Cryptography, Information Security, and Mathematics: Recent Advances

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情報セキュリティにおける数学的方法とその実践
2016 年 12 月 19 日　@北海道大学

For SCIS People

- The (extended) deadline for paper submission has passed (almost now, at 10:00)
- Please take a rest before the next talk :-)

A Few Years Ago (Maybe)

- (Rough) scenario of a police TV drama:
  - Victim was a mathematician, he found **"ultimate property of primes"**
  - He decided to delete his result, because **it will cause "fatal catastrophe" about cryptography in the whole world**
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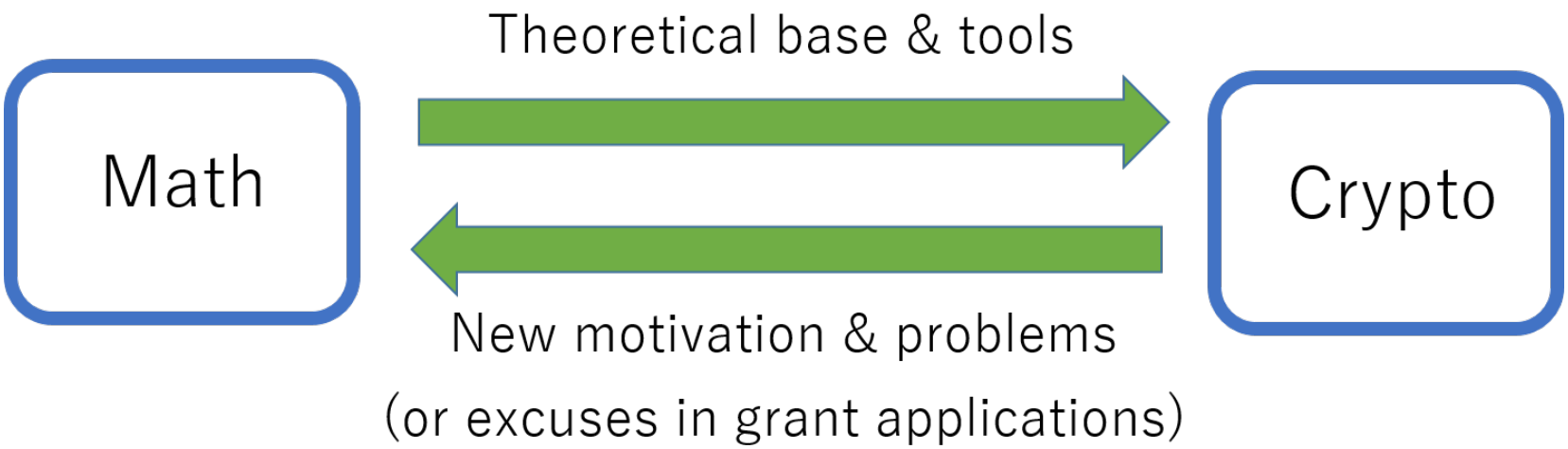
  - He decided to delete his result, because **it will cause "fatal catastrophe" about cryptography in the whole world**
  - (and was killed by his friend who disagreed)
- The "catastrophe" is fictional, but this example shows: It has become recognized so widely that primes and cryptography are related (in some unknown way)

Main Theme of This Talk

- The RSA cryptosystem would have made relation of math. and crypto. so popular, but ...
- **RSA is NOT the only such relation**
  - (and "RSA and elliptic curve crypto." are not the only, too)

Relation between Mathematics and Cryptography



暗号及び情報セキュリティと数学の相関
ワークショップ (workshop on interaction between CRyptography, Information Security and MATHematics)

- Invited talks on math. & crypto. (and more)
- **This year's: Dec. 26th @Tokyo (Odaiba)**
  - Organizers: K. Nuida, T. Abe, S. Kaji, H. Kurihara, T. Maeno, Y. Numata
    - (Underlined: Speakers of the current workshop)

linear-algebraic technique in integer factorization; dynamical systems and sensor networks (Z. Arai); algorithmic randomness and quantum key distribution; network codings and sheaf cohomology; algorithmic randomness and random oracles; algebraic-geometrical approach on LLL algorithm; NP vs. P problem; algebraic surfaces and cryptography (K. Akiyama); cryptography from group theory (N.); group rings and NTRU cryptosystem; quantum computation; isogenies for elliptic curves (K. Takashima); Bernoulli numbers and use of FHE (N.)

CRISMATH Workshop: This Year's Talks

- 川合 豊「鍵が固定された場合の暗号方式の安全性について」
- 品川 和雅「カードとシャッフルから見るカード暗号プロトコル」
- 白勢 政明「楕円曲線，暗号，素因数分解」
  - On applications of elliptic curves to integer factorization
- Taechan Kim「Extended Tower Number Field Sieve」
  - On speed-up for discrete logarithms
- 縫田 光司「秘密計算で他者に迷惑をかけずに疑似乱数を使えるか」

Public Key Encryption (PKE)

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- $\mathrm{Dec}_{\mathrm{sk}}(\mathrm{Enc}_{\mathrm{pk}}(m)) = m$
- pk **should not yield information on** sk

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- Perfectly hiding, **if** k **is used only once**

The RSA Cryptosystem [1977 ?]

- $N = pq$ (distinct primes)
- e, d with $ed \equiv 1 \pmod{(p-1)(q-1)}$

Given message $m \in (\mathbb{Z}/N\mathbb{Z})^\times$,

- $\mathrm{Enc}(m) := m^e$ (public key: (N, e))
- $\mathrm{Dec}(c) := c^d$ (secret key: d)

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Drawback: Enc is deterministic ("textbook RSA")

- Improved variant is practically used

Computational Assumptions

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- E.g. "Factoring N is hard" for the RSA
- Theoretically, just "assumption" (cf. P vs NP)
  - Practically, evaluated by experiments
  - Consensus: "(General) Number Field Sieve" would factorize $N \approx 2^{1024}$ in near future

The Number Field Sieve [Pollard+1990 ?]

State-of-the-art method to factoring integers
($\Longrightarrow$ to break the RSA cryptosystem)

- Using number fields
- Complexity: sub-exponential (not polynomial)
- Current status: Recommended to finish using RSA with $N \approx 2^{1024}$ (1024-bit key)

(Special) Number Field Sieve (1/3)

Choose

- $f(t) \in \mathbb{Z}[t]$, irreducible over $\mathbb{Q}$
- $\alpha \in \mathbb{C}$, with $f(\alpha) = 0$
- $m \in \mathbb{Z}$, with $f(m) \equiv 0 \pmod{N}$, $0 < m \ll N$

and assume the followings: (avoidable, in general)

- $K = \mathbb{Q}(\alpha)$ has integer ring $\mathcal{O}_K = \mathbb{Z}[\alpha]$
- $\mathcal{O}_K$ is UFD

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For $a, b \in \mathbb{Z}$, compute $\varphi(a + b\alpha)$ in two ways:

- φ(factorization of $a + b\alpha$ in $\mathbb{Z}[\alpha]$)
- factorization of $\varphi(a + b\alpha)$ in $\mathbb{Z}$ ($\Rightarrow$ in $\mathbb{Z}/N\mathbb{Z}$)

Then get: ($\prod$ of primes) $\equiv$ ($\prod$ of primes) in $\mathbb{Z}/N\mathbb{Z}$

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Combine them (by observing exponents) to get

$$x^2 \equiv y^2 \pmod{N}$$

yielding (with high probability)

$$1 < \gcd(x+y, N) < N \quad \text{or} \quad 1 < \gcd(x-y, N) < N$$

Prior to RSA — Diffie–Hellman Key Exchange [1976]

Protocol between parties P_1 and P_2

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Getting a common (random) **secret element**

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- Can be converted to PKE [ElGamal 1985]

Is DH Key Exchange Secure?

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$\Rightarrow$ The **discrete logarithm problem** (DL) in G must be computationally hard:

(DL) Given g, h, find x with $h = g^x$ in G

- Remark: (In)sufficiency is still open

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$\Rightarrow$ Difficulty of DL **does depend** on
a "realization" of the same abstract group G!

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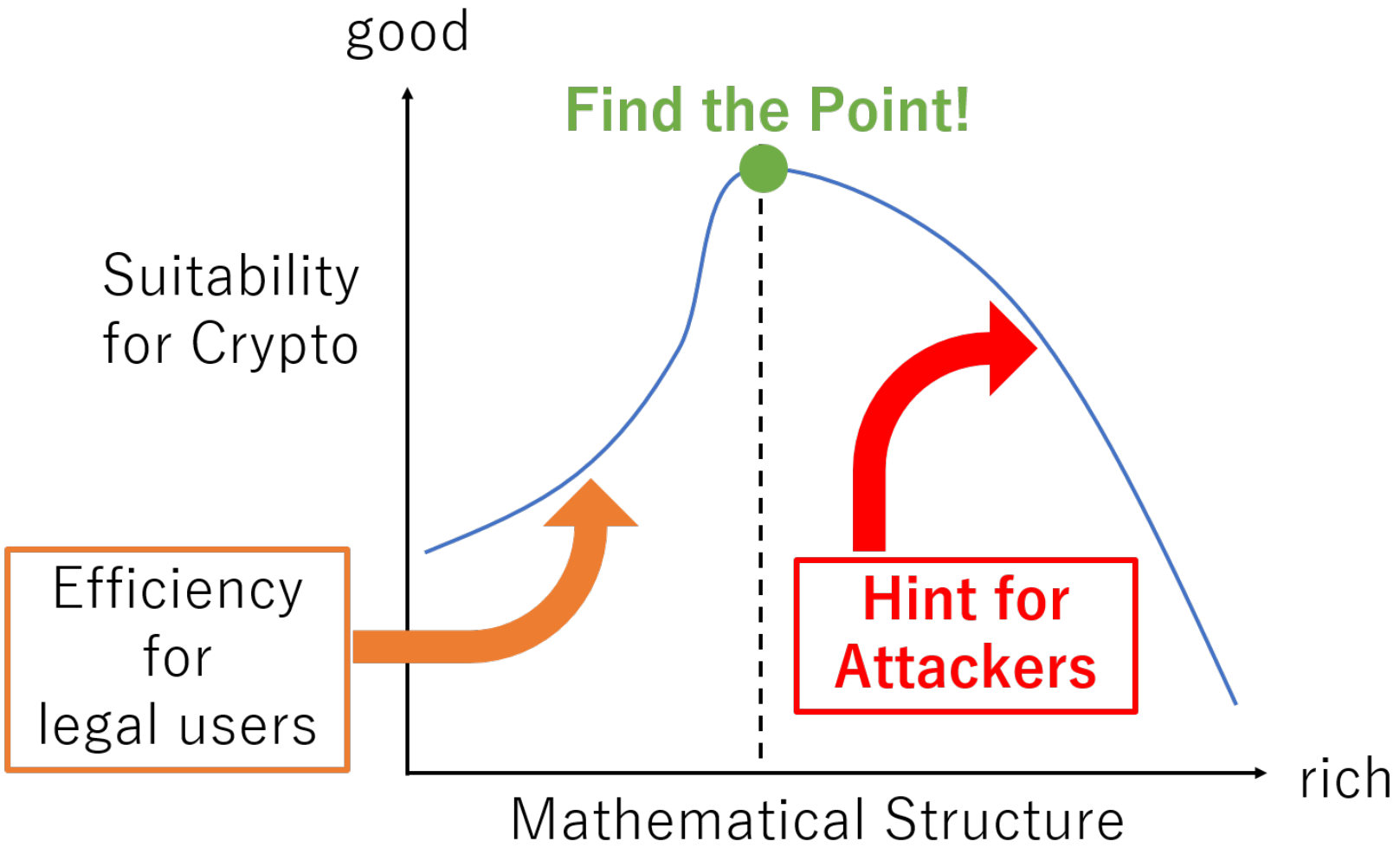
A lesson: Additional structure for group G makes the DL easier ($\Rightarrow$ break of DH Key Exchange)

- Cf. [Maurer 2005] DL is hard in "generic group"
  - "Oracle access to multiplication table only"

A New Viewpoint from Cryptography



(Mathematician: more structures, more happiness)

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Current status: $|G| \gtrapprox 2^{160}$

- Cf. $N \gtrapprox 2^{1024}$ for the RSA

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- integer factoring **in polynomial time**!
- discrete logarithm **in polynomial time**!

(Cf. [Grover 1996]: Search with quadratic speedup)

Shor's main applications: integer factoring and DL

Pinpoint Effect by Shor to Cryptography

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Main tools of PKE: integer factoring and DL

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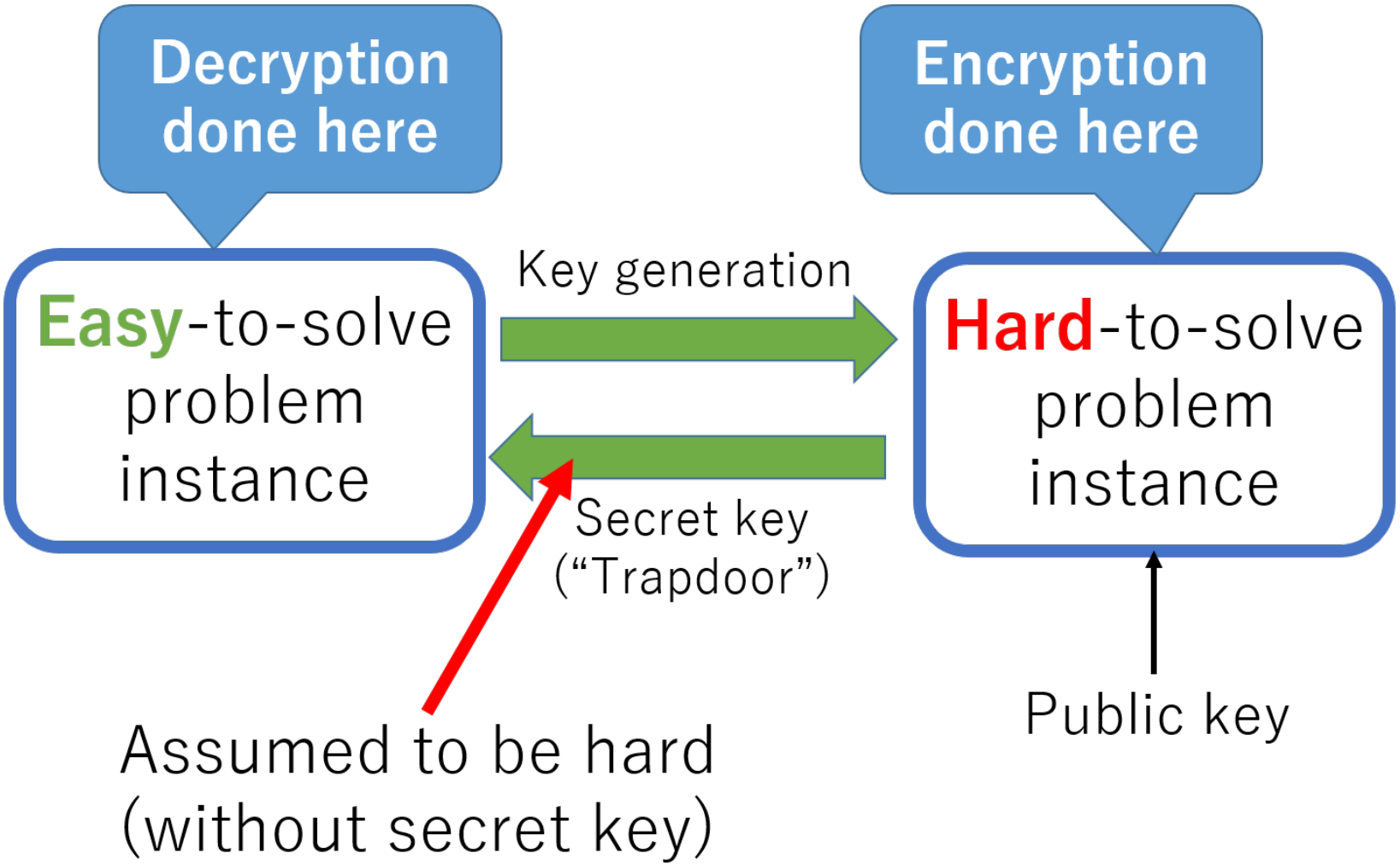
$\longrightarrow$ Importance of "quantum-resistant" PKE

- (Believed to be) unbroken by quantum computer
- Related to the talks by K. Akiyama and K. Takashima

A Major Strategy for PKE



Multi-Party Computation (MPC): Two-Party Case

Given function $f(x, y)$ (e.g., $f(x, y) = \delta_{x,y}$),

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- while **hiding information on each input!**
  - (except those trivially implied from $f(a_1, a_2)$)

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$$\mathrm{Dec}(c_1 \boxplus c_2) = \mathrm{Dec}(c_1) + \mathrm{Dec}(c_2) \in \mathcal{M}$$

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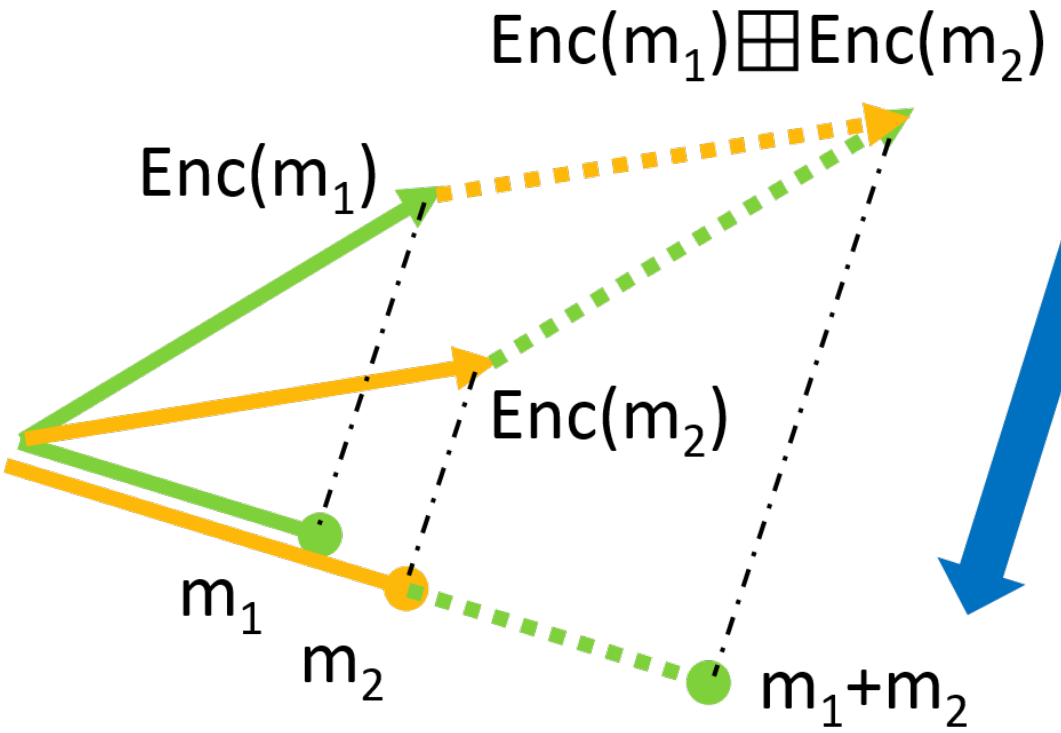
Related to the talks by K. Shimizu, H. Arimura (and K. Akiyama)

A "Rough Idea" for HE



Direction should be hard to find without key (art of cryptographer)

Projection in **this direction** yields decryption

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- Homomorphic operation: multiplication in G^2

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Applications to bioinformatics: Talk by K. Shimizu

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Fully homomorphic encryption (FHE):

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- $\Leftrightarrow$ Ring-HE, when $\mathcal{M} = \mathbb{F}_p$ (p prime)
  - $\rightarrow$ Use of FHE is related to polynomial expressions of functions (talk by T. Maeno)

(Too) Simplified Example [2010] [N. et al. 2015]

$\mathbb{Z}/\ell\mathbb{Z}$ identified with $\{0, \ldots, \ell - 1\}$ by "mod"

Choose $p' \gg p$ primes, $p' \mid N$

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"Bootstrapping": refreshing the ciphertext

- possible, but very inefficient

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$\rightsquigarrow$ hard-to-compute group hom. $\varphi\colon \widetilde{G} \twoheadrightarrow G$

- must be easy-to-compute with secret key
- Public: G and generators of $\ker \varphi$ (for Enc)

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Proof (sketch):

- $\langle\{[*, \sigma_1]\}\rangle$ is normal, hence $= G$
- $\Rightarrow \exists(\text{compositions of } [*, \sigma_1]^{\pm 1}) = \sigma_1$
- When $\sigma_1 \mapsto \sigma_0$, LHS becomes $1 = \sigma_0$ $\square$

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Current problems:

- Knuth-Bendix algorithm may not terminate
- Is it really secure?

Other Topics (1) Set Families and Cryptosystems

- t-cover-free family: Each set is not covered by union of any other t sets
- Application to "stronger" encryption: (Roughly speaking) ciphertext is secure even if other t ciphertexts are decoded by attacker
  - Idea: A set indicates "set of encryption keys for a ciphertext", then at least one key remains safe even when t ciphertexts are stolen

Other Topics (2) Polynomial Interpolation and Secret Sharing

- Fact: Degree-n polynomial is uniquely determined by $n+1$ points
- Applicable to secret sharing [Shamir 1979]
  - Each user holds one point (at outside y-axis)
  - Sufficient # of users can recover the polynomial, whose y-intercept is the secret value
  - Insufficient # of points have no info. on the polynomial (y-intercept can be still arbitrary), hence secret
- Related to the talk by Y. Suga

Other Topics (3) Homomorphic Encryption from Ideal Class Groups

- Additive HE based on ideal class groups of (non-maximal order of) imaginary quadratic fields $\mathbb{Q}(\sqrt{D})$ (in CT-RSA 2015)
- Security under consideration: Efficient algorithm to (approximately) compute the class number will break the cryptosystem

Other Topics (4) Gaussian Quadrature and Content Protection

- Gaussian quadrature: Express integral of polynomial over interval as a finite weighted sum of polynomial values (at some specific points)
- Such choice of points/weights can be applied to good parameter choice in some cryptographic scheme for content protection (digital rights management) [N. et al. 2007]

Other Topics (5) abc Conjecture for Cryptographic Study

- Recent study by Y. Hashimoto, K. Shinagawa, N. et al.: Some kind of cryptographic algorithm by physical cards
- A part of result: A lower bound for such algorithms in certain setting, derived from abc Conjecture, Prime Number Theorem, König's Lemma, etc.
- Affimative result also: Application of "permutations of same type are conjugate to each other"
- To be presented in SCIS 2017

Conclusion

Q. What kind of math. are useful in crypto. (and/or other "practical" topics)?

A. Unpredictable!

- I am now trying to apply techniques from set theory, computability theory, ...