A uniform law of large numbers for ergodic processes under a Lipschitz condition

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In this paper, we give a uniform law of large numbers for ergodic processes, under a smoothness assumption. The proof is simple and easy. However, the claim does not seem to be explicitly written in the literature, so we state it here for references. The theorem is used in Nishiyama (2009).

Theorem 1 *Let $(E, \mathcal{E})$ be a measurable space. Let Θ be a set which is totally bounded with respect to the semimetric ρ. Let a family $\{f(\cdot;\theta); \theta \in \Theta\}$ of measurable functions on E be given. Suppose that there exists a measurable function K such that*

$$|f(x;\theta) - f(x;\theta')| \leq K(x)\rho(\theta,\theta'), \quad \forall \theta, \theta' \in \Theta. \tag{1}$$

(i) Suppose that the E-valued random process $\{X_t\}_{t\in[0,\infty)}$ is ergodic with the invariant law μ, that is, for any μ-integrable function g

$$\frac{1}{T}\int_0^T g(X_t)dt \to^p \int_E g(x)\mu(dx).$$

If all $f(\cdot;\theta)$ and K are μ-integrable, then

$$\sup_{\theta\in\Theta}\left|\frac{1}{T}\int_0^T f(X_t;\theta)dt - \int_E f(x;\theta)\mu(dx)\right| = o_{P^*}(1).$$

(ii) Suppose that the E-valued random process $\{X_i\}_{i=1,2,\ldots}$ is ergodic with the invariant law μ, that is, for any μ-integrable function g

$$\frac{1}{n}\sum_{i=1}^n g(X_i) \to^p \int_E g(x)\mu(dx).$$

If all $f(\cdot;\theta)$ and K are μ-integrable, then

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n} f(X_i;\theta) - \int_E f(x;\theta)\mu(dx)\right| = o_{P^*}(1).$$

Remark. As it is clear from the proof, the smoothness assumption (1) can be replaced by "bracketing". See also Theorem 2.4.1 of van der Vaart and Wellner (1996).

Proof. We only prove (i). The claim (ii) is also proved in the same way.

Given $\varepsilon > 0$, choose an ε-covering $B_1, ..., B_{N(\varepsilon)}$ of Θ, and set

$$\begin{aligned} u_i(x) &= f(x;\theta_i) + K(x)\varepsilon, \\ l_i(x) &= f(x;\theta_i) - K(x)\varepsilon, \end{aligned}$$

where θ_i is any fixed point in B_i. Then

$$l_i(x) \le \inf_{\theta\in B_i} f(x;\theta) \le \sup_{\theta\in B_i} f(x;\theta) \le u_i(x).$$

Observing that for every $\theta \in B_i$

$$\begin{aligned} &\frac{1}{T}\int_0^T f(X_t;\theta)dt - \int_E f(x;\theta)\mu(dx) \\ &\quad\le \frac{1}{T}\int_0^T u_i(X_t)dt - \int_E u_i(x)\mu(dx) + \int_E u_i(x)dt - \int_E f(x;\theta)\mu(dx) \\ &\quad\le \frac{1}{T}\int_0^T u_i(X_t)dt - \int_E u_i(x)\mu(dx) + \int_E \{u_i(x) - l_i(x)\}\mu(dx) \\ &\quad= \frac{1}{T}\int_0^T u_i(X_t)dt - \int_E u_i(x)\mu(dx) + 2\int_E K(x)\mu(dx)\varepsilon, \end{aligned}$$

we have

$$\begin{aligned} &\sup_{\theta\in\Theta}\left\{\frac{1}{T}\int_0^T f(X_t;\theta)dt - \int_E f(x;\theta)\mu(dx)\right\} \\ &\quad\le \max_{1\le i\le N(\varepsilon)}\left\{\frac{1}{T}\int_0^T u_i(X_t)dt - \int_E u_i(x)\mu(dx)\right\} + 2\int_E K(x)\mu(dx)\varepsilon. \end{aligned}$$

Considering the lower bound in the same way, we finally get

$$\begin{aligned} &\sup_{\theta\in\Theta}\left|\frac{1}{T}\int_0^T f(X_t;\theta)dt - \int_E f(x;\theta)\mu(dx)\right| \\ &\quad\le \max_{1\le i\le N(\varepsilon)}\left|\frac{1}{T}\int_0^T u_i(X_t)dt - \int_E u_i(x)\mu(dx)\right| \\ &\qquad+ \max_{1\le i\le N(\varepsilon)}\left|\frac{1}{T}\int_0^T l_i(X_t)dt - \int_E l_i(x)\mu(dx)\right| + 2\int_E K(x)\mu(dx)\varepsilon. \end{aligned}$$

Choose a small $\varepsilon > 0$, and let $T \to \infty$ to obtain the conclusion. □

References

[1] Nishiyama, Y. (2009). Asymptotic theory of semiparametric Z-estimators for stochastic processes, with applications to ergodic diffusions and time series. *Preprint.*

[2] van der Vaart, A.W. and Wellner, J.A. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics.* Springer-Verlag, New York.