Supplement to "Testing the constancy of Spearman's rho in multivariate time series"

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A Proof of Proposition 1

Let us first introduce some additional notation. For integers $1 \leq k \leq l \leq n$, let $H_{k:l}$ denote the empirical c.d.f. of the unobservable sample $\boldsymbol{U}_k, \dots, \boldsymbol{U}_l$ and let $H_{k:l,1}, \dots, H_{k:l,d}$

denote its margins. The corresponding empirical quantile functions are

$$H_{k:l,j}^{-1}(u) = \inf\{v \in [0,1] : H_{k:l,j}(v) \geq u\}, \qquad u \in [0,1], j \in D.$$

Finally, for any $\boldsymbol{u} \in [0,1]^d$, let

$$\boldsymbol{h}_{k:l}(\boldsymbol{u}) = \big(H_{k:l,1}(u_1), \dots, H_{k:l,d}(u_d)\big) \tag{A.1}$$

and

$$\boldsymbol{h}_{k:l}^{-1}(\boldsymbol{u}) = \big(H_{k:l,1}^{-1}(u_1), \dots, H_{k:l,d}^{-1}(u_d)\big). \tag{A.2}$$

By convention, all the quantities defined above are taken equal to zero if $k > l$.

Proof of Proposition 1. Fix $A \subseteq D$, $|A| \geq 1$, and $(s,t) \in \Delta$ such that $\lfloor ns \rfloor < \lfloor nt \rfloor$. On one hand, from (11) and by linearity of ϕ_A defined in (7), we have

$$\mathbb{S}_{n,A}(s,t) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \prod_{j \in A} \{1 - H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij})\} - \sqrt{n}\lambda_n(s,t)\phi_A(C),$$

where we have used the fact that $\hat{U}_{ij}^{\lfloor ns \rfloor+1:\lfloor nt \rfloor} = H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij})$ for all $j \in D$ and all $i \in \{\lfloor ns \rfloor + 1, \dots, \lfloor nt \rfloor\}$. On the other hand,

$$\begin{aligned} \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\} = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \prod_{j \in A} (1 - U_{ij}) - \sqrt{n}\lambda_n(s,t)\phi_A(C) \\ - \int_{[0,1]^d} \sum_{j \in A} \prod_{l \in A \setminus \{j\}} (1 - v_l) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}). \end{aligned}$$

Next, let $\pi(\boldsymbol{u}) = \prod_{j \in A}(1 - u_j)$, $\boldsymbol{u} \in \mathbb{R}^d$. Then, fix $\boldsymbol{u} \in [0,1]^d$, and, for any $x \in [0,1]$, let $\boldsymbol{w}_{\boldsymbol{u}}(x) = \boldsymbol{u} + x\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}$ and let $g(x) = \pi\{\boldsymbol{w}_{\boldsymbol{u}}(x)\}$, where $\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}$ is defined in (A.1). The function g is clearly continuously differentiable on $[0,1]$. By the mean value theorem, there exists $x^*_{\boldsymbol{u},n,s,t} \in (0,1)$ such that $g(1) - g(0) = g'(x^*_{\boldsymbol{u},n,s,t})$, that is, such that

$$\pi\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\} - \pi(\boldsymbol{u}) = \sum_{j \in A} \dot{\pi}_j[\boldsymbol{u} + x^*_{\boldsymbol{u},n,s,t}\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}]\{H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(u_j) - u_j\}.$$

It follows that

$$\begin{aligned} &\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j \in A} \dot{\pi}_j[\boldsymbol{U}_i + x^*_{\boldsymbol{U}_i,n,s,t}\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{U}_i) - \boldsymbol{U}_i\}]\{H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij}) - U_{ij}\} \\ &\qquad - \int_{[0,1]^d} \sum_{j \in A} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}). \end{aligned}$$

Notice that, by the triangle inequality and the fact that $\sup_{\boldsymbol{u}\in[0,1]^d} |\dot{\pi}_j(\boldsymbol{u})| \leq 1$, $j \in D$,

$$\sup_{(s,t)\in\Delta} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| \leq 2|A| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})|.$$

Next, fix $\varepsilon, \eta > 0$. Using the previous inequality and the fact that $\mathbb{B}_n$ vanishes when $s = t$ and is asymptotically uniformly equicontinuous in probability as a consequence of Lemma 2 in Bücher (2014), there exists $\delta \in (0,1)$ such that, for all sufficiently large n,

$$\mathrm{P}\left(\sup_{\substack{(s,t)\in\Delta \\ t-s<\delta}} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| > \varepsilon \right) \\ \leq \mathrm{P}\left(2|A| \sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s<\delta}} |\mathbb{B}_n(s,t,\boldsymbol{u})| > \varepsilon \right) < \eta/2.$$

To show (16), it remains therefore to prove that, for all sufficiently large n,

$$\mathrm{P}\left(\sup_{\substack{(s,t)\in\Delta \\ t-s\geq\delta}} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| > \varepsilon \right) < \eta/2.$$

To show the above, we shall now prove that $\sup_{(s,t)\in\Delta^\delta} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}|$ converges in probability to zero, where $\Delta^\delta = \{(s,t) \in \Delta : t - s \geq \delta\}$. The latter supremum is smaller than $\sum_{j\in A}(I_{n,j} + II_{n,j})$, where

$$I_{n,j} \leq \sup_{(s,t)\in\Delta^\delta} \Bigg| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\dot{\pi}_j[\boldsymbol{U}_i + x^*_{\boldsymbol{U}_i,n,s,t}\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{U}_i) - \boldsymbol{U}_i\}] - \dot{\pi}_j(\boldsymbol{U}_i)\right) \\ \times \{H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(U_{ij}) - U_{ij}\} \Bigg|$$

and

$$II_{n,j} \leq \sup_{(s,t)\in\Delta^\delta} \left| \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v})\mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v})\mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}C(\boldsymbol{v}) \right|.$$

Next, notice that

$$\sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d} |H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})| \\ \leq \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \times n^{-1/2} \times \sup_{(s,t)\in\Delta^\delta} \{\lambda_n(s,t)\}^{-1} \xrightarrow{\mathrm{P}} 0. \quad (A.3)$$

Fix $j \in A$. Since the function $\dot{\pi}_j$ is continuous on $[0,1]^d$, by the continuous mapping theorem, $\sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d} |\dot{\pi}_j[\boldsymbol{u} + x^*_{\boldsymbol{u},n,s,t}\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}] - \dot{\pi}_j(\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0$. Hence,

$$I_{n,j} \leq \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \\ \times \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\in[0,1]^d} |\dot{\pi}_j[\boldsymbol{u} + x^*_{\boldsymbol{u},n,s,t}\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}] - \dot{\pi}_j(\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0.$$

It thus remains to show that $II_{n,j} \xrightarrow{\mathrm{P}} 0$. The latter is mostly a consequence of Lemma 1 below. First, notice that (A.3) implies that $H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor} \xrightarrow{\mathrm{P}} C$ in $\ell^\infty(\Delta^\delta \times [0,1]^d; \mathbb{R})$. Hence, $(\mathbb{B}_n, H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}) \rightsquigarrow (\mathbb{B}_C, C)$ in $\ell^\infty(\Delta^\delta \times [0,1]^d; \mathbb{R})$. Next, combining the previous weak convergence with Lemma 3 in Holmes et al. (2013) and the continuous mapping theorem, we obtain that the finite-dimensional distributions of $(\mathbb{A}_{n,j}, \mathbb{B}_n)$ converge weakly to those of $(\mathbb{A}_{C,j}, \mathbb{B}_C)$, where $\mathbb{A}_{n,j}$ and $\mathbb{A}_{C,j}$ are defined in Lemma 1. The fact that $(\mathbb{A}_{n,j}, \mathbb{B}_n) \rightsquigarrow (\mathbb{A}_{C,j}, \mathbb{B}_C)$ in $\{\ell^\infty(\Delta^\delta \times [0,1]^d; \mathbb{R})\}^2$ then follows from Lemma 1 below and the fact that marginal asymptotic tightness implies joint asymptotic tightness. The latter weak convergence combined with the continuous mapping theorem finally implies that $II_{n,j} \xrightarrow{\mathrm{P}} 0$, which completes the proof. ■

Lemma 1. *For any $j \in D$ and $\delta \in (0,1)$, $\mathbb{A}_{n,j} \rightsquigarrow \mathbb{A}_{C,j}$ in $\ell^\infty(\Delta^\delta; \mathbb{R})$, where*

$$\mathbb{A}_{n,j}(s,t) = \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{v}), \tag{A.4}$$

$$\mathbb{A}_{C,j}(s,t) = \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_C(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}).$$

Proof. Fix $j \in D$ and $\delta \in (0,1)$. To prove the desired result, we shall show that conditions (i) and (ii) of Theorem 2.1 in Kosorok (2008) hold. First, recall that from (A.3), $H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor} \xrightarrow{\mathrm{P}} C$ in $\ell^\infty(\Delta^\delta \times [0,1]^d; \mathbb{R})$. Then, from the fact that $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta \times [0,1]^d; \mathbb{R})$, we obtain that, for any $(s_1,t_1), \dots, (s_k,t_k) \in \Delta^\delta$,

$$\big(\mathbb{B}_n(s_1,t_1,\cdot), H_{\lfloor ns_1 \rfloor+1:\lfloor nt_1 \rfloor}, \dots, \mathbb{B}_n(s_k,t_k,\cdot), H_{\lfloor ns_k \rfloor+1:\lfloor nt_k \rfloor}\big) \rightsquigarrow \big(\mathbb{B}_C(s_1,t_1,\cdot), C, \dots, \mathbb{B}_C(s_k,t_k,\cdot), C\big)$$

in $\{\ell^\infty([0,1]^d; \mathbb{R})\}^{2k}$. From Lemma 3 in Holmes et al. (2013) and the continuous mapping theorem, this implies that $\big(\mathbb{A}_{n,j}(s_1,t_1), \dots, \mathbb{A}_{n,j}(s_k,t_k)\big) \rightsquigarrow \big(\mathbb{A}_{C,j}(s_1,t_1), \dots, \mathbb{A}_{C,j}(s_k,t_k)\big)$ in $\mathbb{R}^k$. Hence, we have convergence of the finite-dimensional distributions, that is, condition (i) of Theorem 2.1 in Kosorok (2008) holds.

It remains to prove condition (ii) of Theorem 2.1 in Kosorok (2008). Specifically, we shall now show that $\mathbb{A}_{n,j}$ is $\|\cdot\|_1$-asymptotically uniformly equicontinuous in probability, which will complete the proof since Δ^δ is totally bounded by $\|\cdot\|_1$. By Problem 2.1.5 in van der Vaart and Wellner (2000), we need to show that, for any positive sequence $a_n \downarrow 0$,

$$\sup_{\substack{(s,t),(s',t') \in \Delta^\delta \\ |s-s'|+|t-t'| \le a_n}} |\mathbb{A}_{n,j}(s,t) - \mathbb{A}_{n,j}(s',t')| \xrightarrow{\mathrm{P}} 0. \tag{A.5}$$

We bound the supremum on the left of the previous display by $I_n + II_n$, where

$$I_n = \sup_{\substack{(s,t),(s',t') \in \Delta^\delta \\ |s-s'|+|t-t'| \le a_n}} \bigg| \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s',t',\boldsymbol{v}^{\{j\}}) \mathrm{d}H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{v}) \bigg|$$

and

$$II_n = \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \int_{[0,1]^d} \dot\pi_j(\boldsymbol{v})\mathbb{B}_n(s',t',\boldsymbol{v}^{\{j\}}) \mathrm{d}H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) \right.$$
$$\left. - \int_{[0,1]^d} \dot\pi_j(\boldsymbol{v})\mathbb{B}_n(s',t',\boldsymbol{v}^{\{j\}}) \mathrm{d}H_{\lfloor ns'\rfloor+1:\lfloor nt'\rfloor}(\boldsymbol{v}) \right|.$$

Now,

$$I_n \le \sup_{\boldsymbol{u}\in[0,1]^d} |\dot\pi_j(\boldsymbol{u})| \times \sup_{\substack{(s,t),(s',t')\in\Delta^\delta, \boldsymbol{u}\in[0,1]^d \\ |s-s'|+|t-t'|\le a_n}} |\mathbb{B}_n(s,t,\boldsymbol{u}) - \mathbb{B}_n(s',t',\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0,$$

since $\mathbb{B}_n$ is asymptotically uniformly equicontinuous in probability as a consequence of Lemma 2 in Bücher (2014). Furthermore, II_n is smaller than

$$\sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \left\{ \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \dot\pi_j(\boldsymbol{U}_i)\mathbb{B}_n(s',t',\boldsymbol{U}_i^{\{j\}}) - \sum_{i=\lfloor ns'\rfloor+1}^{\lfloor nt'\rfloor} \dot\pi_j(\boldsymbol{U}_i)\mathbb{B}_n(s',t',\boldsymbol{U}_i^{\{j\}}) \right\} \right|$$
$$+ \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \left(\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} - \frac{1}{\lfloor nt'\rfloor - \lfloor ns'\rfloor} \right) \sum_{i=\lfloor ns'\rfloor+1}^{\lfloor nt'\rfloor} \dot\pi_j(\boldsymbol{U}_i)\mathbb{B}_n(s',t',\boldsymbol{U}_i^{\{j\}}) \right|,$$

which is smaller than

$$2 \times \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \frac{|\lfloor nt\rfloor - \lfloor nt'\rfloor| + |\lfloor ns\rfloor - \lfloor ns'\rfloor|}{\lfloor nt\rfloor - \lfloor ns\rfloor}$$
$$\times \sup_{\boldsymbol{u}\in[0,1]^d} |\dot\pi_j(\boldsymbol{u})| \times \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0.$$

Hence, $II_n \xrightarrow{\mathrm{P}} 0$ and thus (A.5) holds, which completes the proof. ■

B Proof of Corollary 2

Proof. Starting from (12), using Proposition 1, the linearity of $\psi_{C,A}$ and (13), we obtain that, for any $A \subseteq D$, $|A| \ge 1$,

$$\sup_{s\in[0,1]} |\mathbb{T}_{n,A}(s) - \psi_{C,A}\{\mathbb{B}_n(0,s,\cdot) - \lambda(0,s)\mathbb{B}_n(0,1,\cdot)\}| = o_{\mathrm{P}}(1).$$

Hence, $\mathbb{T}_n$ has the same weak limit as $s \mapsto \psi_C\{\mathbb{B}_n(0,s,\cdot) - \lambda(0,s)\mathbb{B}_n(0,1,\cdot)\}$ and (18) follows from the continuous mapping theorem.

The second to last claim is a consequence of the continuous mapping theorem. To prove the last claim, it suffices to show that the Gaussian process $\sigma_{C,f}^{-1} f\{\mathbb{T}_C(\cdot)\}$ has the same covariance function as $\mathbb{U}$. For any, $s,t \in [0,1]$, we have

$$\mathrm{cov}[\sigma_{C,f}^{-1} f\{\mathbb{T}_C(s)\}, \sigma_{C,f}^{-1} f\{\mathbb{T}_C(t)\}]$$
$$= \sigma_{C,f}^{-2}\, \mathrm{E}[f\circ\psi_C\{\mathbb{B}_C(0,s,\cdot) - s\mathbb{B}_C(0,1,\cdot)\} f\circ\psi_C\{\mathbb{B}_C(0,t,\cdot) - t\mathbb{B}_C(0,1,\cdot)\}]. \quad \text{(B.1)}$$

By linearity of $f \circ \psi_C$ and Fubini's theorem, the expectation in the last display is equal to

$$f \circ \psi_C \left\{ \boldsymbol{u} \mapsto f \circ \psi_C \left(\boldsymbol{v} \mapsto \mathrm{E}[\{\mathbb{B}_C(0,s,\boldsymbol{u}) - s\mathbb{B}_C(0,1,\boldsymbol{u})\}\{\mathbb{B}_C(0,t,\boldsymbol{v}) - t\mathbb{B}_C(0,1,\boldsymbol{v})\}] \right) \right\},$$

that is,

$$(s \wedge t - st) f \circ \psi_C \left[\boldsymbol{u} \mapsto f \circ \psi_C \left\{ \boldsymbol{v} \mapsto \kappa_C(\boldsymbol{u}, \boldsymbol{v}) \right\} \right] = (s \wedge t - st) \operatorname{var}[f \circ \psi_C\{\mathbb{B}_C(0,1,\cdot)\}],$$

where κ_C is defined in (14). Combining the previous display with (B.1), we obtain that $\operatorname{cov}[\sigma_{C,f}^{-1} f\{\mathbb{T}_C(s)\}, \sigma_{C,f}^{-1} f\{\mathbb{T}_C(t)\}] = (s \wedge t - st)$, which completes the proof. ∎

C Proofs of Propositions 4 and 5

Proof of Proposition 4. We only show the first claim as the subsequent claims then mostly follow from the continuous mapping theorem. Also, we only provide the proof under (ii) in the statement of Proposition 3, the proof being simpler under (i). Fix $A \subseteq D$, $|A| \geq 1$. For any $(s,t) \in \Delta$, let $\mathbb{S}_{n,A}^{(m)}(s,t) = \psi_{C,A}\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}$. Using the linearity of the map $\psi_{C,A}$ defined in (17), Proposition 3 and the continuous mapping theorem, we obtain that

$$\left(\mathbb{S}_{n,A}, \mathbb{S}_{n,A}^{(1)}, \dots, \mathbb{S}_{n,A}^{(M)}\right) \rightsquigarrow \left(\mathbb{S}_{C,A}, \mathbb{S}_{C,A}^{(1)}, \dots, \mathbb{S}_{C,A}^{(M)}\right)$$

in $\{\ell^\infty(\Delta; \mathbb{R})\}^{M+1}$. The first claim is thus proved if we show that, for any $m \in \{1, \dots, M\}$, $\sup_{(s,t)\in\Delta} |\check{\mathbb{S}}_{n,A}^{(m)}(s,t) - \mathbb{S}_{n,A}^{(m)}(s,t)|$ is $o_\mathrm{P}(1)$. Fix $m \in \{1, \dots, M\}$ and notice that the latter supremum is smaller than $2|A| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})|$. We can therefore proceed analogously to the proof of Proposition 1. Fix $\varepsilon, \eta > 0$. Using the previous inequality as well as the fact that $\check{\mathbb{B}}_n^{(m)}$ is zero when $s = t$ and is asymptotically uniformly equicontinuous in probability as a consequence of Lemma A.3 in Bücher and Kojadinovic (2014), there exists $\delta \in (0,1)$ such that, for all sufficiently large n,

$$\mathrm{P}\left(\sup_{\substack{(s,t)\in\Delta \\ t-s<\delta}} |\check{\mathbb{S}}_{n,A}^{(m)}(s,t) - \mathbb{S}_{n,A}^{(m)}(s,t)| > \varepsilon \right) < \eta/2.$$

It remains therefore to prove that $\sup_{(s,t)\in\Delta^\delta} |\check{\mathbb{S}}_{n,A}^{(m)}(s,t) - \mathbb{S}_{n,A}^{(m)}(s,t)| \xrightarrow{\mathrm{P}} 0$, where $\Delta^\delta = \{(s,t) \in \Delta : t - s \geq \delta\}$. The latter supremum is smaller than

$$\sum_{j\in A} \sup_{(s,t)\in\Delta^\delta} \left| \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}) \right|,$$

where $\dot{\pi}_j$ is the jth first order partial derivative of the function $\pi(\boldsymbol{u}) = \prod_{j\in A}(1-u_j)$, $\boldsymbol{u} \in \mathbb{R}^d$, introduced in the proof of Proposition 1. Fix $j \in A$. The jth summand in the previous display is smaller than $I_n + II_n$, where

$$I_n = \sup_{(s,t)\in\Delta^\delta} \left| \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{v}) - \check{\mathbb{A}}_{n,j}^{(m)}(s,t) \right|,$$

$$II_n = \sup_{(s,t)\in\Delta^\delta} \left| \check{\mathbb{A}}_{n,j}^{(m)}(s,t) - \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}) \right|,$$

and $\check{\mathbb{A}}^{(m)}_{n,j}$ is defined analogously to the process $\mathbb{A}_{n,j}$ in (A.4) with $\mathbb{B}_n$ replaced by $\check{\mathbb{B}}^{(m)}_n$. In addition, it can be verified that Lemma 1 remains true if $\mathbb{B}_n$ and $\mathbb{B}_C$ are replaced by $\check{\mathbb{B}}^{(m)}_n$ and $\mathbb{B}^{(m)}_C$, respectively, in its statement. It follows that we can proceed as at the end of proof of Proposition 1 to show that II_n above converges to zero in probability.

To show that $I_n \xrightarrow{\mathrm{P}} 0$, we use the fact that $I_n \leq I'_n + I''_n$, where

$$
\begin{aligned}
I'_n = \sup_{(s,t)\in\Delta^\delta} \Bigg| \frac{1}{\lfloor nt \rfloor - \lfloor ns \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} & \left[\dot{\pi}_j\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{U}_i)\} - \dot{\pi}_j(\boldsymbol{U}_i) \right] \\
& \times \check{\mathbb{B}}^{(m)}_n\{s,t,\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{U}_i)^{\{j\}}\} \Bigg|, \\
I''_n = \sup_{(s,t)\in\Delta^\delta} \Bigg| \frac{1}{\lfloor nt \rfloor - \lfloor ns \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} & \dot{\pi}_j(\boldsymbol{U}_i) \left[\check{\mathbb{B}}^{(m)}_n\{s,t,\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{U}_i)^{\{j\}}\} - \check{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{U}_i^{\{j\}}) \right] \Bigg|.
\end{aligned}
$$

For I'_n, we have that

$$
I'_n \leq \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\check{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u})\right| \times \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d} \left|\dot{\pi}_j\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\} - \dot{\pi}_j(\boldsymbol{u})\right| \xrightarrow{\mathrm{P}} 0
$$

as a consequence of the weak convergence of $\check{\mathbb{B}}^{(m)}_n$, (A.3), and the continuous mapping theorem. For I''_n, using the fact that $\sup_{\boldsymbol{u}\in[0,1]^d} |\dot{\pi}_j(\boldsymbol{u})| \leq 1$, we obtain that

$$
I''_n \leq \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d} \left|\check{\mathbb{B}}^{(m)}_n\{s,t,\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})^{\{j\}}\} - \check{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}^{\{j\}})\right| \xrightarrow{\mathrm{P}} 0.
$$

The latter convergence is a consequence of the asymptotic equicontinuity in probability of $\check{\mathbb{B}}^{(m)}_n$ and the fact that $\sup_{(s,t,u)\in\Delta^\delta\times[0,1]} |H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(u) - u| \xrightarrow{\mathrm{P}} 0$ (see e.g. the treatment of the term (B.9) in Bücher et al., 2014, for a detailed proof of a similar convergence). ■

Proof of Proposition 5. We only provide the proof under (ii) in the statement of Proposition 3, the proof being simpler under (i). From Proposition 4, to prove the desired result it suffices to show that, for any $A \subseteq D$, $|A| \geq 1$,

$$
\sup_{(s,t)\in\Delta} |\tilde{\mathbb{S}}^{(m)}_{n,b_n,A}(s,t) - \check{\mathbb{S}}^{(m)}_{n,A}(s,t)| \xrightarrow{\mathrm{P}} 0.
$$

Fix $A \subseteq D$, $|A| \geq 1$. From (24) and (25) and the triangle inequality, the latter will hold if, for any $j \in A$,

$$
\sup_{(s,t,u)\in\Delta\times[0,1]} |\tilde{\mathbb{B}}^{(m)}_{n,b_n,j}(s,t,u) - \check{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}_j)| \xrightarrow{\mathrm{P}} 0.
$$

The previous supremum can actually be restricted to $u \in (0,1)$ as both processes are zero if $u \in \{0,1\}$.

Let $K > 0$ be a constant and let us first suppose that, for any $n \geq 1$ and $i \in \{1,\dots,n\}$, $\xi^{(m)}_{i,n} \geq -K$. Also, fix $j \in A$. The supremum on the right of the previous display is then smaller than $I_n + II_n$, where

$$I_n = \sup_{(s,t,u)\in\Delta\times(0,1)} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} + K) \left| \mathcal{L}_{b_n}(\hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}, u) - \mathbf{1}(\hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le u) \right|,$$

$$II_n = \sup_{(s,t,u)\in\Delta\times(0,1)} \frac{K + \bar\xi_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left| \mathcal{L}_{b_n}(\hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}, u) - \mathbf{1}(\hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le u) \right|.$$

Next, some thought reveals that, for any $(u,v) \in [0,1]\times(0,1)$,

$$\begin{aligned} |\mathcal{L}_{b_n}(u,v) - \mathbf{1}(u \le v)| &\le \mathbf{1}(u_- \le v) - \mathbf{1}(u_+ \le v) \\ &= \mathbf{1}(u - b_n \le v) - \mathbf{1}(u + b_n \le v) \\ &= \mathbf{1}(u \le v_+) - \mathbf{1}(u \le v_-). \end{aligned} \tag{C.1}$$

Then, we write $I_n \le I_{n,1} + I_{n,2}$, where

$$I_{n,1} = \sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} - \bar\xi_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}) \mathbf{1}(u_- < \hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le u_+) \right|,$$

$$I_{n,2} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{K + \bar\xi_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}(u_- < \hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le u_+).$$

For $I_{n,1}$, we have

$$I_{n,1} \le \sup_{\substack{(s,t,\boldsymbol{u},\boldsymbol{v})\in\Delta\times[0,1]^{2d} \\ \|\boldsymbol{u}-\boldsymbol{v}\|_1 \le 2b_n}} \left| \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}) \right| \xrightarrow{\mathrm{P}} 0$$

from the asymptotic uniform equicontinuity in probability of $\check{\mathbb{B}}_n^{(m)}$. Before dealing with $I_{n,2}$, let us first show that

$$I_{n,3} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}(u_- < \hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le u_+) \xrightarrow{\mathrm{P}} 0. \tag{C.2}$$

From the proof of Proposition 3.3 of Bücher et al. (2014), we have that

$$\sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left[\mathbf{1}\{U_{ij} \le H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u)\} - \mathbf{1}(\hat U_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le u) \right] \right| \xrightarrow{\mathrm{P}} 0.$$

Consequently, to prove that $I_{n,3} \xrightarrow{\mathrm{P}} 0$, it suffices to show that

$$\sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left[\mathbf{1}\{U_{ij} \le H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u_+)\} - \mathbf{1}\{U_{ij} \le H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u_-)\} \right] \right| \xrightarrow{\mathrm{P}} 0.$$

The supremum on the left of the previous display is smaller than $J_{n,1} + J_{n,2} + J_{n,3}$, where

$$J_{n,1} = \sup_{(s,t,u)\in\Delta\times[0,1]} \left| \mathbb{B}_n\{s,t,1,H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_+),1\} - \mathbb{B}_n\{s,t,1,H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_-),1\} \right|,$$
$$J_{n,2} = \sup_{(s,t,u)\in\Delta\times[0,1]} \sqrt{n}\lambda_n(s,t) \left| H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_+) - u_+ - H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_-) + u_- \right|,$$
$$J_{n,3} = \sup_{(s,t,u)\in\Delta\times[0,1]} \sqrt{n}\lambda_n(s,t)\,|u_+ - u_-|,$$

with some abuse of notation for $J_{n,1}$. We immediately have $J_{n,3} \leq 2\sqrt{n}b_n \to 0$. The fact $J_{n,2} \xrightarrow{\mathrm{P}} 0$ follows from the asymptotic uniform equicontinuity in probability of the process $(s,t,u) \mapsto \sqrt{n}\lambda_n(s,t)\{H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u) - u\}$, itself following from its weak convergence to $(s,t,u) \mapsto -\mathbb{B}_C(s,t,\boldsymbol{u}_j)$ in $\ell^\infty(\Delta\times[0,1];\mathbb{R})$. The latter is a consequence of the weak convergence of $\mathbb{B}_n$ to $\mathbb{B}_C$ in $\ell^\infty(\Delta\times[0,1]^d;\mathbb{R})$, Lemma B.2 of Bücher and Kojadinovic (2014) and the extended continuous mapping theorem (van der Vaart and Wellner, 2000, Theorem 1.11.1). The fact that $J_{n,2} \xrightarrow{\mathrm{P}} 0$ implies that, for any $\delta \in (0,1)$,

$$\sup_{\substack{(s,t,u)\in\Delta\times[0,1]\\ t-s\geq\delta}} \left| H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_+) - H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_-) \right| \xrightarrow{\mathrm{P}} 0.$$

Combined with the asymptotic uniform equicontinuity in probability of $\mathbb{B}_n$, the latter can be used to prove that $J_{n,1} \xrightarrow{\mathrm{P}} 0$ (see Bücher et al., 2014, page 24, term (B.9), for a similar proof). Hence, $I_{n,3} \xrightarrow{\mathrm{P}} 0$.

Now, $I_{n,2} \leq K \times I_{n,3} + I_{n,4}$, where

$$I_{n,4} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{\bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}(u_- < \hat{U}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}_{ij} \leq u_+).$$

Hence, to show that $I_{n,2} \xrightarrow{\mathrm{P}} 0$, it remains to prove that $I_{n,4} \xrightarrow{\mathrm{P}} 0$. The latter can be shown by proceeding as for the term (B.8) in Bücher et al. (2014).

We therefore have that $I_n \xrightarrow{\mathrm{P}} 0$. The fact that $II_n \xrightarrow{\mathrm{P}} 0$, follows from the fact that $II_n \leq I_{n,2} \xrightarrow{\mathrm{P}} 0$. This completes the proof under the condition $\xi^{(m)}_{i,n} \geq -K$. To show that this condition is not necessary, we use the arguments employed at the end of the proof of Proposition 4.3 of Bücher et al. (2014). ■

D Proofs of Propositions 6 and 7

Lemma 2. *Assume that $\boldsymbol{U}_1,\dots,\boldsymbol{U}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 6$. Then, for any $A \subseteq D$, $|A| \geq 1$ and $j \in A$, $\mathbb{H}_{n,A,j} \rightsquigarrow \mathbb{H}_{A,j}$ in $\ell^\infty([0,1];\mathbb{R})$, where, for any $t\in[0,1]$, $\mathbb{H}_{n,A,j}(t) = n^{-1/2}\sum_{i=1}^n [Y_{i,A,j}(t) - \mathrm{E}\{Y_{1,A,j}(t)\}]$, $Y_{i,A,j}(t) = \prod_{l\in A\setminus\{j\}}(1-U_{il})\mathbf{1}(t \leq U_{ij})$, and $\mathbb{H}_{A,j}$ is a tight process.*

Proof. Fix $A \subseteq D$, $|A| \geq 1$ and $j \in A$. To simplify the notation, we write $\mathbb{H}_n$ instead of $\mathbb{H}_{n,A,j}$ and Y_i instead of $Y_{i,A,j}$ as we continue. To prove the desired result, we mostly adapt the arguments used in the proof of Proposition 2.11 of Dehling and Philipp (2002). From Theorem 2.1 in Kosorok (2008), two conditions are needed to obtain the desired weak convergence. The first condition (which is the weak convergence of the finite-dimensional distributions) is a consequence of Theorem 3.23 of Dehling and Philipp (2002) as $a > 6$ and $Y_i(t) \in [0, 1]$ for all $t \in [0, 1]$. To prove the second condition, we shall show that $\mathbb{H}_n$ is asymptotically $|\cdot|$-equicontinuous in probability. To do so, we shall first prove that, for any $\varepsilon, \delta > 0$, there exists a grid $0 = t_0 < t_1 < \cdots < t_k = 1$ such that, for all n sufficiently large,

$$\mathrm{P}\left\{\max_{1\leq i\leq k} \sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \delta. \tag{D.1}$$

We first note that there exists constants $c \geq 1$ and $\epsilon \in (0, 1)$ such that $\alpha_r \leq cr^{-6-\epsilon}$. Then, using the fact that, for $t, t' \in [0, 1]$,

$$\mathrm{E}[\{Y_1(t) - Y_1(t')\}^2] \leq \mathrm{E}[|Y_1(t) - Y_1(t')|] \leq \mathrm{E}\{\mathbf{1}(t \wedge t' \leq U_{ij} \leq t \vee t')\} = |t - t'|,$$

we apply Lemma 3.22 of Dehling and Philipp (2002) with $\xi_i = Y_i(t) - Y_i(t')$ to obtain that

$$\mathrm{E}[\{\mathbb{H}_n(t) - \mathbb{H}_n(t')\}^4] \leq 10^4 \frac{C}{\epsilon}\left(|t - t'|^\eta + n^{-1}|t - t'|^{\eta/2}\right) = \lambda\left(|t - t'|^\eta + n^{-1}|t - t'|^{\eta/2}\right),$$

where $\eta = 1 + \epsilon/10 > 1$ and $\lambda = 10^4 c/\epsilon$. It follows that, for any $t, t' \in [0, 1]$ such that $|t - t'| \geq n^{-2/\eta}$,

$$\mathrm{E}[\{\mathbb{H}_n(t) - \mathbb{H}_n(t')\}^4] \leq 2\lambda|t - t'|^\eta. \tag{D.2}$$

Next, consider a grid $0 = t_0 < t_1 < \cdots < t_k = 1$ to be specified later. Furthermore, it can be verified that the function $G : t \mapsto \mathrm{E}\{Y_1(t)\}$ is continuous and strictly decreasing on $[0, 1]$. Then, fix $i \in \{1, \ldots, k\}$, let $\tau = \varepsilon n^{-1/2}/4$, let $m = m_i = \lfloor\{G(t_{i-1}) - G(t_i)\}/\tau\rfloor$ and define a subgrid $t_{i-1} = s_0 < s_1 < \cdots < s_m = t_i$ such that $G(s_j) = G(s_0) - j\tau$ for $j \in \{1, \ldots, m - 1\}$. Notice that this ensures that, for any $j \in \{1, \ldots, m\}$, $\tau \leq G(s_{j-1}) - G(s_j) \leq 2\tau$. Now, fix $j \in \{1, \ldots, m\}$. Using the fact that the function $t \mapsto n^{-1}\sum_{i=1}^n Y_i(t)$ is also decreasing, it can be verified that, for any $t \in [s_{j-1}, s_j]$,

$$\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1}) \leq |\mathbb{H}_n(s_{j-1}) - \mathbb{H}_n(t_{i-1})| + \varepsilon/2$$

and

$$-\varepsilon/2 - |\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})| \leq \mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1}).$$

The above inequalities imply that, for any $t \in [t_{i-1}, t_i] = \bigcup_{j=1}^m [s_{j-1}, s_j]$,

$$-\varepsilon/2 + \min_{1\leq j\leq m}\{-|\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})|\} \leq \mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1}) \leq \max_{2\leq j\leq m} |\mathbb{H}_n(s_{j-1}) - \mathbb{H}_n(t_{i-1})| + \varepsilon/2,$$

and thus that

$$\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \leq \max_{1\leq j\leq m} |\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})| + \varepsilon/2.$$

Hence,

$$\mathrm{P}\left\{\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t)-\mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \mathrm{P}\left\{\max_{1\leq j\leq m} |\mathbb{H}_n(s_j)-\mathbb{H}_n(t_{i-1})| \geq \varepsilon/2\right\}. \quad \text{(D.3)}$$

Now, let $\zeta_l = \mathbb{H}_n(s_l)-\mathbb{H}_n(s_{l-1})$, $l \in \{1,\dots,m\}$ with $\zeta_0 = 0$, and let $S_j = \sum_{l=0}^{j} \zeta_l$, $j \in \{0,\dots,m\}$. From (D.2), we then have that, for any $0 \leq j < j' \leq m$ and n sufficiently large,

$$\mathrm{E}\{(S_{j'}-S_j)^4\} = \mathrm{E}\left\{\left(\sum_{l=j+1}^{j'} \zeta_l\right)^4\right\} = \mathrm{E}\left[\{\mathbb{H}_n(s_{j'})-\mathbb{H}_n(s_j)\}^4\right]$$
$$\leq 2\lambda(s_{j'}-s_j)^\eta = 2\lambda\left\{\sum_{j<l\leq j'} (s_l - s_{l-1})\right\}^\eta.$$

Indeed, by construction of the subgrid, for any $0 \leq j < j' \leq m$, $n^{-1/2}\varepsilon/4 \leq G(s_j) - G(s_{j'}) \leq s_{j'} - s_j$, and $n^{-1/2}\varepsilon/4$ can be made larger than $n^{-2/\eta}$ by taking n sufficiently large since $2/\eta > 1/2$. The assumption of Theorem 2.12 of Billingsley (1968) being satisfied (see also Lemma 2.10 in Dehling and Philipp, 2002), we obtain that there exists a constant $K \geq 0$ such that, for any $\nu \geq 0$,

$$\mathrm{P}\left(\max_{1\leq j\leq m} |S_j| \geq \nu\right) \leq \nu^{-4}K(s_m - s_0)^\eta = \nu^{-4}K(t_i - t_{i-1})^\eta.$$

Applying the previous inequality to the right-hand side of (D.3), we obtain that

$$\mathrm{P}\left\{\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t)-\mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \varepsilon^{-4}2^4K(t_i - t_{i-1})^\eta.$$

It follows that

$$\mathrm{P}\left\{\max_{1\leq i\leq k}\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t)-\mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \varepsilon^{-4}2^4K\sum_{i=1}^{k}(t_i - t_{i-1})^\eta$$
$$\leq \varepsilon^{-4}2^4K \times \max_{1\leq i\leq k}(t_i - t_{i-1})^{\eta-1} \times \sum_{i=1}^{k}(t_i - t_{i-1}).$$

By choosing the initial grid such that $\max_{1\leq i\leq k}(t_i - t_{i-1}) \leq \{\delta\varepsilon^4 2^{-4}K^{-1}\}^{1/(\eta-1)}$, we obtain (D.1).

It remains to verify that $\mathbb{H}_n$ is asymptotically $|\cdot|$-equicontinuous in probability. By Problem 2.1.5 in van der Vaart and Wellner (2000), this amounts to showing that for any positive sequence $a_n \downarrow 0$ and any $\varepsilon, \delta > 0$,

$$\mathrm{P}\left\{\sup_{\substack{s,t\in[0,1]\\|t-s|\leq a_n}} |\mathbb{H}_n(s)-\mathbb{H}_n(t)| > 3\varepsilon\right\} \leq \delta \quad \text{(D.4)}$$

for n sufficiently large. Fix $\varepsilon, \delta > 0$ and $a_n \downarrow 0$, and choose a grid $0 = t_0 < \cdots < t_k = 1$ such that (D.1) holds for all n sufficiently large. Furthermore, let $\mu = \min_{1<i<k}(t_i - t_{i-1})$. Then, from Billingsley (1999, Theorem 7.4), we have that, for all n sufficiently large such that $a_n \leq \mu$,

$$\sup_{\substack{s,t\in[0,1]\\|t-s|\leq a_n}} |\mathbb{H}_n(s) - \mathbb{H}_n(t)| \leq 3 \max_{1\leq i\leq k} \sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})|.$$

Finally, (D.4) follows for all n sufficiently large by combining the previous inequality with (D.1). ■

Proof of Proposition 6. We shall only prove the result under (ii), the proof being simpler under (i). Recall $\sigma^2_{n,C,f}$ defined in (33). From (35), we immediately have that $\sigma^2_{n,C,f} \xrightarrow{\mathrm{P}} \sigma^2_{C,f}$. It remains to show that $\check{\sigma}^2_{n,C_{1:n},f} - \sigma^2_{n,C,f} \xrightarrow{\mathrm{P}} 0$.

Recall $\boldsymbol{h}_{1:n}$ defined in (A.1) and that $\hat{\boldsymbol{U}}_i^{1:n} = \boldsymbol{h}_{1:n}(\boldsymbol{U}_i)$ for all $i \in \{1,\dots,n\}$. Then, starting from (30) and (34), it can be verified that

$$\begin{aligned}
|\check{\sigma}^2_{n,C_{1:n},f} - \sigma^2_{n,C,f}| \leq & \left\{ \frac{1}{n} \sum_{i,j=1}^n \varphi\left(\frac{i-j}{\ell_n}\right) \right\} \\
& \times \left[\sup_{\boldsymbol{u}\in[0,1]^d} |f\{\mathcal{I}_C(\boldsymbol{u}) - \psi_C(C)\}| + \sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \psi_{C_{1:n}}(C_{1:n})]| \right] \\
& \times \sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_C(\boldsymbol{u}) - \psi_{C_{1:n}}(C_{1:n}) + \psi_C(C)]| . \qquad \text{(D.5)}
\end{aligned}$$

Some algebra shows that the second term on the right of the previous inequality is smaller than

$$\sup_{\boldsymbol{u}\in[0,1]^d} |f \circ \mathcal{I}_C(\boldsymbol{u})| + |f \circ \psi_C(C)| + 2 \sup_{\boldsymbol{u}\in[0,1]^d} |f \circ \mathcal{I}_{C_{1:n}}(\boldsymbol{u})|.$$

From (22) and (17), we have that, for any $A \subseteq D$, $|A| \geq 1$, $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}(\boldsymbol{u})| \leq 1$, $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C_{1:n},A}(\boldsymbol{u})| \leq 1$ and $|\psi_{C,A}(C)| \leq 1$. Hence, by (20), (28) and linearity of f, we have that the second term (between square brackets) on the right of inequality (D.5) is bounded by $4\sup_{\boldsymbol{x}\in[-1,1]^{2^d-1}} |f(\boldsymbol{x})| < \infty$. Concerning the first term on the right of (D.5), we have

$$\frac{1}{n} \sum_{i,j=1}^n \varphi\left(\frac{i-j}{\ell_n}\right) = \frac{1}{n} \sum_{k=-\ell_n}^{\ell_n} (n - |k|)\varphi\left(\frac{k}{\ell_n}\right) \leq 2\ell_n + 1 = O(n^{1/2-\varepsilon}).$$

We will now show that the last supremum on the right of (D.5) is $O_{\mathrm{P}}(n^{-1/2})$, which will complete the proof. By the triangle inequality,

$$\begin{aligned}
&\sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_C(\boldsymbol{u}) - \psi_{C_{1:n}}(C_{1:n}) + \psi_C(C)]| \\
&\qquad \leq \sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_C(\boldsymbol{u})]| + |f\{\psi_{C_{1:n}}(C_{1:n}) - \psi_C(C)\}| .
\end{aligned}$$

By linearity of f, from (22) and (28), to show that the first term on the right on the previous inequality is $O_{\mathrm{P}}(n^{-1/2})$, it suffices to show that, for any $A \subseteq D$, $|A| \geq 1$,

$$\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C_{1:n},A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}). \tag{D.6}$$

Similarly, for the second term on the right, it suffices to show that, for any $A \subseteq D$, $|A| \geq 1$, $|\psi_{C_{1:n},A}(C_{1:n}) - \psi_{C,A}(C)| = O_{\mathrm{P}}(n^{-1/2})$. Now, from Fubini's theorem, $\psi_{C,A}(C) = \psi_{C,A}[\mathrm{E}\{\mathbf{1}(\boldsymbol{U}_1 \leq \cdot)\}] = \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}$. Hence, $|\psi_{C_{1:n},A}(C_{1:n}) - \psi_{C,A}(C)|$ is smaller than

$$\begin{aligned}
&\left| \frac{1}{n} \sum_{i=1}^n \left\{ \mathcal{I}_{C_{1:n},A}(\hat{\boldsymbol{U}}_i^{1:n}) - \mathcal{I}_{C,A}(\boldsymbol{U}_i) \right\} \right| + \left| \frac{1}{n} \sum_{i=1}^n [\mathcal{I}_{C,A}(\boldsymbol{U}_i) - \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}] \right| \\
&\qquad \leq \sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C_{1:n},A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| + \left| \frac{1}{n} \sum_{i=1}^n [\mathcal{I}_{C,A}(\boldsymbol{U}_i) - \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}] \right|.
\end{aligned}$$

The proof is therefore complete if we show (D.6) and that the second term on the right of the previous inequality is $O_{\mathrm{P}}(n^{-1/2})$. The latter is a consequence of the weak convergence of $n^{-1/2} \sum_{i=1}^n [\mathcal{I}_{C,A}(\boldsymbol{U}_i) - \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}]$ which follows from Theorem 3.23 of Dehling and Philipp (2002) as a consequence of the fact that $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}(\boldsymbol{u})| \leq 1$ and the assumption on the mixing rate.

It remains to prove (D.6). The latter will follow by the triangle inequality if we show that, for any $A \subseteq D$, $|A| \geq 1$,

$$\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}), \tag{D.7}$$

$$\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{H_{1:n},A}(\boldsymbol{u}) - \mathcal{I}_{C,A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}), \tag{D.8}$$

$$\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C_{1:n},A}(\boldsymbol{u}) - \mathcal{I}_{H_{1:n},A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}). \tag{D.9}$$

Fix $A \subseteq D$, $|A| \geq 1$.

Proof of (D.7). We have

$$\begin{aligned}
\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| &\leq \sup_{\boldsymbol{u}\in[0,1]^d} \left| \prod_{l\in A} \{1 - H_{1:n,l}(u_l)\} - \prod_{l\in A} (1 - u_l) \right| \\
&\quad + \sum_{j\in A} \sup_{u\in[0,1]} \left| \int_{[0,1]^d} \prod_{l\in A\setminus\{j\}} (1 - v_l) \left[\mathbf{1}\{H_{1:n,j}(u) \leq v_j\} - \mathbf{1}(u \leq v_j) \right] \mathrm{d}C(\boldsymbol{v}) \right|.
\end{aligned}$$

By an application of the mean value theorem similar to that performed in the proof of Proposition 1, it is easy to verify that the first supremum is $O_{\mathrm{P}}(n^{-1/2})$ since, for any $j \in D$, $\sup_{u\in[0,1]} |H_{1:n,j}(u) - u| = O_{\mathrm{P}}(n^{-1/2})$ as a consequence of the weak convergence of

$\mathbb{B}_n$ defined in (13). The second term is smaller than

$$\sum_{j\in A}\sup_{u\in[0,1]}\int_{[0,1]}|\mathbf{1}\{H_{1:n,j}(u)\leq v\}-\mathbf{1}(u\leq v)|\,\mathrm{d}v$$
$$\leq \sum_{j\in A}\sup_{u\in[0,1]}\int_{[0,1]}\mathbf{1}\{u\wedge H_{1:n,j}(u)\leq v\leq u\vee H_{1:n,j}(u)\}\mathrm{d}v$$
$$=\sum_{j\in A}\sup_{u\in[0,1]}|H_{1:n,j}(u)-u|=O_{\mathrm{P}}(n^{-1/2}).$$

Proof of (D.8)*:* From (22) and the triangle inequality, it suffices to show that, for any $j\in A$,

$$\sup_{u\in[0,1]}\left|\frac{1}{n}\sum_{i=1}^{n}\prod_{l\in A\setminus\{j\}}(1-U_{il})\mathbf{1}(u\leq U_{ij})-\int_{[0,1]^d}\prod_{l\in A\setminus\{j\}}(1-v_l)\mathbf{1}(u\leq v_j)\mathrm{d}C(\boldsymbol{v})\right|=O_{\mathrm{P}}(n^{-1/2}).$$

The latter is an immediate consequence of the weak convergence result stated in Lemma 2 and the continuous mapping theorem.

Proof of (D.9)*:* The supremum on the left of (D.9) is smaller than $I_n+II_n+III_n$, where

$$I_n=\sup_{\boldsymbol{u}\in[0,1]^d}|\mathcal{I}_{C_{1:n},A}(\boldsymbol{u})-\mathcal{I}_{H_{1:n},A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\}|,$$
$$II_n=\sup_{\boldsymbol{u}\in[0,1]^d}|\mathcal{I}_{H_{1:n},A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\}-\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\}-\mathcal{I}_{H_{1:n},A}(\boldsymbol{u})+\mathcal{I}_{C,A}(\boldsymbol{u})|,\tag{D.10}$$
$$III_n=\sup_{\boldsymbol{u}\in[0,1]^d}|\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\}-\mathcal{I}_{C,A}(\boldsymbol{u})|,\tag{D.11}$$

with $\boldsymbol{h}_{1:n}^{-1}$ is defined in (A.2). The term I_n is smaller

$$\sup_{\boldsymbol{u}\in(0,1]^d}\left|\prod_{l\in A}(1-u_l)-\prod_{l\in A}\{1-H_{1:n,l}^{-1}(u_l)\}\right|$$
$$+\sup_{\boldsymbol{u}\in[0,1]^d}\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in A}\prod_{l\in A\setminus\{j\}}\{1-H_{1:n,l}(U_{il})\}\mathbf{1}\{u_j\leq H_{1:n,j}(U_{ij})\}\right.$$
$$\left.-\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in A}\prod_{l\in A\setminus\{j\}}(1-U_{il})\mathbf{1}\{H_{1:n,j}^{-1}(u_j)\leq U_{ij}\}\right|.$$

Since, for any $j\in D$, $\sup_{u\in[0,1]}|H_{1:n,j}^{-1}(u)-u|=\sup_{u\in[0,1]}|H_{1:n,j}(u)-u|$ (for instance, by symmetry arguments on the graphs of $H_{1:n,j}$ and $H_{1:n,j}^{-1}$), and by an application of the mean value theorem as above, we obtain that the first supremum is $O_{\mathrm{P}}(n^{-1/2})$. Using the fact that, for all $u\in[0,1]$, $u\leq H_{1:n,j}(U_{ij})$ is equivalent to $H_{1:n,j}^{-1}(u)\leq U_{ij}$, it can be verified that the second supremum is smaller than

$$\sum_{j\in A}\sup_{u\in[0,1]}\left|\frac{1}{n}\sum_{i=1}^{n}\left[\prod_{l\in A\setminus\{j\}}\{1-H_{1:n,l}(U_{il})\}-\prod_{l\in A\setminus\{j\}}(1-U_{il})\right]\mathbf{1}\{u\le H_{1:n,j}(U_{ij})\}\right|$$
$$\le\sum_{j\in A}\sup_{\boldsymbol{u}\in[0,1]^d}\left|\prod_{l\in A\setminus\{j\}}\{1-H_{1:n,l}(u_l)\}-\prod_{l\in A\setminus\{j\}}(1-u_l)\right|=O_{\mathrm{P}}(n^{-1/2}),$$

where the last equality follows again by an application of the mean value theorem as above. Hence, $I_n=O_{\mathrm{P}}(n^{-1/2})$. For II_n defined in (D.10), we have

$$II_n\le n^{-1/2}\sum_{j\in A}\sup_{u\in[0,1]}\left|\mathbb{H}_{n,A,j}\{H_{1:n,j}^{-1}(u)\}-\mathbb{H}_{n,A,j}(u)\right|=o_{\mathrm{P}}(n^{-1/2}),$$

where $\mathbb{H}_{n,A,j}$ is defined in Lemma 2. The last equality is a consequence of the asymptotic equicontinuity in probability of $\mathbb{H}_{n,A,j}$ and the fact that $\sup_{u\in[0,1]}|H_{1:n,j}^{-1}(u)-u|=\sup_{u\in[0,1]}|H_{1:n,j}(u)-u|\xrightarrow{\text{a.s.}}0$. The latter convergence follows from the almost sure invariance principle established in Berkes and Philipp (1977) and Yoshihara (1979). It implies a functional law of the iterated logarithm for $u\mapsto H_{1:n,j}(u)-u$ as soon as $a>3$, which in turn implies the Glivenko–Cantelli lemma under strong mixing.

It remains to show that III_n defined in (D.11) is $O_{\mathrm{P}}(n^{-1/2})$. The proof of the latter is similar to that of (D.7). ■

Proof of Proposition 7. We only show the result under (ii), the proof being simpler under (i). To prove the desired result, we shall show that $\tilde\sigma^2_{n,b_n,C_{1:n},f}-\check\sigma^2_{n,C_{1:n},f}\xrightarrow{\mathrm{P}}0$. Proceeding as in the proof of Proposition 6 for (D.5), it can be verified that to prove the above, it suffices to show that, for any $A\subseteq D$, $|A|\ge1$,

$$\sup_{\boldsymbol{u}\in[0,1]^d}|\mathcal{I}_{b_n,C_{1:n},A}(\boldsymbol{u})-\mathcal{I}_{C_{1:n},A}(\boldsymbol{u})|=O_{\mathrm{P}}(n^{-1/2}).$$

Fix $A\subseteq D$, $|A|\ge1$. From (22) and (26), we have that the supremum on the right of the previous display is smaller than $\sum_{j\in A}I_{n,j}$, where

$$I_{n,j}=\sup_{u\in[0,1]}\int_{[0,1]^d}|\mathcal{L}_{b_n}(u,v_j)-\mathbf{1}(u\le v_j)|\,\mathrm{d}C_{1:n}(\boldsymbol{v}).$$

Fix $j\in A$. From (C.1), we have that $I_{n,j}\le n^{-1/2}J_{n,j}$, where

$$\begin{aligned}J_{n,j}&=\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\mathbf{1}(u_-\le\hat U_{ij}^{1:n})-\mathbf{1}(u_+\le\hat U_{ij}^{1:n})\}\\&=\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\mathbf{1}(\hat U_{ij}^{1:n}<u_+)-\mathbf{1}(\hat U_{ij}^{1:n}<u_-)\}\\&\le\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\mathbf{1}(\hat U_{ij}^{1:n}\le u_+)-\mathbf{1}(\hat U_{ij}^{1:n}\le u_-)\}+\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}(\hat U_{ij}^{1:n}=u).\end{aligned}$$

Proceeding as for (C.2), we obtain that the first supremum on the right of the previous display converges in probability to zero. The second supremum is smaller than

$$\sup_{u\in[0,1]} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{\mathbf{1}(\hat{U}_{ij}^{1:n} \leq u) - \mathbf{1}(\hat{U}_{ij}^{1:n} \leq u - 1/n)\}$$

and can be dealt with along the same lines. Hence, $J_{n,j} \xrightarrow{\mathrm{P}} 0$, which implies that $I_{n,j} = o(n^{-1/2})$ and completes the proof. ■

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