Proof of Theorem 1 in "Moment convergence of M-estimators"

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This note gives a full proof of Theorem 1 in Nishiyama (2010), which is an adaptation to the proof of Theorem 3.2.5 in van der Vaart and Wellner (1996).

We denote by E^* and P^* the outer expectation and the outer probability, respectively (see van der Vaart and Wellner (1996)).

Theorem 1 (Nishiyama (2010)) *Let* $\mathbb{M}_n$ *be stochastic processes indexed by a semimetric space* (Θ, d) *and* $\mathbb{M} : \Theta \to \mathbb{R}$ *a deterministic function such that for a constant* $c > 0$

$$\mathbb{M}(\theta) - \mathbb{M}(\theta_0) \leq -cd(\theta, \theta_0)^2, \quad \forall \theta \in \Theta.$$

Suppose that there exist functions ϕ_n *such that* $\delta \mapsto \delta^{-\alpha}\phi_n(\delta)$ *is non-increasing for some* $\alpha < 2$ *(not depending on* n*) and that for every* $p \geq 1$ *there exists a constant* $C_p > 0$ *such that for every* $\delta > 0$

$$\left(E^* \left| \sup_{d(\theta,\theta_0)<\delta} |(\mathbb{M}_n - \mathbb{M})(\theta) - (\mathbb{M}_n - \mathbb{M})(\theta_0)| \right|^p \right)^{1/p} \leq C_p \frac{\phi_n(\delta)}{\sqrt{n}}. \tag{1}$$

Let $r_n^2 \phi_n(1/r_n) \leq \sqrt{n}$ *for every* n*. If the sequence* $\widehat{\theta}_n$ *satisfies* $\mathbb{M}_n(\widehat{\theta}_n) \geq \mathbb{M}_n(\theta_0) - r_n^{-2}$*, then* $\sup_n E^*|r_n d(\widehat{\theta}_n, \theta_0)|^p < \infty$ *for every* $p \geq 1$*.*

Proof. Choose $\eta \geq 1$ such that $\alpha - 2 + \eta^{-1} < 0$. For each n, we set $S_{j,n} = \{\theta : 2^{j-1} < r_n d(\theta, \theta_0) \leq 2^j\}$. Notice that

$$\mathbb{M}(\theta) - \mathbb{M}(\theta_0) \leq -cd(\theta, \theta_0)^2 \leq -c\frac{2^{2j-2}}{r_n^2}, \quad \forall \theta \in S_{j,n}.$$

Choose j_0 such that $\frac{c}{2}2^{2j_0-2} \geq 1$. Then, for all $j \geq j_0$ it holds that $c2^{2j-2} - 1 \geq \frac{c}{2}2^{2j-2}$. Now we have

$$\begin{aligned} E^*|r_n d(\widehat{\theta}_n, \theta_0)|^p &\leq 2^{(j_0-1)p} P^*(r_n d(\widehat{\theta}_n, \theta_0) \leq 2^{j_0-1}) + \sum_{j=j_0}^{\infty} 2^{jp} P^*(\widehat{\theta}_n \in S_{j,n}) \\ &\leq 2^{(j_0-1)p} + \sum_{j=j_0}^{\infty} 2^{jp} P^* \left(\sup_{\theta \in S_{j,n}} (\mathbb{M}_n(\theta) - \mathbb{M}_n(\theta_0)) \geq -r_n^{-2} \right). \end{aligned}$$

We denote $W_n = \mathbb{M}_n - \mathbb{M}$. The second term on the right hand side is bounded by

$$\begin{aligned} &\sum_{j=j_0}^{\infty} 2^{jp} P^* \left(\sup_{\theta \in S_{j,n}} (W_n(\theta) - W_n(\theta_0)) \geq -r_n^{-2} + c\frac{2^{2j-2}}{r_n^2} \right) \\ &\leq \sum_{j=j_0}^{\infty} 2^{jp} P^* \left(\sup_{\theta \in S_{j,n}} (W_n(\theta) - W_n(\theta_0)) \geq \frac{\frac{c}{2}2^{2j-2}}{r_n^2} \right) \\ &\leq \sum_{j=j_0}^{\infty} 2^{jp} P^* \left(\sup_{\theta \in S_{j,n}} |W_n(\theta) - W_n(\theta_0)|^{\eta p} \geq \left| \frac{\frac{c}{2}2^{2j-2}}{r_n^2} \right|^{\eta p} \right) \\ &\leq \sum_{j=j_0}^{\infty} 2^{jp} \left| \frac{r_n^2}{\frac{c}{2}2^{2j-2}} \frac{C_p \phi_n(2^j/r_n)}{\sqrt{n}} \right|^{\eta p} \\ &\leq \sum_{j=j_0}^{\infty} 2^{jp} \left| \frac{r_n^2}{\frac{c}{2}2^{2j-2}} \frac{C_p 2^{j\alpha} \phi_n(1/r_n)}{\sqrt{n}} \right|^{\eta p} \\ &\leq \sum_{j=j_0}^{\infty} 2^{jp} \left| C_p \frac{2^{j\alpha}}{\frac{c}{2}2^{2j-2}} \right|^{\eta p} \\ &= \sum_{j=j_0}^{\infty} \left| C_p \frac{2^{j(\alpha-2+\eta^{-1})}}{\frac{c}{8}} \right|^{\eta p}. \end{aligned}$$

Since $\alpha - 2 + \eta^{-1} < 0$, this series is finite. The proof is completed. □

References

[1] Nishiyama, Y. (2010). Moment convergence of M-estimators. *Statist. Neerlandica.* **64** 505-507.

[2] van der Vaart, A.W. and Wellner, J.A. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics.* Springer-Verlag, New York.