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The degrees of freedom of partly smooth regularizers

Samuel Vaiter[1] · **Charles Deledalle**[3] · **Jalal Fadili**[2] · **Gabriel Peyré**[1] · **Charles Dossal**[3]

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Abstract We study regularized regression problems where the regularizer is a proper, lower-semicontinuous, convex and partly smooth function relative to a Riemannian submanifold. This encompasses several popular examples including the Lasso, the group Lasso, the max and nuclear norms, as well as their composition with linear operators (e.g., total variation or fused Lasso). Our main sensitivity analysis result shows that the predictor moves locally stably along the same active submanifold as the observations undergo small perturbations. This plays a pivotal role in getting a closed-form expression for the divergence of the predictor w.r.t. observations. We also show that, for many regularizers, including polyhedral ones or the analysis group Lasso, this divergence formula holds Lebesgue a.e. When the perturbation is random (with an appropriate continuous distribution), this allows us to derive an unbiased estimator of the degrees of freedom and the prediction risk. Our results unify and go beyond those already known in the literature.

Keywords Degrees of freedom · Partial smoothness · Manifold · Sparsity · Model selection · O-minimal structures · Semi-algebraic sets · Group Lasso · Total variation

✉ Jalal Fadili
Jalal.Fadili@greyc.ensicaen.fr

1 CEREMADE, CNRS, Université Paris-Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris Cedex 16, France

2 Normandie Univ, ENSICAEN, CNRS, GREYC, 6, Bd du Maréchal Juin, 14050 Caen Cedex, France

3 IMB, CNRS, Université Bordeaux 1, 351, Cours de la libération, 33405 Talence Cedex, France