Supplementary Material of "Parameter Change Test for Autoregressive Conditional Duration Models" by Lee and Oh

1 Proofs of Theorems 1-2 and Propositions 1-2

Proof of Theorem 1. We first show that $||\hat{L}_n - L_n||_K/n \xrightarrow{a.s.} 0$. Since $\psi_i,\ \hat{\psi}_i \geq \underline{g} > 0$, an application of the mean value theorem leads to $||(\hat{\psi}_i)^{-1} - (\psi_i)^{-1}||_K \leq \underline{g}^{-2}||\hat{\psi}_i - \psi_i||_K$ and $||\log\hat{\psi}_i - \log\psi_i||_K \leq \underline{g}^{-1}||\hat{\psi}_i - \psi_i||_K$. Thus, there exists $C > 0$ with

$$||\hat{L}_n - L_n||_K \leq C\sum_{i=1}^{n}(1+x_i)||\hat{\psi}_i - \psi_i||_K \leq C\sum_{i=1}^{\infty}(1+x_i)||\hat{\psi}_i - \psi_i||_K.$$

Owing to Proposition 3.12 of Straumann and Mikosch (2006), we have $||\hat{\psi}_i - \psi_i||_K \xrightarrow{e.a.s.} 0$. Also, due to (C.1), the fact that $\mathbb{E}\epsilon_0 = 1$, and Lemma 2.2 of Straumann and Mikosch (2006), we have $\mathbb{E}[\log^+(1+x_0)] < \infty$. Hence, $||\hat{L}_n - L_n||_K/n \xrightarrow{a.s.} 0$.

For the uniqueness of the maximum of L on K, we need to prove that $L(\boldsymbol{\theta}) < L(\boldsymbol{\theta}_0)$ for all $\boldsymbol{\theta} \in K\backslash\{\boldsymbol{\theta}_0\}$. Since $\mathbb{E}[\log\psi_{0,0}]$ is finite and does not depend on the parameter $\boldsymbol{\theta}$, we can demonstrate that

$$Q(\boldsymbol{\theta}) = \mathbb{E}\left(\log\frac{\psi_{0,0}}{\psi_o(\boldsymbol{\theta})} - \frac{x_0}{\psi_o(\boldsymbol{\theta})}\right) = \mathbb{E}\left(\log\frac{\psi_{0,0}}{\psi_o(\boldsymbol{\theta})} - \frac{\psi_{0,0}}{\psi_o(\boldsymbol{\theta})}\right), \qquad \boldsymbol{\theta} \in K,$$

is uniquely maximized at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. Since $\log(x) - x \leq -1$ for all $x > 0$ with equality if and only if $x = 1$, we should have $Q(\boldsymbol{\theta}) \leq -1 = Q(\boldsymbol{\theta}_0)$ with equality if and only if $\psi_{0,0}/\psi_0(\boldsymbol{\theta}) \equiv 1$ and also if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, which shows that Q and L are uniquely maximized at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.

By (C.3) and Proposition 3.12 of Straumann and Mikosch (2006), it can be seen that the function

$$\boldsymbol{\theta} \mapsto l_i(\boldsymbol{\theta}) = -\left(\frac{x_i}{\psi_i(\boldsymbol{\theta})} + \log\psi_i(\boldsymbol{\theta})\right)$$

is continuous on K with probability 1. Since for every fixed $\boldsymbol{\theta} \in K$, the sequence $\{l_i(\boldsymbol{\theta})\}$ is stationary and ergodic, it holds that $n^{-1}\sum_{i=1}^{n} l_i(\boldsymbol{\theta}) \overset{a.s.}{\to} L(\boldsymbol{\theta}) = \mathrm{E}[l_0(\boldsymbol{\theta})]$ as $n \to \infty$. In case $\mathrm{E}x_0 = \infty$, the latter limit can take the value $-\infty$ at certain point $\boldsymbol{\theta}$, but $\psi_0(\boldsymbol{\theta}) \geq \underline{g} \geq 0$ guarantees $L(\boldsymbol{\theta}) < \infty$ for all $\boldsymbol{\theta} \in K$. Therefore, we can use the same arguments as given in the proof of Lemma 3.11 of Pfanzagl (1969) to show that the function L is upper semicontinuous on K and $\limsup_{n\to\infty}\sup_{\boldsymbol{\theta}\in K'} L_n(\boldsymbol{\theta})/n \leq \sup_{\boldsymbol{\theta}\in K'} L(\boldsymbol{\theta})$ with probability 1 for any compact subset $K' \subset K$. Since $||\hat{L}_n - L_n||_K/n \xrightarrow{a.s.} 0$, we have $\limsup_{n\to\infty}\sup_{\boldsymbol{\theta}\in K'}\hat{L}_n(\boldsymbol{\theta})/n \leq \sup_{\boldsymbol{\theta}\in K'} L(\boldsymbol{\theta})$ a.s.. Further, since $||\hat{L}_n - L_n||_K/n \xrightarrow{a.s.} 0$ and $L_n(\boldsymbol{\theta}_0)/n \overset{a.s.}{\to} L(\boldsymbol{\theta}_0)$, we get $\limsup_{k\to\infty}\hat{L}_n(\boldsymbol{\theta}_0)/n = L(\boldsymbol{\theta}_0)$ a.s..

Let $\epsilon > 0$ be arbitrary and suppose that $\mathbb{P}(\limsup_{n\to\infty}|\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0| \geq \epsilon) > 0$. Let $K' = K \cap \{\boldsymbol{\theta} : |\boldsymbol{\theta} - \boldsymbol{\theta}_0| \geq \epsilon\}$. Since K' is compact, there is an event $D \subset \{\limsup_{n\to\infty}|\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0| \geq \epsilon,\ \limsup_{n\to\infty}\sup_{\boldsymbol{\theta}\in K'}\hat{L}_n(\boldsymbol{\theta})/n \leq \sup_{\boldsymbol{\theta}\in K'} L(\boldsymbol{\theta})\text{and}\ \limsup_{k\to\infty}\hat{L}_n(\boldsymbol{\theta}_0)/n = L(\boldsymbol{\theta}_0)\}$ with a positive probability, such that on D, there exists a convergent subsequence$(\hat{\boldsymbol{\theta}}_{n_k}) \subset K'$ with $\lim_{k\to\infty}\hat{\boldsymbol{\theta}}_{n_k} = \boldsymbol{\theta}$. Note that by the definition of the QMLE, $\limsup_{k\to\infty}\hat{L}_{n_k}(\boldsymbol{\theta}_0)/n_k \leq \limsup_{k\to\infty}\hat{L}_{n_k}(\hat{\boldsymbol{\theta}}_{n_k})/n_k =$

$\limsup_{k\to\infty} \sup_{\boldsymbol{\theta}\in K'} \hat{L}_{n_k}(\boldsymbol{\theta})/n_k$ on D. Further, $\limsup_{k\to\infty} \sup_{\boldsymbol{\theta}\in K'} \hat{L}_{n_k}(\boldsymbol{\theta})/n_k \leq \sup_{\boldsymbol{\theta}\in K'} L(\boldsymbol{\theta})$ and $\limsup_{k\to\infty} \hat{L}_{n_k}(\boldsymbol{\theta}_0)/n_k = L(\boldsymbol{\theta}_0)$ on D. Since any upper semicontinuous function L attains its maximum on compact sets and D is not an empty set, there exist at least one point $\boldsymbol{\theta} \in K'$ with $L(\boldsymbol{\theta}) \geq L(\boldsymbol{\theta}_0)$. This, however, contradicts the fact that L is uniquely maximized at $\boldsymbol{\theta}_0$. Subsequently, with probability 1, we get $|\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0| < \epsilon$ for all but finitely many n's. Since $\epsilon > 0$ is arbitrary, we conclude that $\hat{\boldsymbol{\theta}}_n \xrightarrow{a.s.} \boldsymbol{\theta}_0$ as $n \to \infty$. This completes the proof. □

Proof of theorem 2. An inspection of the proof of Theorem 1 shows that condition (N.1) also implies $\tilde{\boldsymbol{\theta}}_n \overset{a.s.}{\to} \boldsymbol{\theta}_0$, where $\tilde{\boldsymbol{\theta}} = \mathrm{argmax}_{\boldsymbol{\theta}\in K} L_n(\boldsymbol{\theta})$. Subsequently, for sufficiently large n, we can express

$$L_n'(\tilde{\boldsymbol{\theta}}_n) = L_n'(\boldsymbol{\theta}_0) + L_n''(\boldsymbol{\zeta}_n)(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0), \tag{1}$$

where $|\boldsymbol{\zeta}_n - \boldsymbol{\theta}_0| < |\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0|$. Since $\tilde{\boldsymbol{\theta}}_n$ is the maximizer of L_n and $\boldsymbol{\theta}_0$ lies in the interior of K, one has $L_n'(\tilde{\boldsymbol{\theta}}_n) = 0$, and thus, (1.1) is rewritten as

$$n^{-1} L_n''(\boldsymbol{\zeta}_n)(\tilde{\boldsymbol{\theta}_n} - \boldsymbol{\theta}_0) = -n^{-1} L_n'(\boldsymbol{\theta}_0). \tag{2}$$

Due to the fact that $\mathbb{E}||l_0''||_K < \infty$ and the stationarity and ergodicity of $\{l_i''\}$, we can apply Theorem 2.7 of Straumann and Mikosch to obtain $L_n''/n \overset{a.s.}{\to} L''$ in $\mathbb{C}(K, \mathbf{R}^{d\times d})$ as $n \to \infty$, where $L''(\boldsymbol{\theta}) = \mathbb{E}[l_0''(\boldsymbol{\theta})]$, $\boldsymbol{\theta} \in K$. This uniform convergence result and the fact that $\boldsymbol{\zeta}_n \overset{a.s.}{\to} \boldsymbol{\theta}_0$ imply

$$L_n''(\boldsymbol{\zeta}_n)/n \overset{a.s.}{\to} \mathbb{E}[l_0''(\boldsymbol{\theta}_0)] = \mathbf{B}_0, \qquad n \to \infty.$$

Owing to Propositions 3.12, 6.1 and 6.2 of Straumann and Mikosch (2006), it can be seen that ψ_0, ψ_0' and ψ_0'' are $\mathcal{F}_{-1}$-measurable. Also, since $\psi_0(\boldsymbol{\theta}_0) = \psi_{0,0}$ a.s., $x_0 = \psi_{0,0}\epsilon_0$ and ϵ_0 is independent of $\mathcal{F}_{-1}$, it holds that

$$\mathbf{B}_0 = -\mathbb{E}[(\psi_0'(\boldsymbol{\theta}_0))^T \psi_0'(\boldsymbol{\theta}_0)/\psi_{0,0}^2], \tag{3}$$

which is invertible due to Lemma 1 below. Thus, the matrix $L_n''(\boldsymbol{\zeta}_n)/n$ has an inverse of the form: $\mathbf{B}_0^{-1}(1 + o_P(1))$, $n \to \infty$, and (2) can be reexpressed as

$$\sqrt{n}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) = -\mathbf{B}_0^{-1}(1 + o_P(1)) L_n'(\boldsymbol{\theta}_0)/\sqrt{n}, \qquad n \to \infty.$$

Since $\psi_i(\boldsymbol{\theta}_0) = \psi_{i,0}$ a.s. and $x_i = \psi_{i,0}\epsilon_i$, we can write

$$L_n'(\boldsymbol{\theta}_0) = \sum_{i=1}^{n} l_i'(\boldsymbol{\theta}_0) = \sum_{i=1}^{n} \frac{\psi_i'(\boldsymbol{\theta}_0)}{\psi_{i,0}}(\epsilon_i - 1).$$

Further, since the random element $\psi_i'/\psi_{i,0}$ is $\mathcal{F}_{i-1}$-measurable, $\mathcal{F}_{i-1}$ is independent of ϵ_i, and $\mathbb{E}\epsilon_i = 1$, the sequence $\{l_i'(\boldsymbol{\theta}_0)\}$ forms a stationary and ergodic zero mean martingale difference sequence with respect to the filtration $\{\mathcal{F}_i\}$: owing to (N.3), the sequence $l_i'(\boldsymbol{\theta}_0)$ is square integrable. Then, applying a central limit theorem for square integrable stationary and ergodic martingale difference sequences (cf. Theorem 18.3 of Billingsley (1999)), we get $n^{-1/2}L_n'(\boldsymbol{\theta}_0) \stackrel{d}{\to} N(\mathbf{0}, \mathbb{E}[(l_0'(\boldsymbol{\theta}_0))^T l_0'(\boldsymbol{\theta}_0)])$, $n \to \infty$. Hence, due to (3), we have

$$\sqrt{n}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \stackrel{d}{\to} N(\mathbf{0}, \mathbf{V}_0), \qquad n \to \infty.$$

Now that $\sqrt{n}|\hat{\boldsymbol{\theta}}_n - \tilde{\boldsymbol{\theta}}_n| \stackrel{a.s.}{\to} 0$ owing to Lemma 3 below, an application of Slutsky's lemma finalizes the proof. □

Lemma 1. *If (N.1)-(N.4) hold, then* $\mathbf{B}_0 = \mathbb{E}[l_0''(\boldsymbol{\theta}_0)]$ *is negative definite.*

Proof. Note that $\mathbf{B}_0$ is negative definite if and only if $\mathbf{C}_0 = \mathbb{E}[(\psi_0'(\boldsymbol{\theta}_0))^T \psi_0'(\boldsymbol{\theta}_0)/\psi_{0,0}^2]$ is positive definite. Assume that $\mathbf{x}_0^T \mathbf{C}_0 \mathbf{x}_0 = 0$ for some $\mathbf{x}_0 \in \mathrm{R}^d$. In this case, we get

$$\mathbb{E}\left|\frac{\psi_0'(\boldsymbol{\theta}_0)\mathbf{x}_0}{\psi_{0,0}}\right|^2 = 0,$$

which in turn implies $\psi_0'(\boldsymbol{\theta}_0)\mathbf{x}_0 = 0$ a.s. and $\Psi_0'(\boldsymbol{\theta}_0)\mathbf{x}_0 = 0$ a.s. owing to the stationary of $\{\Psi_i'\}$. Now, observe that

$$\psi_1'(\boldsymbol{\theta}_0) = \frac{\partial g_{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}}(X_0, \Psi_{0,0})\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} + \frac{\partial g_{\boldsymbol{\theta}}}{\partial \Psi}(X_0, \Psi_{0,0})\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \Psi_0'(\boldsymbol{\theta}_0).$$

From this, since $\psi_1'(\boldsymbol{\theta}_0)\mathbf{x}_0 = 0$, it follows that $(\partial g_{\boldsymbol{\theta}}(X_0, \Psi_{0,0})/\partial \boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\mathbf{x}_0 = 0$ a.s.. This together with (N.4) implies $\mathbf{x}_0 = 0$. This validates the lemma. □

Lemma 2. *If (N.1) and (N.2) hold, we have*

$$n^{-1/2}||\hat{L}_n' - L_n'||_K \stackrel{a.s.}{\to} 0, \qquad n \to \infty.$$

Proof. Note that (C.3) implies $\hat{\psi}_i(\boldsymbol{\theta}), \psi_i(\boldsymbol{\theta}) \geq \underline{g} > 0$ for all $\boldsymbol{\theta} \in K$. By applying the mean value theorem to the function $f(a, b) = ab^{-1}(1 - x_i/b)$, $a \in \mathrm{R}$, $b \geq \underline{g}$, we can express

$$\begin{aligned} ||\hat{l}_i' - l_i'||_K &= \left\| \frac{\hat{\psi}_i'}{\hat{\psi}_i}\left(1 - \frac{x_i}{\hat{\psi}_i}\right) - \frac{\psi_i'}{\psi_i}\left(1 - \frac{x_i}{\psi_i}\right) \right\|_K \\ &\leq C(1 - x_i)\{||\hat{\psi}_i' - \psi_i'||_K + ||\hat{\psi}_i - \psi_i||_K ||\psi_i'||_K + ||\hat{\psi}_i - \psi_i||_K ||\hat{\psi}_i' - \psi_i'||_K\} \end{aligned} \tag{4}$$

for some $C > 0$. Then, using (4) and Lemmas 2.1 and 2.2 of Straumann and Mikosch (2006), we can see that $||\hat{L}_n' - L_n'||_K \leq \sum_{i=1}^{\infty} ||\hat{l}_i' - l_i'||_K < \infty$ a.s. This establishes the lemma. □

Lemma 3. *If (N.1)-(N.4) hold, we have*

$$\sqrt{n}|\tilde{\boldsymbol{\theta}}_n - \hat{\boldsymbol{\theta}}_n| \stackrel{a.s.}{\to} 0, \qquad n \to \infty.$$

Proof. Using the mean value theorem, we can express

$$L_n'(\tilde{\boldsymbol{\theta}}_n) - L_n'(\hat{\boldsymbol{\theta}}_n) = L_n''(\tilde{\boldsymbol{\zeta}}_n)(\tilde{\boldsymbol{\theta}}_n - \hat{\boldsymbol{\theta}}_n), \tag{5}$$

where $\tilde{\boldsymbol{\zeta}}_n$ lies on the line segment connecting $\hat{\boldsymbol{\theta}}_n$ and $\tilde{\boldsymbol{\theta}}_n$: this line segment is completely contained in the interior of K when n is large enough. Since $L_n'(\tilde{\boldsymbol{\theta}}_n) = \hat{L}_n'(\hat{\boldsymbol{\theta}}_n) = 0$, (5) is equivalent to

$$n^{-1/2}(\hat{L}_n'(\hat{\boldsymbol{\theta}}_n) - L_n'(\hat{\boldsymbol{\theta}}_n)) = n^{-1}L_n''(\tilde{\boldsymbol{\zeta}}_n)\{n^{1/2}(\tilde{\boldsymbol{\theta}}_n - \hat{\boldsymbol{\theta}}_n)\}. \tag{6}$$

Owing to Lemma 2, we can easily see that both the RHS and LHS sides of (6) should tend to 0 a.s. as $n \to \infty$. Further, using the fact that $\mathbb{E}||l_0''||_K < \infty$ and $\tilde{\boldsymbol{\zeta}} \stackrel{a.s.}{\to} \boldsymbol{\theta}_0$ and applying Theorem 2.7 of Straumman and Mikosch (2006) to L_n''/n, $L_n''(\tilde{\boldsymbol{\zeta}}_n)/n \stackrel{a.s.}{\to} \mathbf{B}_0$, which in turn implies $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \tilde{\boldsymbol{\theta}}_n) \stackrel{a.s.}{\to} 0$. This validates the lemma. □

Proof of Proposition 1. To verify consistency, we first check that (C.1), (C.3) and (C.4) hold. Note that (C.3) holds due to (L.4). To show (C.1), we consider the SRE:

$$\log\psi_{i+1} = \phi_i'(\log\psi_i),$$

where $\phi_i'(x) = \omega_0 + \alpha_0\log\epsilon_i + (\alpha_0+\beta_0)x$. Note that $\Lambda(\phi_0'^{(r)}) = |\alpha_0+\beta_0|^r < 1$ for all r owing to (L.2) and $\mathbb{E}[\log^+|\phi_0'(0)|] = \mathbb{E}(\log^+|\omega_0+\alpha_0\log\epsilon_0|) < \infty$ owing to Lemma 2.2 of Straumann and Mikosch (2006) and the fact that $\mathbb{E}|\log\epsilon_0|^\nu < \infty$. Thus, (C.1) holds by virtue of Theorem 2.8 of Straumann and Mikosch (2006). Next, we verify (C.4). For this, we only have to show that $\log\psi_i(\boldsymbol{\theta}) = \log\psi_{i,0}$ implies $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. Suppose that $\log\psi_i(\boldsymbol{\theta}) = \log\psi_{i,0}$ a.s.. Then, by the stationarity,

$$(\omega-\omega_0) + (\alpha-\alpha_0)\log\epsilon_{i-1} + (\alpha-\alpha_0+\beta-\beta_0)\log\psi_{i-1,0} = 0 \ a.s..$$

If $\alpha-\alpha_0+\beta-\beta_0 \neq 0$, $\log\psi_{i-1,0}$ is a measurable function of ϵ_{i-1} but at the same time must be independent of ϵ_{i-1}. This implies that $\log\psi_{i-1,0}$ is deterministic. However, taking the variance of $\log\psi_{i-1,0}$ gives $Var(\log\psi_{i-1,0}) = \sum_{k=0}^\infty(\alpha_0+\beta_0)^{2k}Var(\alpha_0\log\epsilon_0) > 0$, owing to (L.2) and (L.3). Thus, we should have $\alpha+\beta = \alpha_0+\beta_0$, which indicates that $\omega = \omega_0$ and $\alpha = \alpha_0$ owing to (L.3), so that (C.4) holds.

To establish the proposition, we need to verify that (C.2) holds. However, in the log-ACD case, (C.2) does not hold, and we directly verify that

$$x_i||\hat{\psi}_i^{-1} - \psi_i^{-1}||_K \stackrel{e.a.s.}{\longrightarrow} 0, \ i \to \infty,$$

since this will complete the proof as seen in Theorem 1. To this end, we introduce the SRE:

$$\log\psi_{i+1} = \phi_i(\log\psi_i), \tag{7}$$

where

$$[\phi_i(a)](\boldsymbol{\theta}) = \omega + \alpha\log x_i + \beta a(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in K.$$

Since $\Lambda(\phi_0^{(r)}) = |\beta|^r < 1$ for all r and $\mathbb{E}\log^+ ||\phi_0(0)||_K = \mathbb{E}\log^+ \sup_{\omega,\alpha} |\omega + \alpha \log x_0| < \infty$ owing to (L.1) and (L.3), Theorem 2.8 of Straumann and Mikosch (2006) allows SRE (4.4) to admit an ergodic stationary solution

$$\log \psi_i = \sum_{k=0}^{\infty} \beta^k (\omega + \alpha \log x_{i-k}), \tag{8}$$

where $\log \psi_i$ is $\mathcal{F}_{i-1}$-measurable and $|| \log \hat{\psi}_i - \log \psi_i||_K \xrightarrow{e.a.s.} 0$ as $i \to \infty$. Then, due to the fact that $\mathbb{E}\log^+ x_0 < \infty$, which is true because $\mathbb{E}\log \psi_{0,0} < \infty$ and $\mathbb{E}\epsilon_0 = 1$, we obtain $x_i||\hat{\psi}_i^{-1} - \psi_i^{-1}||_K \xrightarrow{e.a.s.} 0$ as $i \to \infty$. This asserts the strong consistency.

Next, we prove the asymptotic normality. Obviously, (N.1) is satisfied. Instead of (N.2), we verify that $\log \psi_i$ is twice continuously differentiable on K, $(\log \psi_i)'$ and $(\log \psi_i)''$ are $\mathcal{F}_{i-1}$-measurable, and

$$||(\log \hat{\psi}_i)' - (\log \psi_i)'||_K \xrightarrow{e.a.s.} 0, \quad i \to \infty, \tag{9}$$

$$||(\frac{1}{\hat{\psi}_i})' - (\frac{1}{\psi})'||_K \xrightarrow{e.a.s.} 0. \tag{10}$$

Taking the first and second derivatives in $\boldsymbol{\theta}$ of both the sides of

$$\log \hat{\psi}_{i+1} = \omega + \alpha \log x_i + \beta \log \hat{\psi}_i,$$

we get

$$(\log \hat{\psi}_{i+1})' = \hat{\dot{\phi}}_i\big((\log \psi_i)'\big) = \beta(\log \hat{\psi}_i)' + (1, \log x_i, \log \hat{\psi}_i)^T,$$
$$(\log \hat{\psi}_{i+1})'' = \hat{\ddot{\phi}}_i\big((\log \psi_i)''\big) = \beta(\log \hat{\psi}_i)'' + (0,0,1)^T(\log \hat{\psi}_i)' + \big((\log \hat{\psi}_i)'\big)^T(0,0,1).$$

Further, replacing $\log \hat{\psi}_i$ with $\log \psi_i$ and $(\log \hat{\psi}_i)'$ with $(\log \psi_i)'$, we obtain the following SREs on $\mathbf{C}(K, \mathbf{R}^3)$ and $\mathbf{C}(K, \mathbf{R}^{3\times 3})$, respectively:

$$(\log d_{i+1})' = \dot{\phi}_i\big((\log d_i)'\big) = \beta(\log d_i)' + (1, \log x_i, \log \psi_i)^T,$$
$$(\log e_{i+1})'' = \ddot{\phi}_i\big((\log e_i)''\big) = \beta(\log e_i)'' + (0,0,1)^T(\log \psi_i)' + \big((\log \psi_i)'\big)^T(0,0,1).$$

First, note that $\Lambda(\hat{\dot{\phi}}_i - \dot{\phi}_i) \xrightarrow{e.a.s.} 0$ and $||\hat{\dot{\phi}}_i(0) - \dot{\phi}_i(0)||_K \xrightarrow{e.a.s.} 0$ since $|| \log \hat{\psi}_i - \log \psi_i||_K \xrightarrow{e.a.s.} 0$ as $i \to \infty$. Also, $\mathbb{E}|| \log \psi_{i-k}||_K^{\nu} < \infty$ owing to (8) and the fact that $|| \log \psi_i||_K \le |\log \psi_{i,0}| + || \log \psi_i - \log \psi_{i,0}||_K$, $\mathbb{E}|\log \psi_{i,0}|^{\nu} < \infty$ and $\mathbb{E}|\log \epsilon_i|^{\nu} < \infty$. Further, we have $\mathbb{E}[\log^+ ||\dot{\phi}_0(0)||_K] = \mathbb{E}[\log^+ ||(1, \log x_0, \log \psi_0)^T||_K] < \infty$ since $\mathbb{E}|| \log \psi_0||_K^{\nu} < \infty$ and $\mathbb{E}|\log x_0|^{\nu} < \infty$. Hence, using Theorem 2.10 and the identical argument used in the proof of Proposition 6.1 of Straumann and Mikosch (2006) and the fact that $\mathbb{E}[\log \Lambda(\dot{\phi}_0)] \le \log \sup_{\theta \in K} |\beta| < 0$, we can see that $\log \psi_i$ is $\mathcal{F}_{i-1}$-measurable and differentiable and (9) is satisfied. Manifestly, (10) holds because $(\frac{1}{\psi_i})' = -\frac{1}{\psi_i}(\log \psi_i)'$.

Next, note that since $(\log \psi_i)'$ is a linear combination of $\log x_{i-k}$ and $\log \psi_{i-k}$, $\mathbb{E}||(\log \psi_i)'||_K^{\nu} < \infty$, which implies $\mathbb{E}[\log^+ ||\ddot{\phi}_i(0)||_K]=\mathbb{E}\log^+ ||(0,0,1)^T(\log \psi_i)' + \big((\log \psi_i)'\big)^T(0,0,1)||_K < \infty$. Further, note that $\Lambda(\hat{\ddot{\phi}}_i - \ddot{\phi}_i) \xrightarrow{e.a.s.} 0$ and $||\hat{\ddot{\phi}}_i(0) - \ddot{\phi}_i(0)||_K \xrightarrow{e.a.s.} 0$ owing to (9). In addition, $\mathbb{E}[\log \Lambda(\ddot{\phi}_0)] \le$

$\log\sup_{\theta\in K}|\beta| < 0$. Then, combining all these facts and using the identical argument used in the proof of Proposition 6.2 of Straumann and Mikosch (2006), we can conclude that $(\log\psi_i)'$ is $\mathcal{F}_{i-1}$-measurable and differentiable.

Concerning (N.3), note that $\mathbb{E}||(\log\psi_i)'||_K^\nu < \infty$ implies $\mathbb{E}||(\log\psi_i)''||_K^\nu < \infty$. Since ϵ_0 is independent of $\{\log\psi_0, (\log\psi_0)', (\log\psi_0)''\}$, $\mathbb{E}\epsilon_0^\nu < \infty$, and $\mathbb{E}|\psi_{0,0}|^\nu < \infty$, we obtain $\mathbb{E}||x_0\big(\frac{1}{\psi_0}\big)'||_K^{\nu/2} < \infty$ and $\mathbb{E}||x_0\big(\frac{1}{\psi_0}\big)''||_K^{\nu/2} < \infty$. Henceforth, $\mathbb{E}||l_0'||_K < \infty$, $\mathbb{E}||l_0''||_K < \infty$ and $\mathbb{E}|(\log\psi_0)'(\boldsymbol{\theta}_0)|^2 < \infty$, which asserts (N.3).

Finally, note that $(\log\psi_0)'(\boldsymbol{\theta}_0)\mathbf{x} = 0$ a.s. implies $(1, \log x_0, \log\psi_{0,0})\mathbf{x}$ a.s. for every $\mathbf{x}\in\mathbf{R}^3$, which holds if and only if $\mathbf{x} = 0$.

Combining this and all the results obtained thus far (that is, (N.1), (N.3), (9), (10), and the $\mathcal{F}_{i-1}$-adaptivity and twice differentiability of $\log\psi_i$), one can establish the proposition following the same lines as in Theorems 1 and 2. This completes the proof. $\square$

Proof of Proposition 2. Note that $\mathbb{E}||\psi_0'/\psi_0||_K^2 = \mathbb{E}||(\log\psi_0)'||_K^2 < \infty$. The proposition is then validated if $\mathbb{E}(\log^+||\psi_0''||_K) < \infty$, which, however, is difficult to show in our case. Thus, we follow another approach. Notice that $\mathbb{E}(\log^+ x_0) < \infty$. Further, it can be shown similarly to (9) that

$$||(\log\hat{\psi}_i)'' - (\log\psi_i)''||_K \overset{e.a.s.}{\longrightarrow} 0, \quad i\to\infty,$$

so that $||(\frac{1}{\hat{\psi}_i})'' - (\frac{1}{\psi_i})''||_K \overset{e.a.s.}{\longrightarrow} 0$ owing to the fact that $(\frac{1}{\psi})'' = (\frac{1}{\psi_i})^2(\log\psi_i)'((\log\psi_i)')^T - \frac{1}{\psi_i}(\log\psi_i)''$. Since these assert essentially the same result as stated in Lemma 2 in the Appendix, the proposition is validated. $\square$

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