Supplementary Material for "Estimation of the Tail Exponent of Multivariate Regular Variation"

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We prove some lemmas appearing in Section 5.

Proof of Lemma 1. Let $\epsilon > 0$ and take $x_0 > 0$ such that for any $x > x_0$,

$$\frac{x^\alpha}{C_{\boldsymbol{\lambda}}(1+\epsilon)} \le \frac{1}{1-F_{\boldsymbol{\lambda}}(x)} \le \frac{x^\alpha}{C_{\boldsymbol{\lambda}}(1-\epsilon)} \quad \text{for all } \boldsymbol{\lambda} \in \mathbb{S}^{d-1},$$

so that for $x > x_0^\alpha / \{(1-\epsilon)\inf_{\boldsymbol{\lambda}} C_{\boldsymbol{\lambda}}\}$,

$$C_{\boldsymbol{\lambda}}^{1/\alpha}(1-\epsilon)^{1/\alpha} x^{1/\alpha} \le b_{\boldsymbol{\lambda}}(x) \le C_{\boldsymbol{\lambda}}^{1/\alpha}(1+\epsilon)^{1/\alpha} x^{1/\alpha} \quad \text{for each } \boldsymbol{\lambda} \in \mathbb{S}^{d-1},$$

that is, as $x \to \infty$,

$$b_{\boldsymbol{\lambda}}(x) = C_{\boldsymbol{\lambda}}^{1/\alpha} x^{1/\alpha} L_{\boldsymbol{\lambda}}(x), \quad \text{where } L_{\boldsymbol{\lambda}}(x) = 1 + o(1) \text{ uniformly in } \boldsymbol{\lambda} \in \mathbb{S}^{d-1}.$$

This in turn implies that, owing to the continuity of $F_{\boldsymbol{\lambda}}$, as $x \to \infty$,

$$\begin{aligned} x = \frac{1}{1-F_{\boldsymbol{\lambda}}(b_{\boldsymbol{\lambda}}(x))} &= x\{L_{\boldsymbol{\lambda}}(x)\}^\alpha \left\{1 - C_{\boldsymbol{\lambda}}^{\gamma_{\boldsymbol{\lambda}}/\alpha} D_{\boldsymbol{\lambda}} x^{\gamma_{\boldsymbol{\lambda}}/\alpha} \{L_{\boldsymbol{\lambda}}(x)\}^{\gamma_{\boldsymbol{\lambda}}} + o\left(\{b_{\boldsymbol{\lambda}}(x)\}^{\gamma_{\boldsymbol{\lambda}}}\right)\right\} \\ &= x\{L_{\boldsymbol{\lambda}}(x)\}^\alpha \left\{1 - C_{\boldsymbol{\lambda}}^{\gamma_{\boldsymbol{\lambda}}/\alpha} D_{\boldsymbol{\lambda}} x^{\gamma_{\boldsymbol{\lambda}}/\alpha} + o(x^{\gamma_{\boldsymbol{\lambda}}/\alpha})\right\} \quad \text{uniformly in } \boldsymbol{\lambda} \in \mathbb{S}^{d-1}. \end{aligned}$$

Hence, as $x \to \infty$,

$$L_{\boldsymbol{\lambda}}(x) = \left\{1 + C_{\boldsymbol{\lambda}}^{\gamma_{\boldsymbol{\lambda}}/\alpha} D_{\boldsymbol{\lambda}} x^{\gamma_{\boldsymbol{\lambda}}/\alpha} + o(x^{\gamma_{\boldsymbol{\lambda}}/\alpha})\right\}^{1/\alpha} = 1 + \frac{C_{\boldsymbol{\lambda}}^{\gamma_{\boldsymbol{\lambda}}/\alpha} D_{\boldsymbol{\lambda}}}{\alpha} x^{\gamma_{\boldsymbol{\lambda}}/\alpha} + o(x^{\gamma_{\boldsymbol{\lambda}}/\alpha})$$

uniformly in $\boldsymbol{\lambda} \in \mathbb{S}^{d-1}$. This validates the lemma. □

Proof of Lemma 2. Let

$$L_{\boldsymbol{\lambda}}(x) = 1 + D_{\boldsymbol{\lambda}} x^{\gamma_{\boldsymbol{\lambda}}} + x^{\gamma_{\boldsymbol{\lambda}}} \Delta_{\boldsymbol{\lambda}}(x), \quad \tilde{L}_{\boldsymbol{\lambda}}(y) = \frac{L_{\boldsymbol{\lambda}}\left(y^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right)}{L_{\boldsymbol{\lambda}}(b_{\boldsymbol{\lambda}}(n/k))}.$$

Then, we have that $\tilde{L}_{\boldsymbol{\lambda}}(y) = 1 + o(1)$ uniformly in $0 < y < y_0$ and $\boldsymbol{\lambda} \in \mathbb{S}^{d-1}$, and as $x \to \infty$,

$$\begin{aligned} \frac{L_{\boldsymbol{\lambda}}(yx)}{L_{\boldsymbol{\lambda}}(x)} - 1 &= x^{\gamma_{\boldsymbol{\lambda}}} \frac{D_{\boldsymbol{\lambda}}(y^{\gamma_{\boldsymbol{\lambda}}} - 1) + (y^{\gamma_{\boldsymbol{\lambda}}} - 1)\Delta_{\boldsymbol{\lambda}}(yx) + \Delta_{\boldsymbol{\lambda}}(yx) - \Delta_{\boldsymbol{\lambda}}(x)}{L_{\boldsymbol{\lambda}}(x)} \qquad \text{(S.1)} \\ &= \frac{x^{\gamma_{\boldsymbol{\lambda}}}}{L_{\boldsymbol{\lambda}}(x)} \{D_{\boldsymbol{\lambda}}(y^{\gamma_{\boldsymbol{\lambda}}} - 1) + (y^{\gamma_{\boldsymbol{\lambda}}} - 1)\Delta_{\boldsymbol{\lambda}}(yx) + \\ &\quad \frac{\partial \Delta_{\boldsymbol{\lambda}}}{\partial x}(\tilde{y}x)(y-1)x\mathrm{I}(1 \le y < 2) + (\Delta_{\boldsymbol{\lambda}}(yx) - \Delta_{\boldsymbol{\lambda}}(x))\mathrm{I}(y \ge 2)\Big\} \\ &\sim x^{\gamma_{\boldsymbol{\lambda}}} D_{\boldsymbol{\lambda}}(y^{\gamma_{\boldsymbol{\lambda}}} - 1) \quad \text{uniformly in } y > 1 \text{ and } \boldsymbol{\lambda} \in \mathbb{S}^{d-1}, \end{aligned}$$

where the mean value theorem is used with $\tilde{y} \in (1, y)$. Thus, we have

$$\left| \frac{n}{k} \frac{\bar{F}_{\boldsymbol{\lambda}}(y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)) - \bar{F}_{\boldsymbol{\lambda}}(y_1^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))}{y_2 - y_1} - 1 \right| = \left| \frac{n}{k} \bar{F}_{\boldsymbol{\lambda}}(b_{\boldsymbol{\lambda}}(n/k)) \frac{y_2 \tilde{L}_{\boldsymbol{\lambda}}(y_2) - y_1 \tilde{L}_{\boldsymbol{\lambda}}(y_1)}{y_2 - y_1} - 1 \right|$$

$$= \left| \frac{(y_2 - y_1)\tilde{L}_{\boldsymbol{\lambda}}(y_2) + y_1 \left(\tilde{L}_{\boldsymbol{\lambda}}(y_2) - \tilde{L}_{\boldsymbol{\lambda}}(y_1)\right)}{y_2 - y_1} - 1 \right| = \left| \tilde{L}_{\boldsymbol{\lambda}}(y_2) - 1 + \frac{y_1}{y_2 - y_1} \left(\tilde{L}_{\boldsymbol{\lambda}}(y_2) - \tilde{L}_{\boldsymbol{\lambda}}(y_1)\right) \right|,$$

and further,

$$\begin{aligned} \frac{y_1}{y_2 - y_1} \left(\tilde{L}_{\boldsymbol{\lambda}}(y_2) - \tilde{L}_{\boldsymbol{\lambda}}(y_1)\right) &= \frac{y_1}{y_2 - y_1} \tilde{L}_{\boldsymbol{\lambda}}(y_2) \left(1 - \frac{\tilde{L}_{\boldsymbol{\lambda}}(y_1)}{\tilde{L}_{\boldsymbol{\lambda}}(y_2)}\right) \\ &= \frac{y_1}{y_2 - y_1} \tilde{L}_{\boldsymbol{\lambda}}(y_2) \left(1 - \frac{L_{\boldsymbol{\lambda}}\left(y_1^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right)}{L_{\boldsymbol{\lambda}}\left(y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right)}\right) \\ &= \frac{y_1}{y_2 - y_1} \tilde{L}_{\boldsymbol{\lambda}}(y_2) \left(1 - \frac{L_{\boldsymbol{\lambda}}\left((y_1/y_2)^{-1/\alpha} y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right)}{L_{\boldsymbol{\lambda}}\left(y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right)}\right) \\ &\sim -D_{\boldsymbol{\lambda}} \left\{y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right\}^{\gamma_{\boldsymbol{\lambda}}} \frac{(y_2/y_1)^{\gamma_{\boldsymbol{\lambda}}/\alpha} - 1}{(y_2/y_1 - 1)} \end{aligned}$$

uniformly in $0 < y_1 < y_2 < y_0$ and $\boldsymbol{\lambda} \in \mathbb{S}^{d-1}$. Thus, since

$$\sup_{\boldsymbol{\lambda} \in \mathbb{S}^{d-1}} \sup_{0 < y_1 < y_2} \frac{(y_2/y_1)^{\gamma_{\boldsymbol{\lambda}}/\alpha} - 1}{y_2/y_1 - 1} < \infty, \quad \text{and} \quad \lim_{n \to \infty} \sup_{\boldsymbol{\lambda} \in \mathbb{S}^{d-1}} \sup_{0 < y_2 \le y_0} \left\{y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right\}^{\gamma_{\boldsymbol{\lambda}}} = 0,$$

we have

$$\lim_{n \to \infty} \sup_{\boldsymbol{\lambda} \in \mathbb{S}^{d-1}} \sup_{0 < y_1 < y_2 \le y_0} \left| \frac{y_1}{y_2 - y_1} \left(\tilde{L}_{\boldsymbol{\lambda}}(y_2) - \tilde{L}_{\boldsymbol{\lambda}}(y_1)\right) \right| = 0.$$

On the other hand, it can be easily seen that

$$\lim_{n \to \infty} \sup_{0 < y \le y_0} \left| \frac{n}{k} \frac{\bar{F}_{\boldsymbol{\lambda}}\left(y^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k)\right)}{y} - 1 \right| = 0.$$

Hence, (19) is established.

Now, let $p > 0$ be an integer. Observe that

$$\begin{aligned}
&\mathrm{E}\{Y_n(\boldsymbol{\lambda}, y_2) - Y_n(\boldsymbol{\lambda}, y_1)\}^p \\
&= \left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^p \bar{F}_{\boldsymbol{\lambda}}\left(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)\right) + \mathrm{E}Y_n^p(\boldsymbol{\lambda}, y_2)\mathrm{I}\left(\boldsymbol{U}^{(\boldsymbol{\lambda})} \le y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)\right) \\
&= \bar{F}_{\boldsymbol{\lambda}}(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)) \int_0^{\left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^p} \exp\left(-\alpha u^{1/p}\right) \frac{y_2}{y_1} \frac{L_{\boldsymbol{\lambda}}(\exp\left(u^{1/p}\right) y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))}{L_{\boldsymbol{\lambda}}(y_1^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))} du,
\end{aligned}$$

since

$$\begin{aligned}
&\mathrm{E}Y_n^p(\boldsymbol{\lambda}, y_2)\mathrm{I}\left(\boldsymbol{U}^{(\boldsymbol{\lambda})} \le y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)\right) \\
&= \bar{F}_{\boldsymbol{\lambda}}(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)) \int_0^{\left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^p} \exp\left(-\alpha u^{1/p}\right) \frac{y_2}{y_1} \frac{L_{\boldsymbol{\lambda}}(\exp\left(u^{1/p}\right) y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))}{L_{\boldsymbol{\lambda}}(y_1^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))} du \\
&\quad - \left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^p \bar{F}_{\boldsymbol{\lambda}}\left(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)\right).
\end{aligned}$$

Moreover,

$$\frac{L_{\boldsymbol{\lambda}}(\exp\left(u^{1/p}\right) y_2^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))}{L_{\boldsymbol{\lambda}}(y_1^{-1/\alpha} b_{\boldsymbol{\lambda}}(n/k))} = 1 + o(1) \quad \text{as } n \to \infty$$

uniformly in $0 \le u < \infty$, $0 \le y_1 < y_2 \le y_0$, and $\boldsymbol{\lambda} \in \mathbb{S}^{d-1}$. Meanwhile, note that

$$\begin{aligned}
\bar{F}_{\boldsymbol{\lambda}}(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)) \int_0^{\left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^2} e^{-\alpha u^{1/2}} \frac{y_2}{y_1} du &= \frac{2}{\alpha^2} \bar{F}_{\boldsymbol{\lambda}}(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)) \left(\frac{y_2}{y_1} - 1 - \log\frac{y_2}{y_1}\right) \\
&\sim \frac{2k}{\alpha^2 n}\left(y_2 - y_1 - y_1\log\frac{y_2}{y_1}\right)
\end{aligned}$$

and

$$\begin{aligned}
\bar{F}_{\boldsymbol{\lambda}}(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)) \int_0^{\left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^3} e^{-\alpha u^{1/3}} \frac{y_2}{y_1} du &= \frac{3}{\alpha^3} \bar{F}_{\boldsymbol{\lambda}}(y_1^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k)) \left\{2\frac{y_2}{y_1} - 1 - \left(\log\frac{y_2}{y_1} + 1\right)^2\right\} \\
&\sim \frac{3k}{\alpha^2 n}\left\{2y_2 - y_1 - y_1\left(\log\frac{y_2}{y_1} + 1\right)^2\right\}
\end{aligned}$$

uniformly in $0 \le y_1 < y_2 \le 1$, so that (20) and (21) hold. This completes the proof. □

Proof of Lemma 3. By using Hölder's inequality, we get

$$\begin{aligned}
\mathrm{E}\{g_n^*(\boldsymbol{U})\}^2 \mathrm{I}(g_n^*(\boldsymbol{U}) > \eta\sqrt{n}) &\le \left[\mathrm{E}\{g_n^*(\boldsymbol{U})\}^3\right]^{2/3} \left[P(g_n^*(\boldsymbol{U}) > \eta\sqrt{n})\right]^{1/3} \\
&\le K\left\{\frac{n}{k} P(g_n^*(\boldsymbol{U}) > \eta\sqrt{n})\right\}^{1/3} \le K\left\{\frac{n}{k}\frac{\mathrm{E}g_n^*(\boldsymbol{U})}{\eta\sqrt{n}}\right\}^{1/3},
\end{aligned}$$

where the last term converges to 0 as $n \to \infty$. Since the others can be easily verified, we complete the proof without detailing algebras. $\square$

Proof of Lemma 4. Lemma 1 implies $b_{\boldsymbol{\lambda}}(n/k)(k/n)^{1/\alpha} = C_{\boldsymbol{\lambda}}^{1/\alpha}(1+o(1))$ uniformly in $\boldsymbol{\lambda}$. Combining this and Lemma 2, we establish the lemma. $\square$

Proof of Lemma 5. It suffices to prove that provided $|\boldsymbol{u}| \le A$ and $\boldsymbol{u}^{(\boldsymbol{\lambda}^*)} > w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}$,

$$\frac{\boldsymbol{u}^{(\boldsymbol{\lambda}^*)}}{w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}} \le \frac{\boldsymbol{u}^{(\boldsymbol{\lambda})}}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)},$$

and further, when $|\boldsymbol{u}| \le A$ and $\boldsymbol{u}^{(\boldsymbol{\lambda})} > y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)$,

$$\frac{\boldsymbol{u}^{(\boldsymbol{\lambda}^*)}}{w_2^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}} \ge \frac{\boldsymbol{u}^{(\boldsymbol{\lambda})}}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)}.$$

Here, we only prove the first inequality since the second can be handled similarly.

Note that

$$\begin{aligned}
&\frac{\boldsymbol{u}^{(\boldsymbol{\lambda}^*)}}{w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}} - \frac{\boldsymbol{u}^{(\boldsymbol{\lambda})}}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)} \\
&= \boldsymbol{u}^{(\boldsymbol{\lambda}^*)}\left(\frac{1}{w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}} - \frac{1}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)}\right) + \frac{(\boldsymbol{\lambda}^* - \boldsymbol{\lambda})'\boldsymbol{u}}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)} =: I_1 + I_2.
\end{aligned}$$

It can be seen that

$$I_1 \le 1 - \frac{w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)}$$

since $\boldsymbol{u}^{(\boldsymbol{\lambda}^*)} > w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}$ and

$$\frac{1}{w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}} - \frac{1}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)} < 0.$$

Further,

$$I_2 \le \frac{w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}}{y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)} - 1$$

since

$$\left|(\boldsymbol{\lambda}^* - \boldsymbol{\lambda})'\boldsymbol{u}\right| \le w_1^{-\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}} - y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k).$$

These assert the lemma. $\square$

Proof of Lemma 6. We only provide the proof of (23) since (24) can be handled similarly. According to Lemmas 1 and 4, we can find $n_0 \in \mathbb{N}$ and $K_0 > 0$, such that

$$\underline{w} < \frac{y(n/k)}{\{b_{\boldsymbol{\lambda}}(n/k)\}^{\alpha}} < \bar{w} \quad \text{for all } y \in [\underline{y}, \bar{y}],\ \boldsymbol{\lambda} \in \mathbb{S}^{d-1},\ n \ge n_0,$$

and

$$\|s_n(\boldsymbol{U};\boldsymbol{\lambda},v_1,b(n,\epsilon))-l_n(\boldsymbol{U};\boldsymbol{\lambda},v_2,b(n,\epsilon))\|_2^2\le K_0(v_1-v_2)+\frac{\epsilon^2}{2},$$

$$\frac{(n/k)^{\frac{1}{\alpha}}}{b(n,\epsilon)}\ge\frac{\epsilon^{2/\alpha}}{K_0},\quad v_2^{-\frac{1}{\alpha}}-v_1^{-\frac{1}{\alpha}}\ge\frac{v_1-v_2}{K_0},$$

whenever $\epsilon>0$, $\underline{w}<v_2<v_1<\bar{w}$, $\boldsymbol{\lambda}\in\mathbb{S}^{d-1}$ and $n\ge n_0$. For a given $\epsilon>0$, we set

$$w_i=\frac{\epsilon^2 i}{6K_0}\quad\text{for } i\ge 0,$$

and putting $m=\left\lceil\frac{6K_0^3\sqrt{d(d-1)}}{\epsilon^{2+2/\alpha}}\right\rceil$, we denote

$$\mathbb{S}_\epsilon^{d-1}=\left\{\left(\frac{i_1}{m},\frac{i_2}{m},\cdots,\frac{i_d}{m}\right):i_1,\dots,i_d\text{ are nonnegative integers with } i_1+\cdots+i_d=m\right\}.$$

For given $n\ge n_0$, $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_d)\in\mathbb{S}^{d-1}$ and $y\in[\underline{y},\bar{y}]$, we choose w_i,w_{i+3} and $\boldsymbol{\lambda}^*=(\lambda_1^*,\dots,\lambda_d^*)\in\mathbb{S}_\epsilon^{d-1}$, such that $w_{i+2}^{-\frac{1}{\alpha}}\le y^{-\frac{1}{\alpha}}b_{\boldsymbol{\lambda}}(n/k)(k/n)^{\frac{1}{\alpha}}\le w_{i+1}^{-\frac{1}{\alpha}}$ and $|\lambda_j-\lambda_j^*|\le m^{-1}$ for each $j=1,2,\dots,d-1$. Then, it can be seen that

$$l_n(\boldsymbol{u};\boldsymbol{\lambda}^*,w_i,b(n,\epsilon))\le f_n(\boldsymbol{u};\boldsymbol{\lambda},y)\le s_n(\boldsymbol{u};\boldsymbol{\lambda}^*,w_{i+3},b(n,\epsilon))\quad\text{for each }\boldsymbol{u}$$

(cf. Lemma 5) and

$$\|s_n(\boldsymbol{U};\boldsymbol{\lambda}^*,w_{i+3},b(n,\epsilon))-l_n(\boldsymbol{U};\boldsymbol{\lambda}^*,w_i,b(n,\epsilon))\|_2\le\epsilon.$$

Thus, if we put

$$\mathcal{B}_n=\left\{[l_n(\cdot\,;\boldsymbol{\lambda}^*,w_i,b(n,\epsilon)),s_n(\cdot\,;\boldsymbol{\lambda}^*,w_{i+3},b(n,\epsilon))]^f:i=0,1,\cdots,\left\lceil\frac{6K_0\bar{w}}{\epsilon^2}\right\rceil,\ \boldsymbol{\lambda}^*\in\mathbb{S}_\epsilon^{d-1}\right\},$$

we can see that $T\subset\bigcup\mathcal{B}_n$ and each member in $\mathcal{B}_n$ is an ϵ-bracket. Hence, for some $K>0$,

$$\limsup_{n\to\infty}\int_0^\delta\sqrt{\log N_{[]}^f(\epsilon\,;n)}d\epsilon\le\int_0^\delta\sqrt{\log\frac{K}{\epsilon^{(2+2/\alpha)(d-1)+2}}}d\epsilon<\infty$$

for sufficiently small $\delta>0$. This validates the lemma. □

Proof of Lemma 7. We only provide the proof of (25). Let $n_0\in\mathbb{N}$ and $K_0>0$ be the constants in the proof of Lemma 6. For given $\epsilon>0$, assume that

$$n\ge n_0,\quad|\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2|\le\frac{\epsilon^{2+2/\alpha}}{6K_0^3},\quad\frac{\epsilon^2}{6K_0}\le\left|\frac{y_1(n/k)}{\{b_{\boldsymbol{\lambda}_1}(n/k)\}^\alpha}-v_1\right|\le\frac{\epsilon^2}{3K_0}.$$

Then, by Lemmas 4 and 5, we can have

$$\|f_n(\boldsymbol{U};\boldsymbol{\lambda}_1,y_1)-s_n(\boldsymbol{U};\boldsymbol{\lambda}_2,v_1,b(n,\epsilon))\|_2\le\epsilon.$$

Moreover, for $y_2 \in [\underline{y}, \bar{y}]$,

$$\begin{aligned}
&\|s_n(\boldsymbol{U};\boldsymbol{\lambda}_2, v_1, b(n,\epsilon)) - f_n(\boldsymbol{U};\boldsymbol{\lambda}_2, y_2)\|_2^2 \\
&\leq \frac{\epsilon^2}{2} + K_0\left|v_1 - \frac{y_2(n/k)}{\{b_{\boldsymbol{\lambda}_2}(n/k)\}^\alpha}\right| \leq \frac{\epsilon^2}{2} + K_0\left(\left|\frac{y_1(n/k)}{\{b_{\boldsymbol{\lambda}_2}(n/k)\}^\alpha} - \frac{y_2(n/k)}{\{b_{\boldsymbol{\lambda}_2}(n/k)\}^\alpha}\right| + \frac{\epsilon^2}{3K_0}\right) \\
&\leq \frac{\epsilon^2}{2} + K_0\left(\left|\frac{y_1}{C_{\boldsymbol{\lambda}_1}} - \frac{y_2}{C_{\boldsymbol{\lambda}_2}}\right| + \frac{\epsilon^2}{3K_0}\right) + o(1)
\end{aligned}$$

uniformly in $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{S}^{d-1}$ and $y_1, y_2 \in [\underline{y}, \bar{y}]$. Hence, we obtain (25) from the Lipschitz continuity of $\boldsymbol{\lambda} \mapsto 1/C_{\boldsymbol{\lambda}}$. This completes the proof. □

Proof of Lemma 8. Observe that $W_n(\boldsymbol{\lambda}, y) > \zeta$ if and only if

$$\begin{aligned}
\frac{1}{\sqrt{k}}\sum_{i=1}^n \left\{\mathrm{I}\left(\boldsymbol{U}_i^{(\boldsymbol{\lambda})} > e^{\zeta/\sqrt{k}} b_{\boldsymbol{\lambda}}\left(\frac{n}{yk}\right)\right) - P\left(\boldsymbol{U}_i^{(\boldsymbol{\lambda})} > e^{\zeta/\sqrt{k}} b_{\boldsymbol{\lambda}}\left(\frac{n}{yk}\right)\right)\right\} \qquad \text{(S.2)} \\
> \sqrt{k}\left\{y - ye^{-\frac{\alpha\zeta}{\sqrt{k}}} + o\left(\frac{1}{\sqrt{k}}\right)\right\} = \alpha y \zeta + o(1)
\end{aligned}$$

uniformly in $(\boldsymbol{\lambda}, y) \in T$ and $\zeta \in [-K, K]$. Further, the asymptotical uniform equicontinuity of M_n follows from Proposition 1, and thus, the left-hand side of (S.2) is asymptotically equal to $M_n(\boldsymbol{\lambda}, y)$ owing to the fact that

$$e^{\zeta/\sqrt{k}} b_{\boldsymbol{\lambda}}\left(\frac{n}{yk}\right) \sim y^{-\frac{1}{\alpha}} b_{\boldsymbol{\lambda}}(n/k) \quad \text{uniformly in } (\boldsymbol{\lambda}, y) \in T \text{ and } \zeta \in [-K, K]$$

as $n \to \infty$ (cf. Lemma 1). This validates the lemma. □

Proof of Lemma 9. Let $L_{\boldsymbol{\lambda}}(x) = x^\alpha\{1 - F_{\boldsymbol{\lambda}}(x)\}$. Then

$$\begin{aligned}
&\frac{n}{yk}\mathrm{E}\left(\log \boldsymbol{U}^{(\boldsymbol{\lambda})} - \log b_{\boldsymbol{\lambda}}\left(\frac{n}{yk}\right) - \frac{\zeta}{\alpha\sqrt{k}}\right)_+ = e^{-\frac{\zeta}{\sqrt{k}}}\int_0^\infty e^{-\alpha x}\frac{L_{\boldsymbol{\lambda}}(e^{x+\zeta/\alpha\sqrt{k}} b_{\boldsymbol{\lambda}}(n/yk))}{L_{\boldsymbol{\lambda}}(b_{\boldsymbol{\lambda}}(n/yk))}dx \\
&= e^{-\frac{\zeta}{\sqrt{k}}}\left\{\frac{1}{\alpha} + \int_0^\infty e^{-\alpha x}\left(\frac{L_{\boldsymbol{\lambda}}(e^{x+\zeta/\alpha\sqrt{k}} b_{\boldsymbol{\lambda}}(n/yk))}{L_{\boldsymbol{\lambda}}(b_{\boldsymbol{\lambda}}(n/yk))} - 1\right)dx\right\} \\
&= \frac{1}{\alpha} - \frac{\zeta}{\alpha\sqrt{k}} + \frac{\gamma_{\boldsymbol{\lambda}} D_{\boldsymbol{\lambda}} M_{\boldsymbol{\lambda}}}{\sqrt{k}\alpha(\alpha - \gamma_{\boldsymbol{\lambda}})} y^{-\gamma_{\boldsymbol{\lambda}}/\alpha} + o\left(\frac{1}{\sqrt{k}}\right)
\end{aligned}$$

uniformly in $\zeta \in [-K, K]$ and $(\boldsymbol{\lambda}, y) \in T$. This validates the lemma. □