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Statistical problems related to irrational rotations

Received: 15 April 2004 / Revised: 5 July 2005 / Published online: 17 June 2006
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Abstract Let \( \xi_i := \lfloor i\alpha + \beta \rfloor - \lfloor (i-1)\alpha + \beta \rfloor \) \((i = 1, 2, \ldots, m)\) be random variables as functions of \(\beta\) in the probability space \([0, 1)\) with the Lebesgue measure, where \(\alpha \in [0, 1]\) is considered to be an unknown parameter which we want to estimate from the observation \(\xi := \xi_1, \xi_2 \ldots \xi_m\). Let an observation \(\xi\) be given, which is a finite Sturmian sequence. We determine the likelihood function \(P_{\alpha}(\xi)\) as a function of parameter \(\alpha\), and obtain the maximum likelihood estimator \(\hat{\alpha}(\xi)\) as the relative frequency of 1s in a minimal cycle of \(\xi\), where a factor \(\eta\) of \(\xi\) is called a minimal cycle if \(\xi\) is a factor of \(\eta^\infty\) and \(\eta\) has the minimum length among them. We also obtain a minimum sufficient statistics. The sample mean \((\xi_1 + \xi_2 + \cdots + \xi_m)/m\) which is an unbiased estimator of \(\alpha\) is not admissible if \(m = 6\) or \(m \geq 8\) since it is not based on the minimum sufficient statistics.

Keywords Sturmian sequence · Irrational rotations · Minimum sufficient statistics · Admissible estimator · UMVUE

1 Introduction

Let \(\xi = \xi_1, \xi_2, \ldots, \xi_m\) be a finite 0-1-sequence. We denote the length \(m\) of \(\xi\) by \(|\xi|\) and the number of 1s in \(\xi\) by \(|\xi|_1\). We also denote \(\rho(\xi) := |\xi|_1/|\xi|\), the relative frequency of 1s in \(\xi\). Let \(\eta = \eta_1, \eta_2 \ldots \eta_n\) be a finite 0-1-sequence. We say that \(\eta\) is a factor of \(\xi\) if there exists an integer \(i\) with \(0 \leq i \leq m - n\) such that \(\eta_j = \xi_{i+j}\) \((j = 1, 2, \ldots, n)\). In this case, we denote \(\eta \prec \xi\). We say that \(\eta\) is a prefix of \(\xi\) if the above holds with \(i = 0\).