Supplementary Material for "Statistical Inference based on Bridge Divergences"

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S.1 Consistency of the Minimum Bridge Divergence Estimator

In this section, we give a proof of Theorem 1 of the main paper. To begin with, let us define two quantities:

$$M_n(\theta) = \frac{1}{\bar{\lambda}}\log\left(\lambda + \bar{\lambda}\int f_\theta^{1+\alpha}\right) - \frac{1}{\bar{\lambda}}\left(\frac{1+\alpha}{\alpha}\right)\log\left(\lambda + \bar{\lambda}\frac{1}{n}\sum_{i=1}^n f_\theta^\alpha(X_i)\right),$$
$$M(\theta) = \frac{1}{\bar{\lambda}}\log\left(\lambda + \bar{\lambda}\int f_\theta^{1+\alpha}\right) - \frac{1}{\bar{\lambda}}\left(\frac{1+\alpha}{\alpha}\right)\log\left(\lambda + \bar{\lambda}\int g f_\theta^\alpha\right).$$

We have the estimator and the target as

$$\hat{\theta}_n^{(\alpha,\lambda)} = \arg\min_{\theta\in\Theta} M_n(\theta) \quad \text{and} \quad \theta_g^{(\alpha,\lambda)} = \arg\min_{\theta\in\Theta} M(\theta).$$

In order to conclude that $\hat{\theta}_n^{(\alpha,\lambda)} \stackrel{a.s.}{\to} \theta_g^{(\alpha,\lambda)}$, we apply Theorem 5.7 of van der Vaart (1998). To this end, we prove almost sure uniform convergence of the (random) functions $M_n(\theta)$ to $M(\theta)$. Note that

$$\begin{aligned}\sup_{\theta\in\Theta}|M_n(\theta) - M(\theta)| &= \frac{1}{\bar{\lambda}}\left(\frac{1+\alpha}{\alpha}\right)\sup_{\theta\in\Theta}\left|\log\left(\lambda + \bar{\lambda}\frac{1}{n}\sum_{i=1}^n f_\theta^\alpha(X_i)\right) - \log\left(\lambda + \bar{\lambda}\int g f_\theta^\alpha\right)\right| \\ &\le \frac{1}{\lambda}\left(\frac{1+\alpha}{\alpha}\right)\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^n f_\theta^\alpha(X_i) - \int g f_\theta^\alpha\right|.\end{aligned}$$

The last inequality follows from the fact that if $\lambda \le a < b < \infty$, then there exists $\xi \in (a,b)$ such that $\log b - \log a = \frac{1}{\xi}(b-a) \le \frac{1}{\lambda}(b-a)$. By **(C2)**, we know that $\int g f_\theta^\alpha \le \int g(x)K(x)dx < \infty$ for all θ. Applying Theorem 16(a) of Ferguson (1996) with $U(x,\theta) = f_\theta^\alpha(x)$, we conclude that the quantity on the right hand side above converges almost surely to zero. Note that the conditions (1), (2) and (3) of Theorem 16(a) of Ferguson (1996) are exactly the same as our conditions **(C1)**,**(C3)** and **(C2)**, respectively, and we are done.

Remark 1. For the above proof to hold, the assumption $\lambda > 0$ is crucial to bound the terms $\lambda + \bar{\lambda}\frac{1}{n}\sum_{i=1}^n f_\theta^\alpha(X_i)$ and $\lambda + \bar{\lambda}\int g f_\theta^\alpha$ away from 0, so that the mean value theorem can be applied on the log function over its domain $(0,\infty)$. Hence the proof, as it is, cannot be applied on the LDPD, but it works for every other fixed member of the bridge family.

S.2 Proof of Theorem 3.3

First consider the case $0 \le \lambda < 1$. By definition of $d_{\lambda,\alpha}$, we have, using $\ell(h) = \frac{1}{\bar{\lambda}(1+\alpha)}\log\left(\lambda + \bar{\lambda}\int h^{1+\alpha}\right)$,

$$\begin{aligned}d_{\lambda,\alpha}(g,h) &= \ell(h) - \frac{1}{\bar{\lambda}\alpha}\log\left(\lambda + \bar{\lambda}\int\{(1-\varepsilon)f + \varepsilon\delta\}h^\alpha\right) \\ &= \ell(h) - \frac{1}{\bar{\lambda}\alpha}\log\left(\lambda + \bar{\lambda}\int(1-\varepsilon)fh^\alpha + \bar{\lambda}\int\varepsilon\delta h^\alpha\right) \\ &= \ell(h) - \frac{1}{\bar{\lambda}\alpha}\log\left(\varepsilon\left[\lambda + \bar{\lambda}\int\delta h^\alpha\right] + (1-\varepsilon)\left[\lambda + \bar{\lambda}\int f h^\alpha\right]\right).\end{aligned}$$

Now by a simple Taylor series expansion, we get that

$$d_{\lambda,\alpha}(g,h) = \ell(h) - \frac{1}{\alpha\bar{\lambda}} \log\left((1-\varepsilon)\left[\lambda + \bar{\lambda}\int fh^{\alpha}\right]\right) + O(T_{\varepsilon,\delta}),$$

from which the result follows. For the case $\lambda = 1$, observe that

$$\begin{aligned} d_{1,\alpha}(g,h) &= \frac{1}{1+\alpha}\int h^{1+\alpha} - \frac{1}{\alpha}\int gh^{\alpha} \\ &= \frac{1}{1+\alpha}\int h^{1+\alpha} - \frac{1}{\alpha}\int fh^{\alpha} + \frac{\varepsilon}{\alpha}\left[\int fh^{\alpha} - \int \delta h^{\alpha}\right] \\ &= d_{1,\alpha}(f,h) + \frac{\varepsilon}{\alpha}\left[\int \{f-\delta\}h^{\alpha}\right]. \end{aligned}$$

□

S.3 Unboundedness of the LDPD Objective based on Simulated Data

In Section 5 of the main article, we plotted the LDPD objective with $\alpha = 0.5$ based on an artificial sample of size 4, the underlying model being the family $\{N(\mu,\sigma^2) : \mu \in \mathbb{R}, \sigma > 0\}$. Here, we demonstrate the unboundedness of the LDPD objective with $\alpha = 0.5$ based on 100 observations simulated from $N(0,1)$. The interactive MATLAB figure can be found in the file `hundnormfinal.fig` (a part of the online supplement), and the simulated data can be found in `norhundfin.xls` (a part of the online supplement).

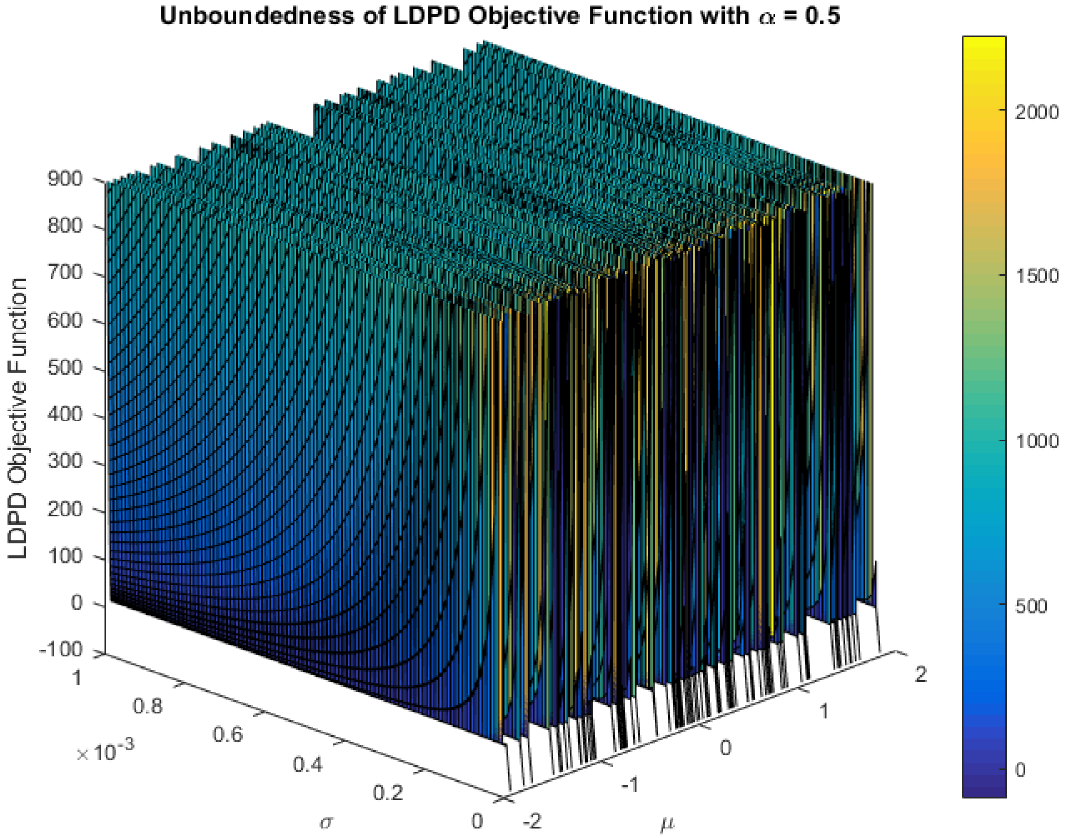

Figure 1: LDPD Objective Based on 100 Observations Simulated From $N(0,1)$

The figure shows sharp dips of the LDPD objective when the mean parameter is one of the 100 data points and the standard deviation approaches 0. From the proof of Theorem 5 of the main article, we see that the Bridge divergence objective function goes to $+\infty$ as σ goes to 0 for all those μ which do not equal any data point, at a fast rate. That is why, the plot looks steep when μ equals a data point. However, putting $\lambda = 0$ in the same proof, we see that the LDPD objective function goes to $-\infty$ for all those μ which equal one of the data points, at rate $\log \sigma$, which is a slow rate. That is why, we need a very small starting value in the σ-axis, to observe the drop in the LDPD objective function.

S.4 Bias and Mean-Squared Error of the Bridge Estimators

In Secion 7 of the main paper, only the plots of the mean-squared errors of the bridge estimators for the exponential scale model case with 5% and 20% contaminations are included. We now present the exact numerical values of the (scaled up) bias and the (scaled up) mean-squared error of the bridge estimators (in Tables 1 to 6) for all the three levels of contamination: $0\%, 5\%$ and 20%, for both the exponential and the normal scale family cases. To gain more numerically significant digits, we scaled up the bias and MSE by 10 $(= \sqrt{n})$ and 100 $(= n)$, respectively. In each table, along each row, the bias/MSE of the bridge estimators is noted, as the chain algorithm proceeds from $\lambda = 1$ to $\lambda = 0$ in steps of 0.1.

In the normal scale family case with contamination level 0.2, it is seen that for each λ, the maximum likelihood estimator (the estimator with $\alpha = 0$) has the smallest mean squared error. The reason behind this is that in this case, we have an inner contamination around the mode of the majority distribution.

To elaborate, consider a weighted estimating equation of the form:

$$\sum_{i=1}^{n} w_i^{\alpha} u_\theta(X_i) = 0. \tag{1}$$

Suppose that $w_i = f_\eta(X_i)$, where f_θ is the density function of $N(5, \theta^2)$, and η is an initial value of θ. Suppose that the true value of θ is 1. For $\alpha > 0$, w_i^α is large if X_i is close to 5. So, observations coming from the contaminating distribution get more weight relative to observations from the true majority distribution. Consequently, the solution

$$\hat{\theta}^{(\alpha)} = \sqrt{\frac{\sum_{i=1}^{n} w_i^{\alpha}(X_i - 5)^2}{\sum_{i=1}^{n} w_i^{\alpha}}}$$

to (1) shrinks more towards 0 when $\alpha > 0$, than when $\alpha = 0$.Thus, the mean squared error around 1, i.e. $\mathbb{E}(\hat{\theta}^{(\alpha)} - 1)^2$ is inversely related to shrinkage towards 0. That is why, we should expect

$$\textbf{MSE}\left(\hat{\theta}^{(\alpha)}\right) \gg \textbf{MSE}\left(\hat{\theta}^{(0)}\right)$$

for $\alpha > 0$. The estimating equation (1) is similar to that of the DPD. Similar heuristic argument also applies to the other bridge divergences, as they downweight using powers of densities.

S.5 Asymptotic Variance Plot of the Bridge Estimators

The plot of the estimated asymptotic variance of the bridge estimators across λ for $\alpha = 0.8$, using the normal data of size 20 presented in Section 4 of the main article, is given in Figure 2. The quantity plotted is the estimated asymptotic variance using the formula given in Theorem 2, evaluated at $\theta = \hat{\theta}^{(\alpha,\lambda)}$ obtained through the chain algorithm for these data. It is seen that the estimated asymptotic variance decreases monotonically with λ.

We also plot the estimated asymptotic variance of the bridge estimators across both λ and α, based on the normal data of size 20 presented in Section 4 of the main article (in Figure 3), and based on a sample of size 1000 simulated from $N(0, 1)$ (in Figure 4). The interactive MATLAB figures can be found in the files `twodplotfinal.fig` and `thousandnewpdf.fig`, and the simulated sample of size 1000 can be found in `x.xls` (which are parts of the online supplement). The plot in Figure 2 represents the cross section of the plot in Figure 3 along $\alpha = 0.8$.

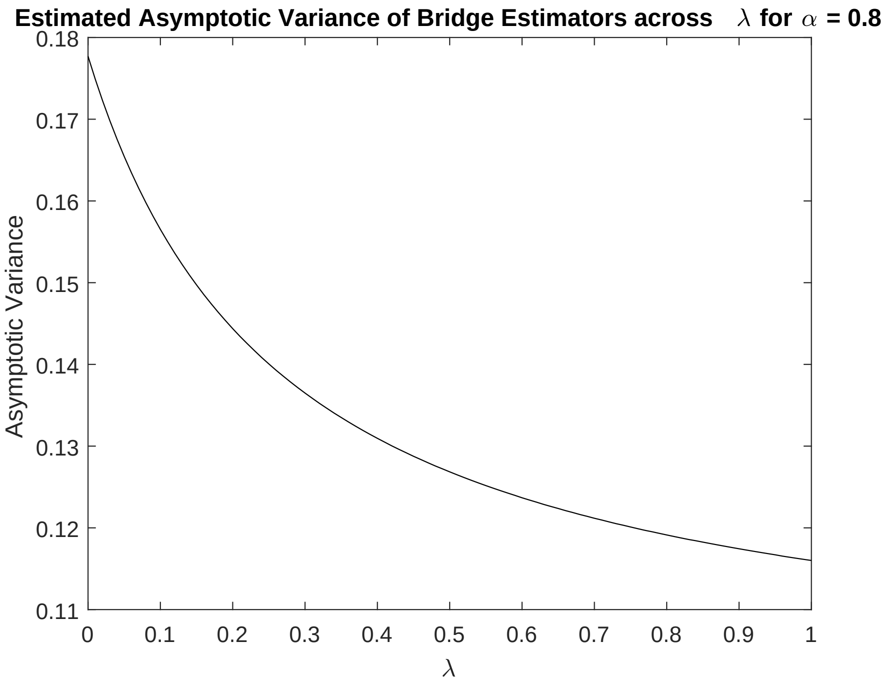

Figure 2: Plot of Estimated Asymptotic Variance of Bridge Estimators across λ for $\alpha = 0.8$, based on the normal data of size 20

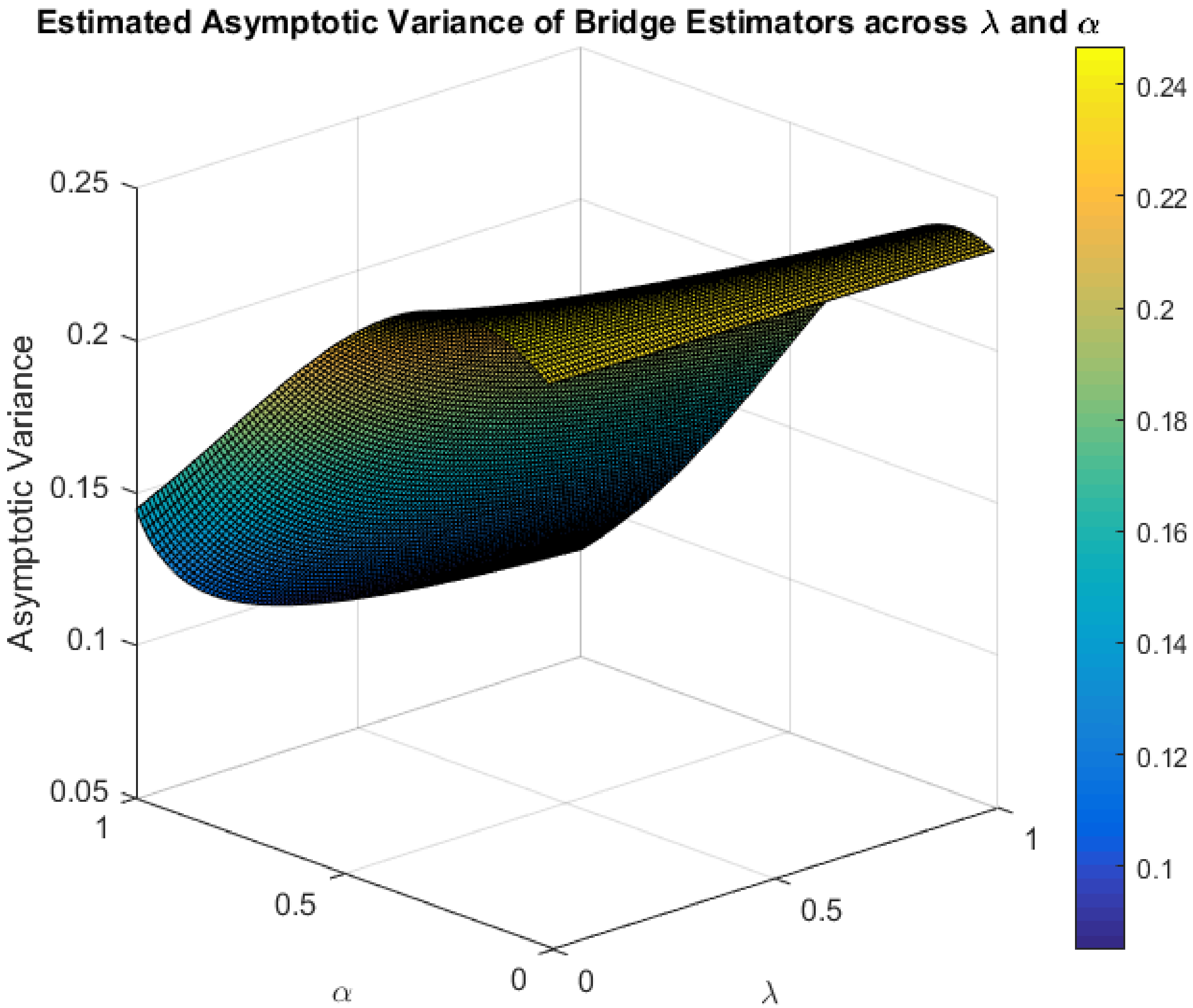

Figure 3: Plot of Estimated Asymptotic Variance of Bridge Estimators across λ and α, based on the normal data of size 20

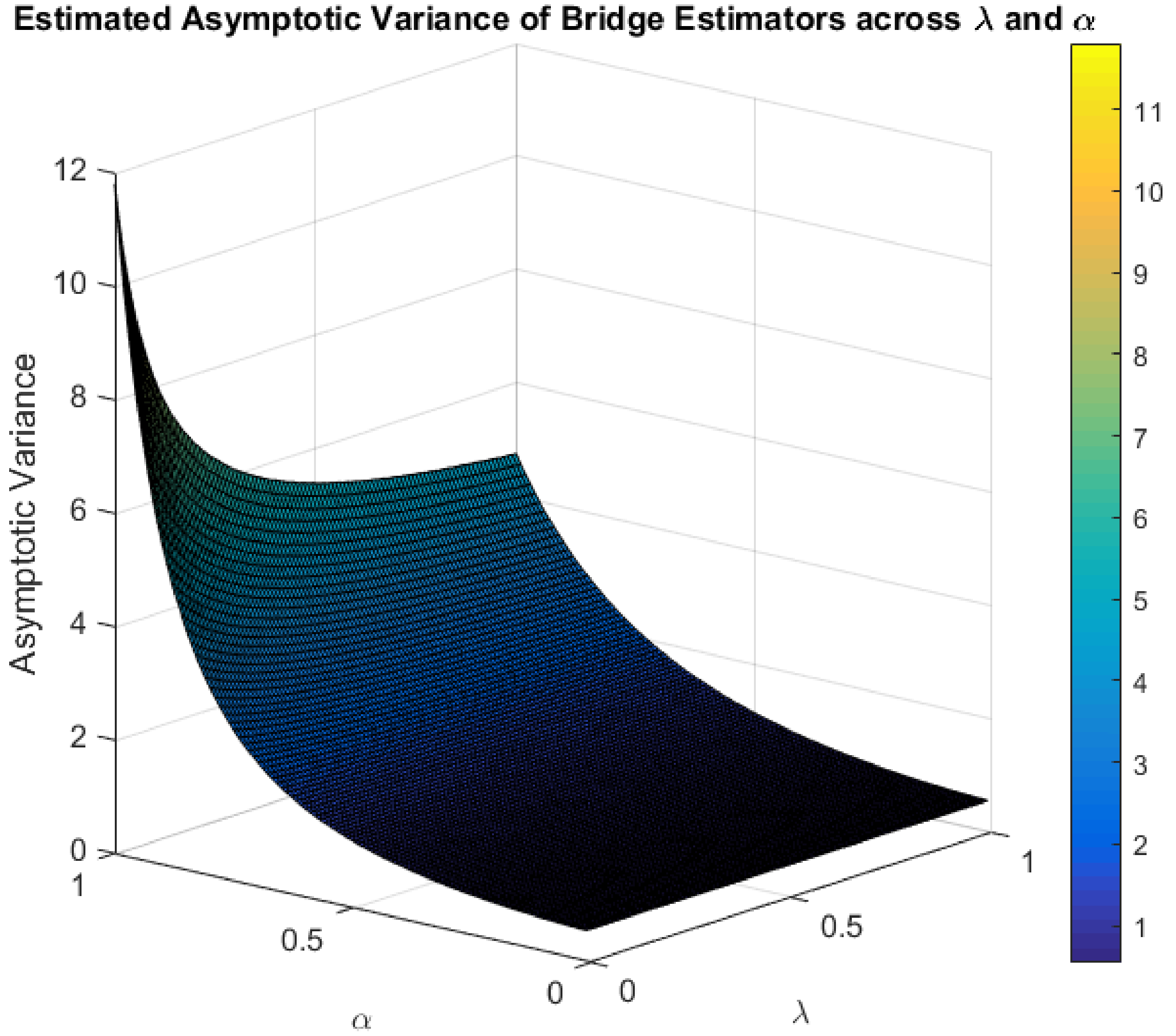

Figure 4: Plot of Estimated Asymptotic Variance of Bridge Estimators across λ and α, based on the simulated normal data of size 1000

S.6 The Determinant is a closeness measure

Theorem 1. *The determinant of a matrix is a closeness measure.*

Proof. Before getting into the proof, note that for any two positive definite matrices A and B of order $p \times p$ with a positive semi-definite difference $A - B$, we have

$$|A| = |A - B + B| = |B| \left| I + B^{-1/2}(A - B)B^{-1/2} \right| = |B| \prod_{i=1}^{p} (1 + \lambda_i),$$

where $\lambda_i \geq 0$ for all $1 \leq i \leq p$ represents the eigenvalues of $B^{-1/2}(A - B)B^{-1/2}$.

(Consistency). The equality above implies that $|A| \geq |B|$ for $A \geq B$. To show that equality $|A| = |B|$ for $A \geq B$ holds if and only if $A = B$, first note that $A = B$ will trivially imply that $|A| = |B|$. If $|A| = |B|$, then by the equality above, we get $\prod_{i=1}^{p}(1 + \lambda_i) = 1$ which implies that $\lambda_i = 0$ for all $1 \leq i \leq p$.

(Continuity). To show that $\|A_n - B\|_\infty \to 0$ if and only if $|A_n| \to |B|$ as $n \to \infty$ under the assumption that $A_n \geq B$ for all n, first note that $\|A_n - B\|_\infty$ converging to zero as $n \to \infty$ implies that $|A_n| - |B| \to 0$. This is because, the determinant is a continuous function of the elements of the matrix. So, it is now enough to prove the opposite direction. For proving this, define $\lambda_{in}, 1 \leq i \leq p$ as the eigenvalues of $B^{-1/2}(A_n - B)B^{-1/2}$. Then by the equality above, we get that as $n \to \infty$,

$$\prod_{i=1}^{p}(1 + \lambda_{in}) \to 1.$$

This ensures that $\lambda_{in} \to 0$ as $n \to \infty$ for all $1 \leq i \leq p$. Therefore $\|A_n - B\|_\infty$ converges to zero as $n \to \infty$. $\square$

References

Ferguson, T. S. (1996). *A course in large sample theory*. Texts in Statistical Science Series. Chapman & Hall, London.

van der Vaart, A. W. (1998). *Asymptotic statistics*. Cambridge University Press, Cambridge.

Table 1: Bias of the minimum bridge divergence estimators for $\epsilon = 0$ in the exponential scale family case. The mean squared errors are given in the parentheses.

α \ λ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286	-0.0286
	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)	(0.9963)
0.2	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0131	-0.0131
	(1.0915)	(1.0918)	(1.0922)	(1.0925)	(1.0929)	(1.0933)	(1.0937)	(1.0942)	(1.0946)	(1.0951)	(1.0956)
0.4	-0.0050	-0.0051	-0.0051	-0.0052	-0.0052	-0.0052	-0.0052	-0.0053	-0.0052	-0.0052	-0.0051
	(1.2918)	(1.2950)	(1.2984)	(1.3021)	(1.3063)	(1.3108)	(1.3158)	(1.3214)	(1.3277)	(1.3348)	(1.3428)
0.6	0.0028	0.0026	0.0023	0.0020	0.0017	0.0014	0.0010	0.0008	0.0006	0.0005	0.0007
	(1.5066)	(1.5155)	(1.5257)	(1.5372)	(1.5505)	(1.5660)	1.5842	1.6060	1.6324	1.6651	1.7068
0.8	0.0113	0.0107	0.0101	0.0094	0.0086	0.0077	0.0068	0.0057	0.0047	0.0041	0.0047
	(1.6968)	(1.7123)	(1.7304)	(1.7517)	(1.7773)	(1.8085)	(1.8473)	(1.8970)	(1.9627)	(2.0535)	(2.1873)
1.0	0.0198	0.0190	0.0180	0.0168	0.0155	0.0138	0.0119	0.0095	0.0068	0.0044	0.0056
	(1.8563)	(1.8770)	(1.9016)	(1.9315)	(1.9686)	(2.0156)	(2.0771)	(2.1609)	(2.2813)	(2.4684)	(2.7981)

Table 2: Bias of the minimum bridge divergence estimators for $\epsilon = 0.05$ in the exponential scale family case. The mean squared errors are given in the parentheses.

α \ λ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	2.5006	2.5006	2.5006	2.5006	2.5006	2.5006	2.5006	2.5006	2.5006	2.5006	2.5006
	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)	(8.3365)
0.2	1.5970	1.5954	1.5937	1.5920	1.5901	1.5882	1.5861	1.5840	1.5817	1.5793	1.5768
	(4.5538)	(4.5490)	(4.5439)	(4.5385)	(4.5329)	(4.5270)	(4.5208)	(4.5142)	(4.5072)	(4.4999)	(4.4921)
0.4	0.9389	0.9306	0.9215	0.9117	0.9008	0.8888	0.8756	0.8608	0.8442	0.8256	0.8044
	(2.8677)	(2.8564)	(2.8441)	(2.8307)	(2.8162)	(2.8002)	(2.7826)	(2.7632)	(2.7417)	(2.7177)	(2.6907)
0.6	0.6599	0.6459	0.6301	0.6122	0.5918	0.5682	0.5406	0.5080	0.4690	0.4215	0.3626
	(2.4616)	(2.4543)	(2.4465)	(2.4382)	(2.4294)	(2.4202)	(2.4105)	(2.4007)	(2.3911)	(2.3826)	(2.3766)
0.8	0.5923	0.5769	0.5590	0.5382	0.5134	0.4836	0.4471	0.4015	0.3428	0.2652	0.1585
	(2.4580)	(2.4572)	(2.4569)	(2.4574)	(2.4595)	(2.4637)	(2.4715)	(2.4853)	(2.5093)	(2.5517)	(2.6287)
1.0	0.6000	0.5857	0.5689	0.5487	0.5240	0.4933	0.4540	0.4021	0.3307	0.2272	0.0670
	(2.5751)	(2.5800)	(2.5865)	(2.5955)	(2.6080)	(2.6261)	(2.6532)	(2.6956)	(2.7659)	(2.8918)	(3.1419)

Table 3: Bias of the minimum bridge divergence estimators for $\epsilon = 0.2$ in the exponential scale family case. The mean squared errors are given in the parentheses.

α \ λ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	9.9310	9.9310	9.9310	9.9310	9.9310	9.9310	9.9310	9.9310	9.9310	9.9310	9.9310
	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)	(103.1433)
0.2	8.5802	8.5776	8.5748	8.5719	8.5688	8.5655	8.5619	8.5581	8.5541	8.5496	8.5449
	(80.1781)	(80.1383)	(80.0956)	(80.0508)	(80.0029)	(79.9513)	(79.8964)	(79.8376)	(79.7743)	(79.7059)	(79.6319)
0.4	6.3748	6.3504	6.3232	6.2927	6.2583	6.2192	6.1744	6.1225	6.0618	5.9898	5.9034
	(49.0325)	(48.7654)	(48.4686)	(48.1369)	(47.7637)	(47.3413)	(46.8588)	(46.3030)	(45.6555)	(44.8928)	(43.9815)
0.6	4.4240	4.3646	4.2958	4.2153	4.1202	4.0060	3.8666	3.6934	3.4734	3.1869	2.8039
	(27.5293)	(27.0756)	(26.5580)	(25.9626)	(25.2711)	(24.4596)	(23.4958)	(22.3361)	(20.9208)	(19.1671)	(16.9600)
0.8	3.5066	3.4379	3.3566	3.2591	3.1402	2.9922	2.8036	2.5565	2.2216	1.7511	1.0688
	(18.6063)	(18.1708)	(17.6661)	(17.0746)	(16.3735)	(15.5316)	(14.5061)	(13.2397)	(11.6579)	(9.6848)	(7.3460)
1.0	3.2200	3.1591	3.0861	2.9967	2.8852	2.7422	2.5529	2.2921	1.9147	1.3370	0.4157
	(15.6965)	(15.3395)	(14.9194)	(14.4182)	(13.8115)	(13.0645)	(12.1278)	(10.9318)	(9.3892)	(7.4499)	(5.4069)

Table 4: Bias of the minimum bridge divergence estimators for and $\epsilon = 0$ in the normal scale family case. The mean squared errors are given in the parentheses.

α \ λ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191	-0.0191
	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)	(2.1467)
0.2	-0.0263	-0.0263	-0.0263	-0.0263	-0.0263	-0.0263	-0.0263	-0.0263	-0.0262	-0.0262	-0.0262
	(0.5846)	(0.5846)	(0.5847)	(0.5848)	(0.5849)	(0.5850)	(0.5851)	(0.5852)	(0.5853)	(0.5854)	(0.5855)
0.4	-0.0266	-0.0267	-0.0267	-0.0268	-0.0269	-0.0269	-0.0270	-0.0271	-0.0272	-0.0273	-0.0274
	(0.6812)	(0.6818))	(0.6826)	(0.6834)	(0.6843)	(0.6853)	(0.6864)	(0.6877)	(0.6893)	(0.6910)	(0.6931)
0.6	-0.0294	-0.0296	-0.0298	-0.0301	-0.0304	-0.0307	-0.0311	-0.0315	-0.0321	-0.0327	-0.0335
	(0.7944)	(0.7964)	(0.7987)	(0.8013)	(0.8043)	(0.8080)	(0.8124)	(0.8178)	(0.8245)	(0.8332)	(0.8449)
0.8	-0.0309	-0.0313	-0.0317	-0.0323	-0.0329	-0.0337	-0.0346	-0.0359	-0.0374	-0.0395	-0.0423
	(0.9041)	(0.9078)	(0.9121)	(0.9172)	(0.9234)	(0.9312)	(0.9410)	(0.9540)	(0.9718)	(0.9978)	(1.0393)
1.0	-0.0302	-0.0307	-0.0313	-0.0321	-0.0331	-0.0343	-0.0358	-0.0379	-0.0410	-0.0455	-0.0525
	(0.9987)	(1.0037)	(1.0097)	(1.0170)	(1.0262)	(1.0380)	(1.0538)	(1.0760)	(1.1093)	(1.1647)	(1.2741)

Table 5: Bias of the minimum bridge divergence estimators for $\epsilon = 0.05$ in the normal scale family case. The mean squared errors are given in the parentheses.

α \ λ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136	-0.5136
	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)	(2.3361)
0.2	-0.3604	-0.3605	-0.3606	-0.3606	-0.3607	-0.3608	-0.3609	-0.3610	-0.3611	-0.3612	-0.3613
	(0.7260)	(0.7260)	(0.7262)	(0.7264)	(0.7265)	(0.7267)	(0.7269)	(0.7270)	(0.7272)	(0.7275)	(0.7277)
0.4	-0.4476	-0.4481	-0.4486	-0.4493	-0.4500	-0.4508	-0.4516	-0.4526	-0.4537	-0.4550	-0.4564
	(0.9104)	(0.9118)	(0.9132)	(0.9149)	(0.9167)	(0.9187)	(0.9209)	(0.9235)	(0.9264)	(0.9297)	(0.9336)
0.6	-0.5335	-0.5350	-0.5366	-0.5385	-0.5406	-0.5432	-0.5462	-0.5498	-0.5542	-0.5597	-0.5667
	(1.1352)	(1.1396)	(1.1448)	(1.1506)	(1.1574)	(1.1654)	(1.1750)	(1.1865)	(1.2008)	(1.2190)	(1.2427)
0.8	-0.6092	-0.6117	-0.6147	-0.6182	-0.6223	-0.6274	-0.6338	-0.6421	-0.6530	-0.6683	-0.6911
	(1.3591)	(1.3679)	(1.3783)	(1.3907)	(1.4057)	(1.4242)	(1.4476)	(1.4782)	(1.5198)	(1.5794)	(1.6720)
1.0	-0.6716	-0.6750	-0.6790	-0.6839	-0.6900	-0.6977	-0.7079	-0.7219	-0.7422	-0.7742	-0.8314
	(1.5545)	(1.5673)	(1.5827)	(1.6016)	(1.6253)	(1.6558)	(1.6967)	(1.7543)	(1.8413)	(1.9851)	(2.2646)

Table 6: Bias of the minimum bridge divergence estimators for $\epsilon = 0.2$ in the normal scale family case. The mean squared errors are given in the parentheses.

α \ λ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778	-1.9778
	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)	(5.7870)
0.2	-2.7379	-2.7381	-2.7385	-2.7387	-2.7391	-2.7394	-2.7397	-2.7401	-2.7404	-2.7408	-2.7413
	(18.1322)	(18.1329)	(18.1339)	(18.1348)	(18.1358)	(18.1368)	(18.1378)	(18.1389)	(18.1401)	(18.1412)	(18.1426)
0.4	-2.3770	-2.3794	-2.3821	-2.3850	-2.3882	-2.3918	-2.3957	-2.4001	-2.4050	-2.4106	-2.4170
	(11.0261)	(11.0370)	(11.0486)	(11.0614)	(11.0755)	(11.0910)	(11.1082)	(11.1275)	(11.1493)	(11.1739)	(11.2020)
0.6	-2.2992	-2.3084	-2.3187	-2.3307	-2.3445	-2.3608	-2.3831	-2.4134	-2.4445	-2.4831	-2.5295
	(6.8577)	(6.9119)	(6.9739)	(7.0458)	(7.1303)	(7.2318)	(7.4016)	(7.6577)	(7.8807)	(8.1785)	(8.5203)
0.8	-2.6605	-2.6814	-2.7062	-2.7382	-2.7941	-2.8534	-2.9425	-3.0508	-3.2000	-3.4096	-3.6874
	(8.6585)	(8.8162)	(9.0067)	(9.2747)	(9.8667)	(10.4664)	(11.4595)	(12.6542)	(14.3434)	(16.7522)	(19.9901)
1.0	-2.9865	-3.0278	-3.0800	-3.1562	-3.2333	-3.3812	-3.5566	-3.8028	-4.1377	-4.6124	-5.3932
	(10.7030)	(11.1038)	(11.6395)	(12.5220)	(13.3224)	(15.1724)	(17.2957)	(20.3636)	(24.4783)	(30.3887)	(40.3177)