(x)\right|\geq\frac{\epsilon}{2}n\int_0^{C_n}xdF(x)\right\}\leq Kn\{1-\Phi_F(C_n)\}$$

where K *is a constant.*

Proof. Put

$$a_n=\int_0^{C_n}xdF(x),$$

then from (F. I. 6), we see

$$n|f'(t/C_n)-1|\leq n(t^2+2|t|+4)\{1-\Phi_F(C_n)\},$$

where

$$f'(t/C_n)\equiv f(t/C_n)\exp\{-ia_nt/C_n\}.$$

Hence by (6. 2. 4)

$$n|f'(t/C_n)-1|\to 0,\quad (n\to\infty)$$

uniformly for every finite interval. Whence for sufficiently large values of n

$$|n \log f'(t/C_n) - n\{f'(t/C_n) - 1\}| \leq n|f'(t/C_n) - 1|^2.$$

Therefore we easily have

$$\{f'(t/C_n)\}^n = f^n(t/C_n)\exp(-ina_n t/C_n) \to 1, \quad (n \to \infty)$$

uniformly for every finite interval. Now put $F(x+a_n)^{n*} = F_n(x)$ and $\alpha_n = \int_{-C_n}^{C_n} x dF_n(x)$, then paying attention to $|\alpha_n| \leq \frac{1}{2} C_n$ for sufficiently large values of $n (\geq n_0)$, (F. I. 1), (F. I. 2) and (F. I. 4) imply that for $n \geq n_0$

$$P_\gamma\{|S_n - na_n| > C_n\} \leq P_\gamma\{|S_n - na_n - \alpha_n| > C_n - |\alpha_n|\}$$

$$\leq 2\int_{-\infty}^{\infty} \frac{x^2}{(C_n - |\alpha_n|)^2 + x^2} dF_n(x+\alpha_n) \leq 8\{1 - \Phi_{F_n(x+\alpha_n)}(C_n)\}$$

$$\leq 8R\}1 - \Psi_{F_n(x)}(C_n)\} \leq 8Rn\{1 - \Psi_F(C_n)\}$$

$$\leq 16Rn\{1 - \Phi_F(C_n)\}$$

where R is a constant. On the other hand, as

$$\frac{C_n}{na_n} = C_n \frac{1}{n\int_0^{C_n} x dF(x)} \to 0, \quad (n \to \infty)$$

follows from (6. 2. 5), we get, for any $\epsilon > 0$, $C_n/na_n \leq \epsilon/2$ $(n \geq n_0)$. Consequently

$$P_\gamma\left\{\left|\frac{S_n}{na_n} - 1\right| \geq \epsilon/2\right\} \leq Kn\{1 - \Phi(C_n)\}$$

where $K = 16R$.

Lemma 6. 2. 2. *Let* $\{X_k | k = 1, 2, \cdots\cdots, n\}$ *be a system of independent and symmetric variables and put* $S_n = X_1 + X_2 + \cdots\cdots + X_n$. *Then if*

$$P_\gamma\{|S_n| \geq a\} \leq 1/3,$$

we have

$$\sum_{k=1}^n P_\gamma\{|X_k| \geq 2a\} \leq 12 P_\gamma\{|S_n| \geq a\}.$$

This lemma was given by J. Marcinkiewicz(49).

(49) J. Marcinkiewicz, loc. cit. (15).

Proof of Theorem 6. 2. 1. In the first place, choose a sequence $\{n_i \mid i=1, 2, \cdots\cdots\}$ satisfying that for fixed $\gamma(>1)$ and $\delta(>0)$

$$(n_{i+1}-1)/n_i \leq \gamma < n_{i+1}/n_i \quad (\gamma>1\,;\ i=1, 2, \cdots\cdots) \tag{6.2.7}$$

and

$$1/n_i \leq \delta \quad (i=0, 1, 2, \cdots\cdots), \tag{6.2.8}$$

then we see

$$n_{i+1}\kappa(n_{i+1})/n_i\kappa(n_i) > \gamma \quad (i=0, 1, 2, \cdots\cdots) \tag{6.2.9}$$

and

$$\frac{n_{i+1}\kappa(n_{i+1})}{n_i\kappa(n_i)} = \left(\frac{n_{i+1}-1}{n_i}+\frac{1}{n_i}\right)\frac{\kappa(n_{i+1})}{\kappa(n_i)} \leq (\gamma+\delta)\frac{\kappa(\mu n_i)}{\kappa(n_i)} \leq (\gamma+\delta)C(\mu) \tag{6.2.10}$$

where $\mu n_i \geq \gamma n_i + 1 > n_{i+1}(i=0, 1, 2, \cdots\cdots)$ and γ, δ and μ will be determined afterwards.

1°. *Sufficiency.* Put

$$\kappa(n) = \int_0^{C_n} x dF(x),$$

then by (6. 2. 2), we see that (6. 2. 6) is clearly satisfied and

$$n\kappa(n)/C_n = n\int_0^{C_n} x/C_n dF(x) \to \infty, \quad (n\to\infty).$$

Hence, for any $\eta>0$, if n is sufficiently large, we have $C_n \leq n\kappa(n)\eta$. Whence

$$P_\gamma\{X_n \geq n\kappa(n)\eta\} \leq P_\gamma\{X_n \geq C_n\} \to 0, \quad (n\to\infty).$$

Now we have for any $\epsilon>0$

$$P_\gamma\left\{\bigcap_{n=n_0}^{\infty}\left(\left|\frac{S_n}{n\kappa(n)}-1\right| \leq \epsilon\right)\right\}$$

$$= P_\gamma\left\{\bigcap_{i=0}^{\infty}\bigcap_{n=n_i}^{n_{i+1}-1}\left(\left|\frac{S_n}{n\kappa(n)}-1\right| \leq \epsilon\right)\right\}$$

$$= P_\gamma\left\{\bigcap_{i=0}^{\infty}\bigcap_{n=n_i}^{n_{i+1}-1}\left(\frac{S_n}{n\kappa(n)} < 1+\epsilon\right)\cap\left(\frac{S_n}{n\kappa(n)} > 1-\epsilon\right)\right\}$$

$$\geq P_\gamma\left\{\bigcap_{i=0}^{\infty}\left(\frac{S_{n_{i+1}}}{n_i\kappa(n_i)}<1+\epsilon\right)\frown\left(\frac{S_{n_i}}{n_{i+1}\kappa(n_{i+1})}>1-\epsilon\right)\right\}$$

$$=P_\gamma\left\{\bigcap_{i=0}^{\infty}\left(\frac{S_{n_{i+1}}}{n_{i+1}\kappa(n_{i+1})}<\frac{n_i\kappa(n_i)}{n_{i+1}\kappa(n_{i+1})}(1+\epsilon)\right)\frown\left(\frac{S_{n_i}}{n_i\kappa(n_i)}>\frac{n_{i+1}\kappa(n_{i+1})}{n_i\kappa(n_i)}(1-\epsilon)\right)\right\}$$

By (6. 2. 9) ond (6. 2. 10)

$$P_\gamma\left\{\bigcap_{n=n_0}^{\infty}\left(\left|\frac{S_n}{n\kappa(n)}-1\right|\leq\epsilon\right)\right\}$$

$$\geq P_\gamma\left\{\bigcap_{i=0}^{\infty}\left(\frac{S_{ni+1}}{n_{i+1}\kappa(n_{i+1})}<\frac{1+\epsilon}{(\gamma+\delta)C(\mu)}\right)\frown\left(\frac{S_{n_i}}{n_i\kappa(n_i)}>(\gamma+\delta)C(\mu)(1-\epsilon)\right\}.$$

From (6.2.9) we can determine γ, δ and μ such that

$$\frac{1+\epsilon}{(\gamma+\delta)C(\mu)}\geq 1+\frac{\epsilon}{2},\quad (\gamma+\delta)C(\mu)(1-\epsilon)<1-\frac{\epsilon}{2}.$$

Then we have

$$P_\gamma\left\{\bigcap_{n=n_0}^{\infty}\left(\left|\frac{S_n}{n\kappa(n)}-1\right|\leq\epsilon\right)\right\}$$

$$\geq P_\gamma\left\{\bigcap_{n=n_0}^{\infty}\left(\frac{S_{n_{i+1}}}{n_{i+1}\kappa(n_{i+1})}<1+\frac{\epsilon}{2}\right)\frown\left(\frac{S_{ni}}{n_i\kappa(n_i)}>1-\frac{\epsilon}{2}\right)\right\}$$

$$\geq P_\gamma\left\{\bigcap_{i=0}^{\infty}\left(\left|\frac{S_{n_i}}{n_i\kappa(n_i)}-1\right|<\frac{\epsilon}{2}\right)\right\}.$$

Hence, by Lemma 6. 2. 1 we have a constant K so that

$$P_\gamma\left\{\bigcup_{n=n_0}^{\infty}\left(\left|\frac{S_n}{n\kappa(n)}-1\right|>\epsilon\right)\right\}\leq P_\gamma\left\{\bigcup_{i=0}^{\infty}\left(\left|\frac{S_{n_i}}{n_i\kappa(n_i)}-1\right|\geq\frac{\epsilon}{2}\right)\right\}$$

$$\leq K\sum_{i=0}^{\infty}n_i\{1-\Phi_F(C_{n_i})\}.$$

By (6. 2. 7) putting $\gamma=1+\gamma'(\gamma'>0)$, we have

$$P_\gamma\left\{\bigcup_{n=n_0}^{\infty}\left(\left|\frac{S_n}{n\kappa(n)}-1\right|>\epsilon\right)\right\}$$

$$\leq K(\gamma+\delta)\sum_{i=1}^{\infty}n_{i-1}\{1-\Phi_F(C_{n_i})\}+Kn_0\{1-\Phi_F(C_{n_0})\}$$

$$\leq \frac{K(\gamma+\delta)}{\gamma'} \sum_{i=1}^{\infty} (n_i - n_{i-1})\{1-\Phi_F(C_{n_i})\} + Kn_0\{1-\Phi_F(C_{n_0})\}$$

$$\leq \frac{K(\gamma+\delta)}{\gamma'} \sum_{n=n_0}^{\infty} \{1-\Phi_F(C_n)\} + Kn_0\{1-\Phi_F(C_{n_0})\}.$$

Hence (6. 2. 4) implies

$$\lim_{n_0\to\infty} P_\gamma\left\{\bigcup_{n=n_0}^{\infty}\left(\left|\frac{S_n}{n\kappa(n)}-1\right|>\epsilon\right)\right\}=0, \ .$$

which is true that with probability 1 $S_n/n\kappa(n)\underset{n\to\infty}{\to}1$.

2, *Necessity*. Let X_1, $\bar{X}_1$, X_2, $\bar{X}_2$, ... be mutually independent random variables all having the same distribution function $F(x)$ (6. 2. 1) and put $\tilde{X}_k = X_k - \bar{X}_k (k=1, 2, \cdots\cdots)$ and $\tilde{S}_n = \tilde{X}_1 + \tilde{X}_2 + \cdots\cdots + \tilde{X}_n$. Then, by the assumption, we clearly have

$$P_\gamma\{\lim_{n\to\infty} \tilde{S}_n/n\kappa(n) = 0\} = 1.$$

where $\{\kappa(n)\}$ satisfies (6. 2. 6). Then, we easily see that there exists for any $\epsilon>0$ and for infinitely many i's,

$$|\tilde{S}_{n_{i+1}} - \tilde{S}_{n_i}| \leq \epsilon n_i \kappa(n_i)$$

with probability 1, where $\{n_i\}$ is the subsequeuce defined by (6. 2. 7). Hence, by Borel-Cantelli's thearem, we have $\sum\delta_i<\infty$, where $\delta_i = P_\gamma \{|\tilde{S}_{n_{i+1}} - \tilde{S}_{n_i}| > \epsilon n_i \kappa(n_i)\}$. By Lemma 6. 2. 2

$$\sum_{i=0}^{\infty}\delta_i = \sum_{i=0}^{\infty} P_\gamma\{|\tilde{S}_{n_{i+1}} - \tilde{S}_{n_i}| > \epsilon n_i \kappa(n_i)\}$$

$$\geq A\sum_{i=0}^{\infty}\sum_{n_i+1}^{n_{i+1}} P_\gamma\{|\tilde{X}_n| > 4\epsilon n_i \kappa(n_i)\}$$

$$\geq A\sum_{n=n_0+1}^{\infty} P_\gamma\{(\tilde{X}_n| > 4\epsilon n\kappa(n)\}$$

$$\geq A\sum_{n=n_0+1}^{\infty} P_\gamma\{(X_n > \delta\epsilon n\kappa(n)) \frown (\bar{X}_n < 4\epsilon n\kappa(n))\}$$

$$= A\sum_{n=n_0+1}^{\infty} P_\gamma\{X_n > 8\epsilon n\kappa(n)\} P_\gamma\{\bar{X}_n < 4\epsilon n\kappa(n)\},$$

where A is an absolute constant. As we have

$$P_\gamma\{\bar{X}_n < 4\epsilon n\kappa(n)\} = \int_{|x|<4\epsilon n k(n)} dF(x) \geq \frac{1}{2}$$

for sufficiently large values of n $(>n_0)$, we obtain

$$\infty > \sum_{n=n_0+1}^{\infty} \delta_n \geq \frac{A}{2} \sum_{n=n_0+1}^{\infty} P_Y\{X_n > 8\epsilon n\kappa(n)\}.$$

Hence, by Lemma 6. 1. 1, we have

$$\sum_{n=1}^{\infty} \{1-\Phi_F(8\epsilon n\kappa(n))\} < \infty.$$

Whence, for any small $\eta > 0$ we have

$$\sum_{n=1}^{\infty} \{1-\Phi_F(\eta n\kappa(n))\} < \infty. \tag{6.2.11}$$

Furthermore by the monoton increase of $\kappa(n)$ we see clearly $\eta n\kappa(n) \leq \eta(n+1)\kappa(n+1)$ $(n=1, 2, \cdots\cdots)$ for any $\eta > 0$. Next show that

$$\frac{n}{\eta n\kappa(n)} \int_0^{\eta n\kappa(n)} x dF(x) \to \infty. \tag{6.2.12}$$

Suppose that (6. 2. 12) is not true, then there exist two subsequences $\{n_k \mid k=1, 2, \cdots\cdots\}$, $\{\eta_{n_k} \downarrow 0 \mid k=1, 2, \cdots\cdots\}$ and a constant M such that

$$\frac{1}{\eta_{n_k}\kappa(n_k)} \int_0^{\eta_{n_k} n_k \kappa(n_k)} x dF(x) \leq M.$$

Since we easily have $n_k\{1-F(\eta_{n_k} n_k \kappa(n_k))\} = o(1)$, $(k \to \infty)$,

$$n_k \left| f\left(\frac{t}{n_k\kappa(n_k)}\right) - 1 \right| = n_k \left| \int_0^{\infty} \left\{ e^{itx/n_k\kappa(n_k)} - 1 \right\} dF(x) \right|$$

$$\leq 2n_k \int_{|x| > \eta_{n_k} n_k \kappa(n_k)} dF(x) + \frac{t}{\kappa(n_k)} \int_0^{\eta_{n_k} n_k \kappa(n_k)} x dF(x) = o(1), \quad (k \to \infty).$$

As

$$|n \log f(t/n\kappa(n)) - n\{f(t/\eta\kappa(n)) - 1\}| \leq |f(t/n\kappa(n)) - 1|^2$$

for sufficiently large values of n, we clearly see $f^n(t/n\kappa(n)) \to 1$ $(n \to \infty)$, which is contrary to the first assumption. Thus we can conclude (6. 2. 3), (6. 2. 4) and (6. 2 .5) if we put $\eta n\kappa(n) = C_n$.

§ **6. 3. An extension of P. Lévy—J. Marcinkiewicz's theorem.** Let $\{\theta_n\}$ be a sequence of positive numbers monotonically tending to ∞. Then it is called, according to P. Lévy, that $\{\theta_n\}$ belongs to an upper class $\mathfrak{U}$ (a lower class $\mathfrak{L}$) if we have with probability 1

$$|S_n| > \theta_n$$

for at most finitely many n's (infinitely many n's)

Now given a sequence of independent random variables $\{X_n|n=1, 2, \cdots\cdots\}$ having respectively distribution functions $F_n(x)$ $(n=1,2,\cdots\cdots)$ such that

(6.3.1) $$\frac{1}{Z^2}\int_{|x|\leq Z} x^2 dF_n(x) \leq A\int_{|x|>Z} dF_n(x) \quad (n=1, 2, \cdots\cdots, \ Z\geq Z_0),$$

where A and Z_0 are consatnts independent of n. And furthermore suppose that the expectation $E(X_n)$ of X_n is zero when it exists. Then we have

Theorem 6. 3. 1. *Let $\{X_n\}$ be a sequence of the independent random variables having the properties quoted above and let $\{\theta_n\}$ be a sequence of positive numbers monotonically tending to ∞ such that*

(6.3.2) $$\sum_{m=1}^{n}\{1-\Phi_{F_m}(\theta_n)\}\to 0, \quad (n\to\infty),$$

(6.3.3) $$\frac{1}{\theta_n}(a_{1n}+a_{2n}+\cdots\cdots+a_{nm})\to 0, \quad (n\to\infty)$$

and there exist a constant $\delta(1>\delta>0)$ and subsequence $\{\theta_{n_k}|k=0, 1, 2,\cdots\cdots\}$ satisfying

(6.3.4) $$\theta_{n_k}/\theta_{n_{k+1}}<\delta\leq\theta_{n_k+1}/\theta_{n_{k+1}}, \quad k=0, \ 1, 2,\cdots\cdots,$$

then if

(6.3.5) $$\sum_{n=1}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}<\infty,$$

$\{\theta_n\}\in\mathfrak{U}$, and if

(6.3.6) $$\sum_{n=1}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}=\infty,$$

$\{\theta_n\}\in\mathfrak{L}$, where a_{in} $(i=1, 2,\cdots\cdots, n)$ are defin̈ded as follows:

$$a_{in}=\begin{cases} 0(=E(X_i)) \ \textit{if} \ E(X_i) \ \textit{exists,} \\ \int_{|x|\leq\theta_n} x dF_i(x) \ \textit{otherwise.}\end{cases}$$

For the proof of Theorem we need the following Lemma.

Lemma 6. 3. 1. *Let $\{X_n\}$ be a system of random variables such that the expectation $E(X_n)$ of X_n is zero when it exists, then a sufficient condition for the existence of a sequence $\{\theta_n>0|n=1, 2,\cdots\cdots\}$ satisfying for any $\epsilon>0$*

(6.3.7) $$P_\kappa\{|\sum_{m=1}^{n}(X_m - a_{mn})| \geq \epsilon\theta_n\} \to 0, \quad (n \to \infty)$$

is

(6.3.8) $$\sum_{m=1}^{n}\{1 - \Phi_{F_{nm}}(\theta_n)\} \to 0, \quad (n \to \infty),$$

where

$$a_{mn} = \begin{cases} 0 \text{ for } E(X_m) = 0 \\ \int_{-An}^{An} x dF_m(x), \text{ otherwise.} \end{cases}$$

Proof. By (F. I. 6), (6. 3. 5) implies

$$\sum_{m=1}^{n}|f_{nm}(t/\theta_n)\exp(-ita_{mn}/\theta_n) - 1| \leq (t^2 + 2|t| + 4)\sum_{m=1}^{n}\{1 - \Phi_{F_m}(\theta_n)\}$$

for every finite interval of t. And we have

$$|\log \Pi_{m=1}^{n} f_m(t/\theta_n)\exp(-ita_{mn}/\theta_n) - \sum_{m=1}^{n}\{f_m(t/\theta_n)\exp(-ita_{mn}/\theta_n) - 1\}|$$
$$\leq \sum_{m=1}^{n}|f_m(t/\theta_n)\exp(-ita_{mn}/\theta_n) - 1|^2$$

for every finite interval of t and for sufficienly large values of n. Hence we obtain

$$\Pi_{m=1}^{n} f_m(t/\theta_n)\exp(-ita_{mn}/\theta_n) \to 1, \quad (n \to \infty)$$

for every finite interval of t, from which we can get the conclusion.

Proof of Theorem 6. 3. 1. From Lemma 6. 3. 1, (6. 3. 2) implies that

$$\sum_{i=1}^{n}(X_i - a_{in})/\theta_n$$

converges to zero in probablility as $n \to \infty$. Hence by (6. 3. 3)

$$\sum_{i=1}^{n} X_i/\theta_n$$

also converges to zero in probability as $n \to \infty$. Next, for $\{\theta_{n_k}\}$ defined by (6. 3. 4) and for any $\eta(\delta/2 > \eta > 0)$, if k is sufficiciently large, we have

(6.3.9) $$P_\gamma\{S_{n_{k+1}} - S_m \geq -\eta\theta_{n_{k+1}}\} \geq \frac{1}{2}$$

for any fixed m $(n_{k+1} > m)$. Now, by A. Kolmogoroff—G. Ottaviani's method, denote by $E_m(m = n_k + 1, \cdots\cdots, n_{k+1})$ the following events:

$$\bigcap_{n=n_k+1}^{m-1}\{(S_n-S_{n_k})\leq\theta_n/2\}\frown\{(S_m-S_{n_k})>\theta_m/2\}$$

$$(m=n_k+1,\cdots\cdots,n_{k+1}),$$

then $E_m(m=n_k+1,\cdots\cdots,n_{k+1})$ are mutually exclusive and

$$P_\gamma\left\{\bigcup_{n=n_k+1}^{n_{k+1}}(S_n-S_{n_k})>\theta_n/2\right\}=\sum_{m=n_k+1}^{n_{k+1}}P_\gamma\{E_m\}.$$

Hence we have

$$\frac{1}{2}P_\gamma\left\{\bigcup_{n=n_k+1}^{n_{k+1}}(S_n-S_{n_k})>\theta_n/2\right\}$$

$$\leq\sum_{m=n_k+1}^{n_{k+1}}P_\gamma\{E_m\frown(\sum_{n=m+1}^{n_{k+1}}X_n\geq-\eta\theta_{n_{k+1}})\}$$

$$\leq P_\gamma\left\{S_{n_{k+1}}-S_{n_k}\geq\left(\frac{\delta}{2}-\eta\right)\theta_{n_{k+1}}\right\}$$

Whence, putting $\delta/2-\eta=\tau$, we get

$$P_\gamma\left\{\bigcup_{n=n_k+1}^{n_{k+1}}(S_n-S_{n_k})>\theta_n/2\right\}\leq 2P_\gamma\{S_{n_{k+1}}-S_{n_k}\geq\tau\theta_{n_{k+1}}\}.$$

In the same manner, we see

$$P_\gamma\left\{\bigcup_{n=n_k+1}^{n_{k+1}}(S_n-S_{n_k})<-\theta_n/2\right\}\leq 2P_\gamma\{S_{n_{k+1}}-S_{n_k}\leq-\tau\theta_{n_{k+1}}\}.$$

Hence

$$(6.3.10)\quad P_\gamma\left\{\bigcup_{n=n_k+1}^{n_{k+1}}(|S_n-S_{n_k}|>\theta_n/2)\right\}\leq 2P_\gamma\{|S_{n_{k+1}}-S_{n_k}|\geq\tau\theta_{n_{k+1}}\}$$

On the other hand, let $X_1, \bar{X}_1, X_2, \bar{X}_2, \cdots\cdots$ be a sequence of mutually independent random variables and let the distribution of $\bar{X}_k(k=1, 2,\cdots\cdots)$ be the same as that of $X_k(k=1,2,\cdots\cdots)$. Further put $\bar{S}_n=\bar{X}_1+\cdots\cdots+\bar{X}_n$ ane $\tilde{S}_n=\tilde{X}_1+\cdots\cdots+\tilde{X}_n(n=1,2,\cdots\cdots)$. Then, by (6. 3. 6)

$$P_\gamma\{\bar{S}_{n_{k+1}}-\bar{S}_{n_k}\geq-\eta\theta_{n_{k+1}}\}\geq\frac{1}{2}$$

and

$$P_\gamma\{\bar{S}_{n_{k+1}}-\bar{S}_{n_k}\leq\eta\theta_{n_{k+1}}\}\geq\frac{1}{2}$$

for sufficienly large values of k and for any $\eta>0$. Hence we get

$$P_\gamma\{|\tilde{S}_{n_{k+1}}-\tilde{S}_{n_k}|\geq(\tau+\eta)\theta_{n_{k+1}}\}$$

$$\geq P_\gamma\{(S_{n_{k+1}}-S_{n_k}-(\bar{S}_{n_{k+1}}-\bar{S}_{n_k})\geq(\tau+\eta)\theta_{n_{k+1}})\frown(\bar{S}_{n_{k+1}}-\bar{S}_{n_k})\leq\eta\theta_{n_{k+1}}\}$$

$$+P_\gamma\{(S_{n_{k+1}}-S_{n_k}-(\bar{S}_{n_{k+1}}-\bar{S}_{n_k})\leq-(\tau+\eta)\theta_{n_{k+1}})\frown(\bar{S}_{n_{k+1}}-\bar{S}_{n_k})\geq-\eta\theta_{n_{k+1}}\}$$

$$\geq(1/2)P_\gamma\{|S_{n_{k+1}}-S_{n_k}|\geq\tau\theta_{n_{k+1}}\}$$

for sufficiently large values of k. Since $(S_{n_{k+1}}-S_{n_k})/\theta_{n_{k+1}}$ converges to zero in probability as $k\to\infty$ and also $(\tilde{S}_{n_{k+1}}-\tilde{S}_{n_k})/\theta_{n_{k+1}}$ so, the above inequality implies, by (F. I. 4) and (F. I. 1)

$$P_\gamma\{|S_{n_{k+1}}-S_{n_k}|\geq\tau\theta_{n_{k+1}}\}\leq2P_\gamma\{|\tilde{S}_{n_{k+1}}-\tilde{S}_{n_k}|\geq(\tau+\eta)\theta_{n_{k+1}}\}$$

$$\leq2[\{1+(\tau+\eta)^2\}/(\tau+\eta)^2]\{1-\Psi_{F_{k+1}*\cdots\cdots*F_{n_{k+1}}}(\theta_{n_{k+1}})\}\tag{6.3,11}$$

$$\leq2R[\{1+(\tau+\eta)^2\}/(\tau+\eta)^2]\textstyle\sum_{n=n_k+1}^{n_{k+1}}\{1-\Psi_{F_n}(\theta_{n_{k+1}})\}$$

$$\leq K_1\textstyle\sum_{n=n_k+1}^{n_{k+1}}\{1-\Phi_{F_n}(\theta_{n_{k+1}})\}$$

for sufficiently large values of k, where R ia an absolute constant and $K_1=4R\{1+(\tau+\eta)^2\}/(\tau+\eta)^2$. Now, if we take a sufficiently large number n_0, we may assume the simultaneous existence of (6. 3. 4), (6. 3. 10) and (6. 3. 11) for n_k $(k=0, 1, \cdots\cdots)$. Hence in the following inequality:

$$P_\gamma\Big\{\bigcap_{n=n_0}^{\infty}(|S_n|\leq\theta_n)\Big\}=P_\gamma\Big\{\bigcap_{k=0}^{\infty}\bigcap_{n=n_k+1}^{n_{k+1}}(|S_n|\leq\theta_n)\Big\}$$

$$\geq P_\gamma\Big\{\bigcap_{k=0}^{\infty}\bigcap_{n=n_k+1}^{n_{k+1}}(|S_n-S_{n_k}|\leq\theta_n-|S_{n_k}|)\frown\bigcap_{k=0}^{\infty}(|S_{n_k}|\leq\theta_{n_k}/2)\Big\}$$

$$\geq P_\gamma\Big\{\bigcap_{k=0}^{\infty}\bigcap_{n=n_k+1}^{n_{k+1}}(|S_n-S_{n_k}|\leq\theta_n/2)\frown\bigcap_{k=i}^{\infty}(|S_{n_k}|\leq\theta_{n_k}/2)\Big\}$$

$$\geq1-P_\gamma\Big\{\bigcup_{k=0}^{\infty}\bigcup_{n=n_k}^{n_{k+1}}(|S_n-S_{n_k}|>\theta_n/2)\Big\}$$

$$-P_\gamma\left\{\bigcup_{k=0}^{\infty}\left(|S_{n_k}|>\frac{1}{2}\theta_{n_k}\right)\right\}$$

we see, by (6. 3. 10) and (6. 3. 11),

$$P_\gamma\left\{\bigcup_{k=0}^{\infty}\bigcup_{n=n_k+1}^{n_{k+1}}(S_n-S_{n_k}|>\frac{1}{2}\theta_n)\right\}\leq\sum_{k=0}^{\infty}P_\gamma\left\{\bigcup_{n=n_k+1}^{n_{k+1}}(|S_n-S_{n_k}|>\theta_n/2)\right\}$$

$$\leq 2\sum_{k=0}^{\infty}P_\gamma\{|S_{n_{k+1}}-S_{n_k}|>\tau\theta_{n_{k+1}}\}$$

$$\leq K_1\sum_{k=0}^{\infty}\sum_{n=n_k+1}^{n_{k+1}}\{1-\Phi_{F_n}(\theta_{n_{k+1}})\}\leq K_1\sum_{n=n_0}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}.$$

And also by (6. 3. 4)

$$P_\gamma\left\{\bigcup_{k=0}^{\infty}(|S_{n_k}|>\theta_{n_k}/2)\right\}\leq P_\gamma\left\{\bigcup_{k=0}^{\infty}(|S_{n_{k+1}}-S_{n_k}|>\frac{(1-\delta)}{2}\theta_{n_{k+1}})\right\}$$

$$+P_\gamma\{|S_{n_0}|>\theta_{n_0}/2\}$$

$$\leq\sum_{k=0}^{\infty}P_\gamma\left\{|S_{n_{k+1}}-S_{n_k}|>\frac{(1-\epsilon)}{2}\theta_{n_{k+1}}\right\}+P_\gamma\{|S_{n_0}|>\theta_{n_0}/2\}.$$

Since in the same way as (6. 3. 10), we obtain

$$P_\gamma\{|S_{n_{k+1}}-S_{n_k}|\geq(1-\delta)\theta_{n_{k+1}}/2\}$$

$$\leq K_2\sum_{n_k+1}^{n_{k+1}}\{1-\Phi_{F_n}(\theta_n)\},\quad k=0,1,\cdots\cdots,$$

where K_2 is a constant, we have

$$P_\gamma\left\{\bigcup_{k=0}^{\infty}\left(|S_{n_k}|>\frac{1}{2}\theta_{n_k}\right)\right\}\leq K_2\sum_{n=n_0}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}+P_\gamma\{|S_{n_0}|>\theta_{n_0}/2\}.$$

Consequently we see

$$P_\gamma\left\{\bigcup_{n=n_0}^{\infty}(|S_n|>\theta_n)\right\}$$

$$\leq K_1\sum_{n=n_0}^{\infty}\{1-\Phi_{F_n}(\theta)\}+K_2\sum_{n=n_0}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}+P_\gamma\{|S_{n_0}|>\theta_{n_0}/2\}.$$

Thus by appealing to (6. 3. 5) we can conclude

$$\lim_{n_0\to\infty}P_\gamma\left\{\bigcup_{n=n_0}^{\infty}(|S_n|>\theta_n)\right\}=0$$

which shows the fact $\{\theta_n\}\in\mathfrak{U}$.

Next we shall suppose (6.3.6). First we readily see, for any $\omega>0$,

$$\sum_{n=1}^{\infty}\{1-\Phi_{F_n}(\omega\theta_n)\}=\infty. \tag{6.3.12}$$

And (6.3.1) implies that for sufficiently large values of n

$$\begin{aligned}1-\Phi_{F_n}(\omega\theta_n)&\leq\frac{1}{\omega^2\theta_n^2}\int_{|x|\leq\omega\theta_n}x^2dF_n(x)+\int_{|x|>\omega\theta_n}dF_n(x)\\&\leq(A+1)\int_{|x|>\omega\theta_n}dF_n(x),\end{aligned} \tag{6.3.13}$$

where A is a constant. Whereas let $X_1, \bar{X}_1, X_2, \bar{X}_2, \cdots\cdots$ be a sequence of independent random variables and let X_k and $\bar{X}_k$ have the same distribution $F_k(x)$ $(k=1,2,\cdots\cdots)$. Then $\tilde{X}_k=X_k-\bar{X}_k$ depends on the symmetrized distribution $\tilde{F}_k(x)$ of $F_k(x)$ $(k=1,2,\cdots\cdots)$. Now taking into account that for sufficiently large values of n

$$P_\gamma\{|\bar{X}_n|>\omega\theta_n/2\}\leq1/2,$$

which easily follows from (6.3.2), we see

$$\frac{1}{2}\int_{|x|>\omega\theta_n}dF_n(x)=\frac{1}{2}P_\gamma\{X_n>\omega\theta_n\}+\frac{1}{2}P_\gamma\{X_n<-\omega\theta_n\}$$

$$\leq P_\gamma\Big\{(X_n>\omega\theta_n)\frown\Big(\bar{X}_n\leq\frac{1}{2}\omega\theta_n\Big)\Big\}+P_\gamma\Big\{(X_n>-\omega\theta_n)\frown\Big(\bar{X}_n\geq-\frac{1}{2}\omega\theta_n\Big)\Big\}$$

$$\leq P_\gamma\Big\{X_n-\bar{X}_n>\frac{1}{2}\omega\theta_n\Big\}+P_\gamma\Big\{X_n-\bar{X}_n<-\frac{1}{2}\omega\theta_n\Big\}$$

$$=\int_{|x|>\frac{1}{2}\omega\theta}d\tilde{F}_n(x).$$

Hence by Lemma 6.2.1 and (6.3.2) there exists for $n\geq n_0>0$

$$\sum_{n=n_0+1}^{\infty}\int_{|x|>\omega\theta_n}dF_n(x)\leq2\sum_{n=n_0+1}^{\infty}\int_{|x|>\omega\theta_n/2}d\tilde{F}_n(x)$$

$$\leq2\sum_{k=0}^{\infty}\sum_{n=n_k+1}^{n_{k+1}}\int_{|x|>\omega\theta_{n_k+1}/2}d\tilde{F}_n(x)\leq24\sum_{k=0}^{\infty}\int_{|x|>\omega\theta_{n_k+1}/4}d\tilde{F}_{n_k+1}*\cdots\cdots*\tilde{F}_{n_{k+1}}(x)$$

$$=24\sum_{k=0}^{\infty}P_{\gamma}\{|\tilde{S}_{n_{k+1}}-\tilde{S}_{n_k}|>\omega\theta_{n_k+1}/4\}$$

$$=24\sum_{k=0}^{\infty}[P_{\gamma}\{(|S_{n_{k+1}}-S_{n_k}-(\bar{S}_{n_{k+1}}-\bar{S}_{n_k})|>\omega\theta_{n_k+1}/4)\frown(|\bar{S}_{n_{k+1}}-\bar{S}_{n_k}|\leq\omega\theta_{n_k+1}/8)\}$$

$$+P_{\gamma}\{(|S_{n_{k+1}}-S_{n_k}-(\bar{S}_{n_{k+1}}-\bar{S}_{n_k})|>\omega\theta_{n_k+1}/4)\frown(|\bar{S}_{n_{k+1}}-\bar{S}_{n_k}|>\omega\theta_{n_k+1}/8)\}]$$

$$\leq 24\sum_{k=0}^{\infty}[P_{\gamma}\{|S_{n_{k+1}}-S_{n_k}|>\omega\theta_{n_k+1}/8\}+P_{\gamma}\{|\bar{S}_{n_{k+1}}-\bar{S}_{n_k}|>\omega\theta_{n_k+1}/8\}]$$

$$=48\sum_{k=0}^{\infty}P_{\gamma}\{|S_{n_{k+1}}-S_{n_k}|>\omega\theta_{n_k+1}\},$$

where $\tilde{S}_n=S_n-\bar{S}_n=X_1+\cdots\cdots+X_n-(\bar{X}_1+\cdots\cdots+\bar{X}_n)=\tilde{X}+\cdots\cdots+\tilde{X}_n$. Hence, by (6. 3. 12) and (6. 3. 13), we have

$$\sum_{k=0}^{\infty}P_{\gamma}\{|S_{n_{k+1}}-S_{n_k}|>\omega\theta_{n_k+1}/8\}=\infty$$

Whence from (6. 3. 6) we obtain, putting $\delta\omega/8=2$,

$$\sum_{k=0}^{\infty}P_{\gamma}\{|S_{n_{k+1}}-S_{n_k}|>2\theta_{n_{k+1}}\}=\infty. \tag{6.3.14}$$

Now put

$$u_m=P_{\gamma}\left\{\bigcap_{k=0}^{m-1}(|S_{n_k}|\leq\theta_{n_k})\frown(|S_{n_m}|>\theta_{n_m})\right\}$$

$$U_i=P_{\gamma}\left\{\bigcup_{k=0}^{i}(|S_{n_k}|>\theta_{n_k})\right\}$$

and

$$v_i=P_{\gamma}\{|S_{n_i}-S_{n_{i-1}}|>2\theta_{n_i}\},$$

then we easily have $U_i=u_0+u_1+\cdots\cdots+u_i$. Whereas we get

$$u_i(1-U_{i-1})=P_{\gamma}\{(|S_{n_i}-S_{n_{i-1}}|>2\theta_{n_i})\frown\bigcap_{k=0}^{i-1}(|S_{n_k}|\leq\theta_{n_k})\}$$

$$\leq P_{\gamma}\left\{(|S_{n_i}|>\theta_{n_i})\frown\bigcap_{k=0}^{i-1}|S_{n_k}|\leq\theta_{n_k}\right\}$$

$$=u_i=U_i-U_{i-1}.$$

Hence we see $1-U_i\leq(1-U_{i-1})(1-v_i)\leq\Pi_1^i(1-v_k)$. As $\sum v_i=\infty$ follows from (6. 3. 14), we obtain $U_i\to 1$ as $i\to\infty$, which completes the proof of Theorem 6. 3. 1.

Next we shall show that Theorem 6.3.1 is an extension of P. Lévy—J. Marcinkiewicz's theorem.

Let $\{X_n | n=1, 2, \cdots\cdots\}$ be a sequence of independent random variables satisfying

$$(6.3.15) \qquad P_\gamma\{|X_n|>Z\}\leq CZ^{-\alpha} \quad P_\gamma\{|X_n|>Z\}\geq cZ^{-\alpha}$$
$$(0<\alpha<2,\ Z\geq Z_0>0,\ n=1, 2, \cdots\cdots),$$

where C and c are constants both independent of n. And besides, suppose that the expectation $E(X_n)$ of X_n is zero when it exists. Then P. Lévy[50]—J. Marcinkiewicz's[51] theorem can be stated as follows.

Theorem 6. 3. 2. *Let $\{X_n | n=1, 2,\cdots\cdots\}$ be the sequence of independent random variables defined above and let $\lambda(t)$ be a nondecreasing function defined on $(0, \infty)$, tending to ∞ as $t\to\infty$, in such a manner as the oscilation of $\log \lambda(t)$ between t and $2t$ tends to zero as $t\to\infty$. Then if*

$$(6.3.16) \qquad \sum_{k=1}^{\infty}(k\lambda(k))^{-1}<\infty,$$

$\{(n \log \lambda(\log n))^{1/\alpha}\}\in\mathfrak{U}$
and if

$$(6.3.17) \qquad \sum_{k=1}^{\infty}(k\lambda(k))^{-1}=\infty,$$

$\{(n \log \lambda(\log n))^{1/\alpha}\}\in\mathfrak{L}$.

This theorem was first proved by P. Lévy for the case $0<\alpha\leq1$ and next by J. Marcinkiewicz for the case $1<\alpha<2$.

By (6. 3. 15) and partial integration[52], we get

$$\int_{|x|\leq Z} x^2 dF_n(x)= -Z^2\int_{|x|>Z} dF_n(x)+2\int_0^Z vdv\int_{|x|>v} dF_n(x)$$
$$\leq 2\int_{Z_0}^Z vdv\int_{|x|>v} dF_n(x)+Z_0^2\leq 2C\int_{Z_0}^Z v^{1-\alpha}dv+Z_0^2$$
$$=\frac{2C}{2-\alpha}(Z^{2-\alpha}-Z_0^{2-\alpha})+Z_0^2.$$

On the other hand, since we may, from the first, take Z_0 so as 2C/

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(50) P. Lévy, loc. cit. (6).

(51) J. Marcinkiewicz, loc. cit. (15).

(52) H. Cramér, loc. cit. (47).

$(2-\alpha)\geq Z_0^\alpha$, we have

$$\frac{1}{Z^2}\int_{|x|\leq Z} x^2 dF_n(x) \leq \frac{2C}{2-\alpha} Z^{-\alpha} \leq \frac{2C}{(2-\alpha)c}\int_{|x|>Z} dF_n(x). \tag{6.3.16}$$

Hence $\{X_n\}$ satisfies (6.3.1). Next from

$$P_\gamma\{|X_k|>(n\log n\lambda(\log n))\}^{1/\alpha}\begin{cases}\leq C(n\log n\lambda(\log n))^{-1}\\ \geq c(n\log n\lambda(\log n))^{-1}\end{cases}$$

$$k=1, 2, \cdots\cdots,$$

it follows that

$$\sum\int_{|x|>[n\log n\lambda(\log n)]^{1/\alpha}} dF_k(x) = O(\log\lambda(\log n))^{-1}.$$

Whence, put

$$\theta_n=[n\log n\lambda(\log n)]^{1/\alpha},$$

then it clearly satisfies (6.3.4) and we obtain, paying attention to (6.3.17),

$$\sum_{k=1}^n\{1-\Phi_{F_k}(\theta_n)\} = O\Big[\sum_{k=1}^n\int_{|x|>\theta_n} dF_k(x)\Big] = o(1), \quad (n\to\infty)$$

which shows (6.3.2). Now taking into account that for $0<\alpha\leq 1$ all expectations $E(X_k)(k=1, 2, \cdots\cdots)$ do not exist, we can derive from the definition of a_{kn}

$$a_{1n}+a_{2n}+\cdots\cdots+a_{nn}=\sum_{k=1}^n\int_{|x|\leq\theta_n} x\,dF_k(x)$$

for $0<\alpha\leq 1$, and so by partial integration we see

$$\begin{aligned}|\textstyle\sum_{k=1}^n a_{kn}| &\leq \sum_{k=1}^n\int_{|x|\leq\theta_n}|x|\,dF_k(x)\\ &=\sum_{k=1}^n\Big\{-\theta_n\int_{|x|>\theta_n} dF_k(x)+\int_0^{\theta_n} dx\int_{|v|>x} dF_k(v)\Big\}\\ &\leq\sum_{k=1}^n\Big\{\int_{Z_0}^{\theta_n} dx\int_{|v|>x} dF_k(v)+\int_0^{Z_0} dx\int_{|v|>x} dF_k(v)\\ &=O(n\theta_n^{1-\alpha})+O(n).\end{aligned}$$

Accordingly for $0<\alpha\leq 1$

$$\frac{1}{\theta_n}|\sum_{k=1}^{n}a_{kn}|\leq O(n/\theta_n)+O(n/\theta_n)$$

$$=O(1/\log n\lambda(\log n))+O\left(n^{1-\frac{1}{\alpha}}/(\log n\lambda(\log n))^{\frac{1}{\alpha}}\right)$$

$$=o(1),\quad (n\to\infty).$$

On the other hand, for $1<\alpha<2$, all expectations $E(X_k)(k=1,2,\cdots\cdots)$ exist. Hence the definition of a_{kn} implies

$$a_{1n}+a_{2n}+\cdots\cdots+a_{nn}=0.$$

Thus (6. 3. 3) is also satisfied, Finally, from

$$\sum_{n=1}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}=O\left\{\sum_{n=1}^{\infty}\int_{|x|\geq\theta_n}dF_n(x)\right\}$$

$$=O\{\sum_{n=1}^{\infty}(n\log n\lambda(\log n))^{-1}\}$$

$$=O\left\{\sum_{k=1}^{\infty}\sum_{n=2^k+1}^{2^{k+1}}\frac{1}{n\log n\lambda(\log n)}\right\}$$

$$=O\left\{\sum_{k=1}^{\infty}\frac{1}{k\lambda(k)}\right\},$$

it follows that

$$\sum_{n=1}^{\infty}\{1-\Phi_{F_n}(\theta_n)\}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n\lambda(n)}$$

are always simultaneously convergent or divergent. Thus we see that Theorem 6. 3. 1 is an extension of Theorem 5. 3. 2.

Institute of Statistical Mathematics Tokyo.
Tokyo Institute of Technology.