グラフィカルモデルによる推論 ー 確率伝搬法

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Inference with Graphical Model

- Assumption in this part

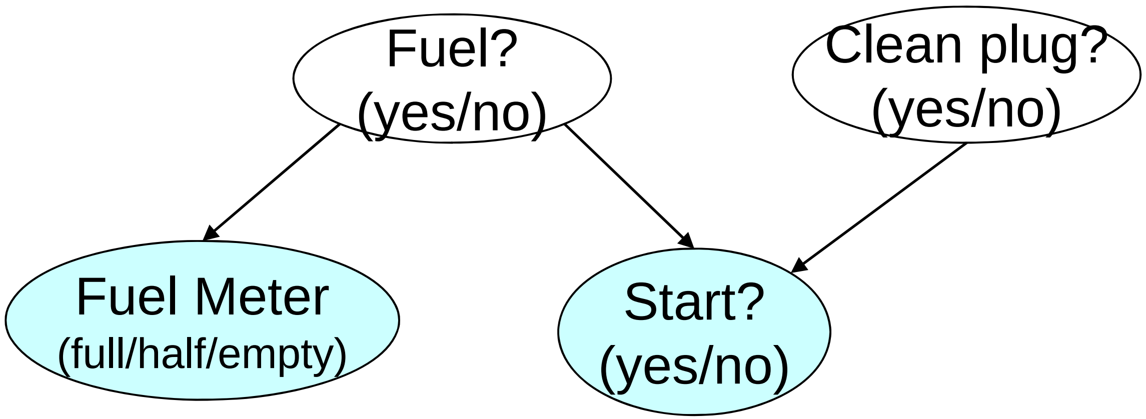
  Every variable takes values in a finite set.

- Probabilistic Inference

  $$p(Y \mid X)$$

  X: observed (evidence)
  Y: variable for inference

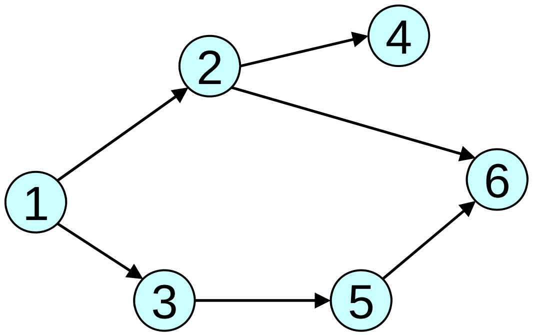
  - Example: diagnosis for car start



P(Clean plug = no | No start, Fuel meter = half)

Inference with Graphical Model

- Probabilistic inference with graphical model



$$p(X) = p(X_1)p(X_2 \mid X_1)p(X_3 \mid X_1)$$
$$\times p(X_4 \mid X_2)p(X_5 \mid X_3)p(X_6 \mid X_2, X_5)$$

Given a value of $X_6 = e$, compute the probability of X_1

$$p(X_1 \mid X_6 = e) = \frac{p(X_1, X_6 = e)}{p(X_6 = e)}$$

Inference with Graphical Model

Assume each variable takes K values

- Naïve method

$$p(X_1, X_6 = e) = \sum_{X_2} \sum_{X_3} \sum_{X_4} \sum_{X_5} p(X_1, X_2, X_3, X_4, X_5, X_6 = e)$$

(K^5 operations)

$$p(X_6 = e) = \sum_{X_1} p(X_1, X_6 = e)$$

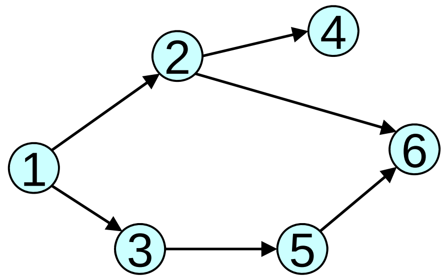
(K operations)

$$p(X_1 \mid X_6 = e) = \frac{p(X_1, X_6 = e)}{p(X_6 = e)}$$

(K operations)

In total: K^5 + 2K operations are needed.

Inference with Graphical Model



- Efficient method:

  Elimination or successive marginalization

$$p(X_1, X_6 = e) = \sum_{X_2, X_3, X_4, X_5} p(X_1) p(X_2 \mid X_1) p(X_3 \mid X_1) p(X_4 \mid X_2) \times p(X_5 \mid X_3) p(X_6 = e \mid X_2, X_5)$$

$$= p(X_1) \sum_{X_2} p(X_2 \mid X_1) \sum_{X_3} p(X_3 \mid X_1) \sum_{X_4} p(X_4 \mid X_2) \boxed{\sum_{X_5} p(X_5 \mid X_3) p(X_6 = e \mid X_2, X_5)}$$

$$= p(X_1) \sum_{X_2} p(X_2 \mid X_1) \sum_{X_3} p(X_3 \mid X_1) m_5(X_2, X_3, X_6 = e) \boxed{\sum_{X_4} p(X_4 \mid X_2)}$$

$$= p(X_1) \sum_{X_2} p(X_2 \mid X_1) \boxed{\sum_{X_3} p(X_3 \mid X_1) m_5(X_2, X_3, X_6 = e)}$$

$$= p(X_1) \boxed{\sum_{X_2} p(X_2 \mid X_1) m_3(X_1, X_2, X_6 = e)}$$

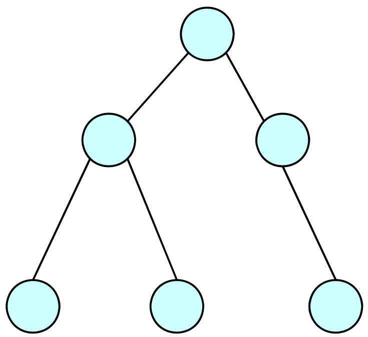
In total: K^3 (+ K) + K^3 + K^2 + 2K operations are needed.

The efficiency depends on the number of variables in the factors.

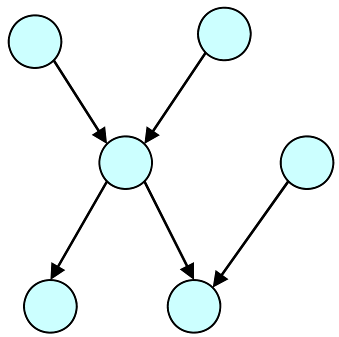
Tree

- The previous elimination method works most efficiently for trees.

- Tree: a (directed or undirected) graph such that for any two nodes there is a unique (undirected) path connecting them.
- Tree is connected, and has no loop.



undirected tree



directed tree

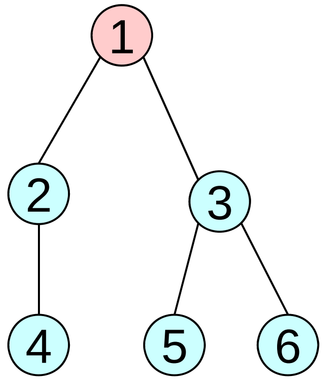
- |E| = |V|-1

Inference with Undirected Tree

Inference with Undirected Tree

- Propagation in a tree

Marginalization in an undirected tree



$$p(X_1) = \frac{1}{Z} \sum_{X_2,X_3,X_4,X_5,X_6} \psi_{12}(X_1,X_2)\psi_{13}(X_1,X_3)\psi_{24}(X_2,X_4) \times \psi_{35}(X_3,X_5)\psi_{36}(X_3,X_6)$$

$$= \frac{1}{Z}\sum_{X_2}\psi_{12}(X_1,X_2)\sum_{X_3}\psi_{13}(X_1,X_3)\underbrace{\boxed{\sum_{X_4}\psi_{24}(X_2,X_4)}}_{m_{42}(X_2)}\underbrace{\boxed{\sum_{X_5}\psi_{35}(X_3,X_5)}}_{m_{53}(X_3)}\underbrace{\boxed{\sum_{X_6}\psi_{36}(X_3,X_6)}}_{m_{63}(X_3)}$$

$$= \frac{1}{Z}\sum_{X_2}\psi_{12}(X_1,X_2)\sum_{X_3}\psi_{13}(X_1,X_3)m_{42}(X_2)m_{53}(X_3)m_{63}(X_3)$$

$$= \frac{1}{Z}\underbrace{\boxed{\sum_{X_2}\psi_{12}(X_1,X_2)m_{42}(X_2)}}_{m_{21}(X_1)}\underbrace{\boxed{\sum_{X_3}\psi_{13}(X_1,X_3)m_{53}(X_3)m_{63}(X_3)}}_{m_{31}(X_1)}$$

$$= \frac{1}{Z}m_{21}(X_1)m_{31}(X_1)$$

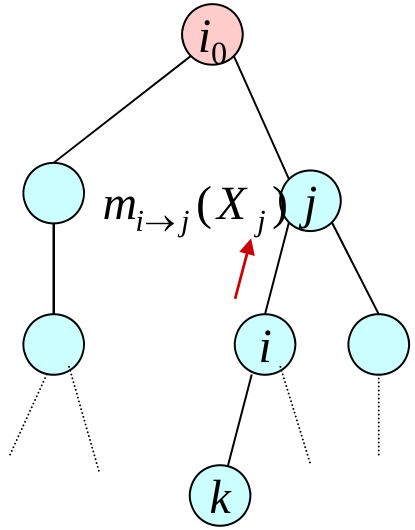
5K² operations

Inference with Undirected Tree

Propagate messages from the bottom nodes to an upper level.

$$m_{i\to j}(X_j) = \sum_{X_i} \psi_{ji}(X_j, X_i) \prod_{k\in ne(i)\setminus\{j\}} m_{k\to i}(X_i)$$

K[2] operations



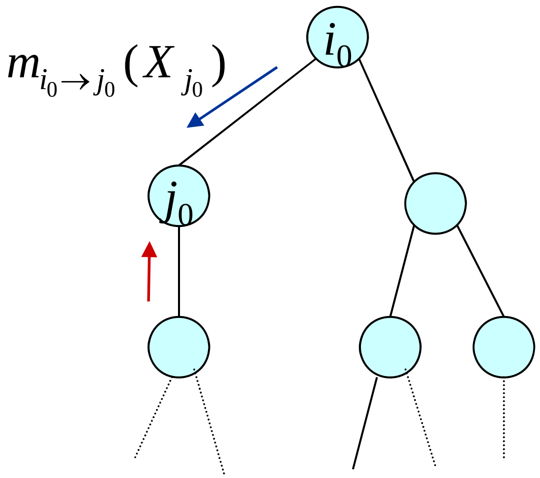
When all the messages are propagated to i_0,
the marginal of X_{i_0} is given by

$$p(X_{i_0}) = \frac{b(X_{i_0})}{\sum_{X_{i_0}} b(X_{i_0})}, \qquad b(X_{i_0}) = \prod_{j\in ne(i_0)} m_{j\to i_0}(X_{i_0})$$

Note: normalization factor 1/Z in the joint probability is not needed.
We can normalize the marginal after the propagation finishes.

Inference with Undirected Tree

- Computation of all the marginals
  - We DO NOT need to repeat the process for every node.
  - Propagate the messages downward after the upward propagations are done.
  - When all the upward and downward messages are computed, every marginal can be obtained.



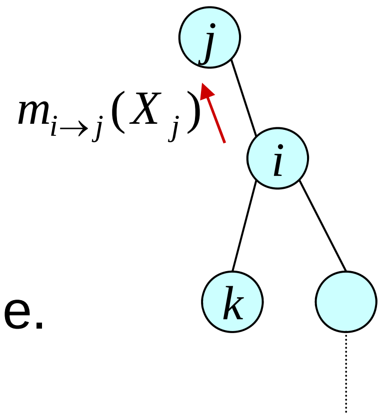
Belief Propagation for Undirected Tree

- Belief propagation algorithm for undirected tree (sum-product algorithm)

(1) Fix a root of the tree

(2) [Upward] Propagate the messages from to bottom nodes to the root according to

$$m_{i\to j}(X_j) = \sum_{X_i} \psi_{ji}(X_j, X_i) \prod_{k\in ne(i)\setminus\{j\}} m_{k\to i}(X_i)$$



(3) [Downward] Propagate the messages from the root to the bottom nodes by the same rule.

(4) The marginals are obtained by

$$p(X_i) = \frac{b(X_i)}{\sum_{X_i} b(X_i)}, \qquad b(X_i) = \prod_{j\in ne(i)} m_{j\to i}(X_i)$$

($b(X_i)$: belief)

Belief Propagation for Undirected Tree

- Message passing protocol
  - The order of updates may be different, but should keep the following message passing protocol:

    "The message to a node must be propagated after the messages from all the other neighbors are received".

- Efficient algorithm
  - Reuse of messages to compute all the marginals.
  - The cost for computing all the marginals

    $= \text{(Upward + Downward)} \times K^2 = 2|E| \times K^2 = 2(|V|-1) \times K^2$

    Linear in the number of nodes or edges

- Use of evidence
  - If some nodes have evidence, just fix the values in computing the messages.

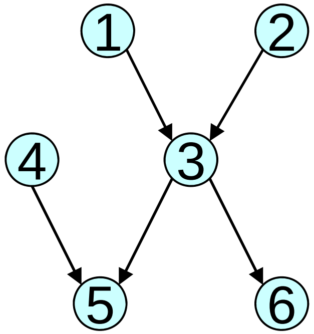
Inference with Directed Tree

(Details are omitted in this course)

Directed Tree

- Directed tree (polytree)

  Example



directed tree
(polytree)

$$p(X) = p(X_1)p(X_2)p(X_3 | X_1, X_2)$$
$$\times p(X_4)p(X_5 | X_3, X_4)p(X_6 | X_3)$$

Belief Propagation for Directed Tree

- πλ-algorithm (Kim & Pearl 1983) two types of messages are used

  Parent to child:

$$\pi_{i\to k}(X_i) = \sum_{X_{pa(i)}} p(X_i \mid X_{pa(i)}) \prod_{j\in pa(i)} \pi_{j\to i}(X_j) \prod_{r\in ch(i)\setminus\{k\}} \lambda_{r\to i}(X_i)$$

  Child to parent:

$$\lambda_{i\to j}(X_j) = \sum_{X_i, X_{pa(i)\setminus\{j\}}} p(X_i \mid X_{pa(i)}) \prod_{k\in ch(i)} \lambda_{k\to i}(X_i) \prod_{n\in pa(i)\setminus\{j\}} \pi_{n\to i}(X_n)$$

  Marginal:

$$p(X_i) \propto \lambda(X_i)\pi(X_i)$$

$$\lambda(X_i) = \prod_{k\in ch(i)} \lambda_{k\to i}(X_i), \quad \pi(X_i) = \sum_{X_{pa(i)}} p(X_i \mid X_{pa(i)}) \prod_{j\in pa(i)} \pi_{j\to i}(X_j)$$

  - πλ-algorithm is the first general belief propagation algorithm.

Mini Summary

- Belief propagation / sum-product algorithm
  - All the marginals are exactly calculated for trees.
    - Undirected tree, factor tree, polytree.
    - Non-tree cases will be discussed later.
  - The computational cost is linear w.r.t. the tree size (number of variables).
    - Basic idea is successive marginal-out, but the messages are reused to compute all the marginals.
    - Messages are passed upward and then downward.
    - In general, the order of the message passing should keep the message passing protocol.
  - The equations of message passing is local:

    product of the messages from the neighbors and sum over local variables.

Mini Summary

- Constant factor is not necessary.

  To given the joint probability density, the form

$$p(X) \propto \prod f_a(X_a)$$

  is sufficient to apply the belief propagation.

  Just normalize after the unnormalized marginal is computed.

- Normalization factor can be computed by belief propagation.

  For

$$p(X) \propto \prod f_a(X_a)$$

  Normalization factor Z is given by marginal-out:

$$Z = \sum_X \prod f_a(X_a)$$